

VISIT...

LANZAROTE
Caliente.COM

Wetland Ecology and Management



Introduction to Management Science Edition

By Bernard W. Taylor III
University of North Carolina at Chapel Hill

Publisher : Prentice Hall

Pub Date : February 2003

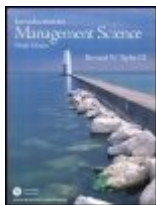
Print ISBN-10 : 0-13-022544-4

Print ISBN-13 : 978-0-13-022544-4

eText ISBN-10 : 0-13-022545-2

eText ISBN-13 : 978-0-13-022545-2

Pages : 800



- [Table of Contents](#)
- [Index](#)

This text focuses on using

and examples with step-by solution techniques to make complex.

Introduction to Management Science 7th Edition

By Bernard W. Taylor III
University of North Carolina at Chapel Hill

Publisher : Prentice Hall

Pub Date : February 2008

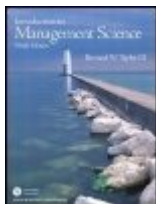
Print ISBN-10 : 0-13-035927-7

Print ISBN-13 : 978-0-13-035927-7

eText ISBN-10 : 0-13-035928-4

eText ISBN-13 : 978-0-13-035928-4

Pages : 800



- [Table of Contents](#)
- [Index](#)

Copyright

<u>Preface</u>	<u>xiii</u>
<u>Learning Features</u>	<u>xiii</u>
<u>Microsoft Project</u>	<u>xvi</u>
<u>Instructors' and Students'</u>	<u>xvii</u>
<u>Supplements</u>	
<u>Acknowledgments</u>	<u>xviii</u>
<u>Chapter 1.</u>	<u>1</u>
<u>Management Science</u>	
<u>The Management Science Approach to Problem Solving</u>	<u>2</u>
<u>Model Building: Break-Even Analysis</u>	<u>7</u>
<u>Computer Solution</u>	<u>12</u>
<u>Management Science Modeling Techniques</u>	<u>16</u>
<u>Business Usage of Management Science Techniques</u>	<u>18</u>

<u>Management Science</u>	
<u>Models in Decision Support Systems</u>	<u>20</u>
<u>Summary</u>	<u>22</u>
<u>References</u>	<u>23</u>
<u>Problems</u>	<u>23</u>
<u>Case Problem</u>	<u>27</u>
<u>Case Problem</u>	<u>27</u>
<u>Case Problem</u>	<u>28</u>
<u>Chapter 2. Linear Programming: Model Formulation and Graphical Solution</u>	
<u>Model Formulation</u>	<u>30</u>
<u>A Maximization Model Example</u>	<u>30</u>
<u>Graphical Solutions of Linear Programming Models</u>	<u>34</u>
<u>A Minimization Model</u>	<u>47</u>

<u>Example</u>	
<u>Irregular Types of</u> <u>Linear Programming</u> <u>Problems</u>	<u>53</u>
<u>Characteristics of</u> <u>Linear Programming</u> <u>Problems</u>	<u>55</u>
<u>Summary</u>	<u>57</u>
<u>References</u>	<u>57</u>
<u>Example Problem</u> <u>Solutions</u>	<u>57</u>
<u>Problems</u>	<u>61</u>
<u>Case Problem</u>	<u>69</u>
<u>Case Problem</u>	<u>70</u>
<u>Case Problem</u>	<u>70</u>
<u>Chapter 3. Linear</u> <u>Programming:</u> <u>Computer Solution and</u> <u>Sensitivity Analysis</u>	<u>71</u>
<u>Computer Solution</u>	<u>72</u>

<u>Sensitivity Analysis</u>	<u>79</u>
<u>Summary</u>	<u>92</u>
<u>References</u>	<u>92</u>
<u>Example Problem</u>	<u>93</u>
<u>Solution</u>	
<u>Problems</u>	<u>95</u>
<u>Case Problem</u>	<u>107</u>
<u>Case Problem</u>	<u>108</u>
<u>Case Problem</u>	<u>108</u>
<u>Chapter 4. Linear</u>	
<u>Programming: Modeling</u>	<u>110</u>
<u>Examples</u>	
<u>A Product Mix</u>	<u>111</u>
<u>Example</u>	
<u>A Diet Example</u>	<u>116</u>
<u>An Investment</u>	
<u>Example</u>	<u>119</u>
<u>A Marketing Example</u>	<u>125</u>
<u>A Transportation</u>	
<u>Example</u>	<u>129</u>

<u>A Blend Example</u>	<u>132</u>
<u>A Multiperiod Scheduling Example</u>	<u>135</u>
<u>A Data Envelopment Analysis Example</u>	<u>140</u>
<u>Summary</u>	<u>144</u>
<u>References</u>	<u>145</u>
<u>Example Problem Solution</u>	<u>145</u>
<u>Problems</u>	<u>147</u>
<u>Case Problem</u>	<u>176</u>
<u>Case Problem</u>	<u>176</u>
<u>Case Problems</u>	<u>177</u>
<u>Case Problem</u>	<u>178</u>
<u>Case Problems</u>	<u>179</u>
<u>Chapter 5. Integer Programming</u>	<u>180</u>
<u>Integer Programming Models</u>	<u>181</u>
<u>Integer Programming</u>	

Graphical Solution 185

Computer Solution of
Integer Programming
Problems with Excel 187
and QM for Windows

01 Integer
Programming 194
Modeling Examples

Summary 203

References 203

Example Problem
Solution 203

Problems 204

Case Problem 216

Case Problems 217

Case Problem 218

Case Problems 219

Case Problem 220

Chapter 6.
Transportation.

<u>Transshipment, and Assignment Problems</u>	<u>222</u>
<u>The Transportation Model</u>	<u>223</u>
<u>Computer Solution of a Transportation Problem</u>	<u>226</u>
<u>The Transshipment Model</u>	<u>230</u>
<u>The Assignment Model</u>	<u>234</u>
<u>Computer Solution of an Assignment Problem</u>	<u>235</u>
<u>Summary</u>	<u>238</u>
<u>References</u>	<u>239</u>
<u>Example Problem Solution</u>	<u>239</u>
<u>Problems</u>	<u>240</u>

<u>Case Problem</u>	<u>266</u>
<u>Case Problem</u>	<u>266</u>
<u>Case Problem</u>	<u>267</u>
<u>Case Problem</u>	<u>268</u>
<u>Case Problem</u>	<u>269</u>
<u>Case Problem</u>	<u>270</u>
<u>Chapter 7. Network Flow Models</u>	<u>272</u>
<u>Network Components</u>	<u>273</u>
<u>The Shortest Route Problem</u>	<u>274</u>
<u>The Minimal Spanning Tree Problem</u>	<u>281</u>
<u>The Maximal Flow Problem</u>	<u>286</u>
<u>Summary</u>	<u>292</u>
<u>References</u>	<u>293</u>
<u>Example Problem Solution</u>	<u>293</u>

<u>Problems</u>	<u>295</u>
<u>Case Problem</u>	<u>314</u>
<u>Case Problem</u>	<u>315</u>
<u>Case Problem</u>	<u>317</u>
<u>Case Problem</u>	<u>319</u>
<u>Chapter 8. Project Management</u>	<u>320</u>
<u>The Elements of Project Management</u>	<u>321</u>
<u>CPM/PERT</u>	<u>329</u>
<u>Probabilistic Activity Times</u>	<u>338</u>
<u>Microsoft Project</u>	<u>345</u>
<u>Project Crashing and TimeCost Trade-off</u>	<u>352</u>
<u>Formulating the CPM/PERT Network as a Linear Programming Model</u>	<u>357</u>
<u>Summary</u>	<u>364</u>

<u>References</u>	<u>364</u>
<u>Example Problem Solution</u>	<u>365</u>
<u>Problems</u>	<u>367</u>
<u>Case Problem</u>	<u>386</u>
<u>Case Problem</u>	<u>387</u>
<u>Chapter 9. Multicriteria Decision Making</u>	<u>389</u>
<u>Goal Programming</u>	<u>390</u>
<u>Graphical Interpretation of Goal Programming</u>	<u>394</u>
<u>Computer Solution of Goal Programming Problems with QM for Windows and Excel</u>	<u>397</u>
<u>The Analytical Hierarchy Process</u>	<u>404</u>
<u>Scoring Models</u>	<u>416</u>

<u>Summary</u>	<u>418</u>
<u>References</u>	<u>418</u>
<u>Example Problems</u>	<u>418</u>
<u>Solutions</u>	
<u>Problems</u>	<u>422</u>
<u>Case Problem</u>	<u>447</u>
<u>Case Problem</u>	<u>448</u>
<u>Case Problem</u>	<u>449</u>
<u>Chapter 10. Nonlinear</u>	
<u>Programming</u>	<u>451</u>
<u>Nonlinear Profit</u>	
<u>Analysis</u>	<u>452</u>
<u>Constrained</u>	
<u>Optimization</u>	<u>455</u>
<u>Solution of Nonlinear</u>	
<u>Programming</u>	<u>458</u>
<u>Problems with Excel</u>	
<u>A Nonlinear</u>	
<u>Programming Model</u>	
<u>with Multiple</u>	<u>461</u>

<u>Constraints</u>	
<u>Nonlinear Model</u>	<u>463</u>
<u>Examples</u>	
<u>Summary</u>	<u>468</u>
<u>References</u>	<u>469</u>
<u>Example Problem</u>	<u>469</u>
<u>Solution</u>	
<u>Problems</u>	<u>469</u>
<u>Case Problem</u>	<u>474</u>
<u>Case Problem</u>	<u>475</u>
<u>Chapter 11. Probability</u>	<u>476</u>
<u>and Statistics</u>	
<u>Types of Probability</u>	<u>477</u>
<u>Fundamentals of</u>	<u>479</u>
<u>Probability</u>	
<u>Statistical</u>	
<u>Independence and</u>	<u>483</u>
<u>Dependence</u>	
<u>Expected Value</u>	<u>490</u>
<u>The Normal</u>	

<u>Distribution</u>	<u>492</u>
<u>Summary</u>	<u>504</u>
<u>References</u>	<u>504</u>
<u>Example Problem Solution</u>	<u>504</u>
<u>Problems</u>	<u>506</u>
<u>Case Problem</u>	<u>513</u>
<u>Chapter 12. Decision Analysis</u>	<u>514</u>
<u>Components of Decision Making</u>	<u>515</u>
<u>Decision Making Without Probabilities</u>	<u>516</u>
<u>Decision Making with Probabilities</u>	<u>523</u>
<u>Decision Analysis with Additional Information</u>	<u>538</u>
<u>Utility</u>	<u>545</u>

<u>Summary</u>	<u>547</u>
<u>References</u>	<u>547</u>
<u>Example Problem</u>	<u>547</u>
<u>Solution</u>	
<u>Problems</u>	<u>551</u>
<u>Case Problem</u>	<u>568</u>
<u>Case Problem</u>	<u>569</u>
<u>Case Problem</u>	<u>569</u>
<u>Case Problem</u>	<u>570</u>
<u>Chapter 13. Queuing</u>	
<u>Analysis</u>	<u>572</u>
<u>Elements of Waiting</u>	
<u>Line Analysis</u>	<u>573</u>
<u>The Single-Server</u>	
<u>Waiting Line System</u>	<u>574</u>
<u>Undefined and</u>	
<u>Constant Service</u>	<u>584</u>
<u>Times</u>	
<u>Finite Queue Length</u>	<u>587</u>

<u>Finite Calling Population</u>	<u>590</u>
<u>The Multiple-Server Waiting Line</u>	<u>593</u>
<u>Additional Types of Queuing Systems</u>	<u>598</u>
<u>Summary</u>	<u>599</u>
<u>References</u>	<u>599</u>
<u>Example Problem Solution</u>	<u>600</u>
<u>Problems</u>	<u>601</u>
<u>Case Problems</u>	<u>609</u>
<u>Case Problem</u>	<u>610</u>
<u>Chapter 14. Simulation</u>	<u>611</u>
<u>The Monte Carlo Process</u>	<u>612</u>
<u>Computer Simulation with Excel</u>	<u>617</u>
<u>Spreadsheets</u>	

<u>Simulation of a Queuing System</u>	<u>623</u>
<u>Continuous Probability Distributions</u>	<u>627</u>
<u>Statistical Analysis of Simulation Results</u>	<u>632</u>
<u>Crystal Ball</u>	<u>634</u>
<u>Verification of the Simulation Model</u>	<u>643</u>
<u>Areas of Simulation Application</u>	<u>643</u>
<u>Summary</u>	<u>645</u>
<u>References</u>	<u>646</u>
<u>Example Problem Solution</u>	<u>647</u>
<u>Problems</u>	<u>649</u>
<u>Case Problem</u>	<u>663</u>
<u>Case Problem</u>	<u>664</u>

<u>Chapter 15.</u> <u>Forecasting</u>	<u>666</u>
 <u>Forecasting</u> <u>Components</u>	 <u>667</u>
 <u>Time Series Methods</u>	 <u>669</u>
<u>Forecast Accuracy</u>	<u>683</u>
<u>Time Series</u> <u>Forecasting Using</u> <u>Excel</u>	 <u>688</u>
 <u>Time Series</u> <u>Forecasting Using</u> <u>QM for Windows</u>	 <u>690</u>
<u>Regression Methods</u>	<u>691</u>
 <u>Summary</u>	 <u>702</u>
<u>References</u>	<u>702</u>
<u>Example Problem</u> <u>Solutions</u>	 <u>703</u>
<u>Problems</u>	<u>705</u>

<u>Case Problem</u>	<u>725</u>
<u>Case Problem</u>	<u>726</u>
<u>Case Problem</u>	<u>726</u>
<u>Chapter 16. Inventory Management</u>	<u>728</u>
<u>Elements of Inventory Management</u>	<u>729</u>
<u>Inventory Control Systems</u>	<u>731</u>
<u>Economic Order Quantity Models</u>	<u>733</u>
<u>The Basic EOQ Model</u>	<u>733</u>
<u>The EOQ Model with Noninstantaneous Receipt</u>	<u>739</u>
<u>The EOQ Model with Shortages</u>	<u>742</u>

<u>EOQ Analysis with QM for Windows</u>	<u>746</u>
<u>EOQ Analysis with Excel and Excel QM</u>	<u>746</u>
<u>Quantity Discounts</u>	<u>747</u>
<u>Reorder Point</u>	<u>752</u>
<u>Determining Safety Stock By Using Service Levels</u>	<u>754</u>
<u>Order Quantity for a Periodic Inventory System</u>	<u>758</u>
<u>Summary</u>	<u>759</u>
<u>References</u>	<u>760</u>
<u>Example Problem Solutions</u>	<u>760</u>
<u>Problems</u>	<u>761</u>
<u>Case Problem</u>	<u>769</u>
<u>Case Problem</u>	<u>769</u>

<u>Case Problem</u>	<u>770</u>
<u>Case Problem</u>	<u>770</u>
<u>Appendix A. Normal and Chi-Square Tables</u>	<u>773</u>
<u>Appendix B. Setting Up and Editing a Spreadsheet</u>	<u>775</u>
<u>Titles and Headings</u>	<u>775</u>
<u>Borders</u>	<u>775</u>
<u>Column Centering</u>	<u>776</u>
<u>Deleting and Inserting Rows and Columns</u>	<u>776</u>
<u>Decimal Places</u>	<u>776</u>
<u>Increasing or Decreasing the Spreadsheet Area</u>	<u>777</u>
<u>Expanding or</u>	

<u>Reducing Column and Row Widths</u>	<u>777</u>
<u>Inserting an Equation or a Formula into a Cell</u>	<u>777</u>
<u>Printing a Spreadsheet</u>	<u>777</u>
<u>Appendix C. The Poisson and Exponential Distributions</u>	<u>779</u>
<u>The Poisson Distribution</u>	<u>779</u>
<u>The Exponential Distribution</u>	<u>780</u>
<u>Solutions to Selected Odd-Numbered Problems</u>	<u>781</u>

<u>Chapter 1</u>	<u>781</u>
<u>Chapter 2</u>	<u>781</u>
<u>Chapter 3</u>	<u>781</u>
<u>Chapter 4</u>	<u>782</u>
<u>Chapter 5</u>	<u>783</u>
<u>Chapter 6</u>	<u>784</u>
<u>Chapter 7</u>	<u>784</u>
<u>Chapter 8</u>	<u>785</u>
<u>Chapter 9</u>	<u>785</u>
<u>Chapter 10</u>	<u>786</u>
<u>Chapter 11</u>	<u>786</u>
<u>Chapter 12</u>	<u>786</u>
<u>Chapter 13</u>	<u>786</u>
<u>Chapter 14</u>	<u>787</u>
<u>Chapter 15</u>	<u>787</u>
<u>Chapter 16</u>	<u>787</u>
<u>Glossary</u>	<u>789</u>

<u>A</u>	<u>789</u>
<u>B</u>	<u>789</u>
<u>C</u>	<u>789</u>
<u>D</u>	<u>790</u>
<u>E</u>	<u>790</u>
<u>F</u>	<u>790</u>
<u>G</u>	<u>790</u>
<u>H</u>	<u>790</u>
<u>I</u>	<u>790</u>
<u>J</u>	<u>791</u>
<u>L</u>	<u>791</u>
<u>M</u>	<u>791</u>
<u>N</u>	<u>791</u>
<u>O</u>	<u>791</u>
<u>P</u>	<u>791</u>
<u>Q</u>	<u>792</u>
<u>R</u>	<u>792</u>

<u>S</u>	<u>792</u>
<u>T</u>	<u>793</u>
<u>U</u>	<u>793</u>
<u>V</u>	<u>793</u>
<u>W</u>	<u>793</u>
<u>Z</u>	<u>793</u>
<u>Photo Credits</u>	<u>803</u>
<u>Module A. The Simplex Solution Method</u>	<u>A-1</u>
<u> Converting the Model into Standard Form</u>	<u>A-2</u>
<u> The Simplex Method</u>	<u>A-5</u>
<u> Summary of the Simplex Method</u>	<u>A-16</u>
<u> Simplex Solution of a Minimization Problem</u>	<u>A-16</u>
<u> A Mixed Constraint</u>	

Problem A-21

Irregular Types of
Linear Programming A-23
Problems

The Dual A-30

Sensitivity Analysis A-34

Problems A-43

Module B.
Transportation and
Assignment Solution B-1
Methods

Solution of the
Transportation Model B-2

Solution of the
Assignment Model B-22

Problems B-25

Module C. Integer

<u>Programming: The Branch and Bound Method</u>	<u>C-1</u>
---	------------

<u>The Branch and Bound Method</u>	<u>C-2</u>
--	------------

<u>Problems</u>	<u>C-11</u>
-----------------	-------------

<u>Module D. Nonlinear Programming Solution Techniques</u>	<u>D-1</u>
--	------------

<u>The Substitution Method</u>	<u>D-2</u>
------------------------------------	------------

<u>The Method of Lagrange Multipliers</u>	<u>D-4</u>
---	------------

<u>Problems</u>	<u>D-7</u>
-----------------	------------

<u>Module E. Game Theory</u>	<u>E-1</u>
----------------------------------	------------

<u>Game Theory</u>	<u>E-2</u>
<u>Types of Game Situations</u>	<u>E-2</u>
<u>A Pure Strategy</u>	<u>E-3</u>
<u>A Mixed Strategy</u>	<u>E-6</u>
<u>Problems</u>	<u>E-10</u>
<u>Module F. Markov Analysis</u>	<u>F-1</u>
<u>The Characteristics of Markov Analysis</u>	<u>F-2</u>
<u>The Transition Matrix</u>	<u>F-5</u>
<u>Steady-State Probabilities</u>	<u>F-8</u>
<u>Additional Examples of Markov Analysis</u>	<u>F-12</u>
<u>Special Types of Transition Matrices</u>	<u>F-13</u>

[Excel Solution of the Debt Example Problems](#) [F-16](#)
[F-17](#)

[Case Problem](#) [F-26](#)
[Case Problem](#) [F-26](#)

[Have You Thought About Customizing This Book?](#) [InsideFrontCover](#)

[The Prentice Hall Just-In-Time Program in Decision Science](#) [InsideFrontCover](#)

[You Can Customize Your Textbook With Chapters From Any Of The Following Prentice Hall Titles](#) [InsideFrontCover](#)

[Site License Agreement and Limited Warranty](#) [InsidebackCover](#)

[Index](#)

Copyright

[Page ii]

Library of Congress Cataloging-in-Publication Data

Taylor, Bernard W.

Introduction to management
p. cm.

Includes bibliographical re
ISBN 0-13-196133-0

1. Management science. I.

T56T38 2007

AVP/Executive Editor:

Mark Pfaltzgraff

Senior Project Manager:

Alana Bradley

Editorial Assistant:

Barbara Witmer

Product Development

Manager, Media: Nancy

Welcher

AVP/Executive

Marketing Manager:

Debbie Clare

Marketing Assistant:

Joanna Sabella

Senior Managing Editor (Production): Cynthia

Regan

Production Editor:

Denise Culhane

Permissions

Supervisor: Charles

Morris

Production Manager:

Arnold Vila

Interior Design: Kevin

Kall

Cover Design: Bruce

Kenselaar

Cover

Illustration/Photo:

Terry Donnelly / Image
Bank / Getty Images, Inc.

Director, Image

Resource Center:

Melinda Reo

Manager, Rights and Permissions: Zina Arabia

Manager: Visual
Research: Beth Brenzel

Manager, Cover Visual Research & Permissions: Karen
Sanatar

Image Permission Coordinator: Jennifer
Puma

Photo Researcher:
Joanne Dippel

**Multimedia Print
Production Manager:**
Christy Mahon

**Composition/Full-
Service Project
Management:** GGS Book
Services

Printer/Binder: RR
Donnelley-Willard

Typeface: 10/12 Minion

Credits and acknowledgments
borrowed from other sources
and reproduced, with

permission, in this textbook appear on appropriate page within text and on page [803](#).

Microsoft® and Windows® are registered trademarks of the Microsoft Corporation in the U.S.A. and other countries. Screen shots and icons reprinted with permission from the Microsoft Corporation. This book is not sponsored or endorsed by or affiliated with the Microsoft Corporation.

**Copyright © 2007, 2004,
2002, 1999, 1996 by
Pearson Education, Inc.,**

Upper Saddle River, New Jersey, 07458. Pearson Prentice Hall. All rights reserved. Printed in the United States of America. This publication is protected by Copyright and permission should be obtained from the publisher prior to any prohibited reproduction, storage in a retrieval system, or transmission in any form or by any means, electronic, mechanical, photocopying, recording, or likewise. For information regarding permission(s), write to: Rights

and Permissions Department.

Pearson Prentice Hall™ is a trademark
Pearson® is a registered trademark
Prentice Hall® is a registered trademark

Pearson Education LTD.

Pearson Education Singapore, Pte Ltd

Pearson Education, Canada, Ltd

Pearson Education-Japan

Pearson Education Australia PTY Ltd

Pearson Education North Asia Ltd

Pearson Educación de México, S. de CV

Pearson Education Malaysia, Pte Ltd

10 9 8 7 6 5 4 3 2 1

Dedication

***To Diane, Kathleen, and
Lindsey***

***To the memory of my
grandfather, Bernard W.
Taylor, Sr.***

Preface

The objective of management science is to solve the decision-making problems that confront and confound managers in both the public and the private sector by developing mathematical models of those problems. These models have traditionally been solved with various mathematical techniques, all of which lend themselves to specific types of problems. Thus, management science as a field of study has always been inherently

mathematical in nature, and as a result sometimes complex and rigorous. When I began writing the first edition of this book in 1979, my main goal was to make these mathematical topics seem less complex and thus more palatable to undergraduate business students. To achieve this goal I started out by trying to provide simple, straightforward explanations of often difficult mathematical topics. I tried to use lots of examples that demonstrated in detail the fundamental

mathematical steps of the modeling and solution techniques. Although in the last two and a half decades the emphasis in management science has shifted away from strictly mathematical to mostly computer solutions, my objective has not changed. I have provided clear, concise explanations of the techniques used in management science to model problems, and provided lots of examples of how to solve these models on the computer, while still including some of the fundamental

mathematics of the techniques.

The stuff of management science can seem abstract, and students sometimes have trouble perceiving the usefulness of quantitative courses in general. I remember when I was a student I could not foresee how I would use such mathematical topics (in addition to a lot of the other things I learned in college) in any job after graduation. Part of the problem is that the examples used in books often do not seem realistic. Unfortunately, examples must

be made simple to facilitate the learning process. Larger, more complex examples reflecting actual applications would be too complex to help the student learn the modeling technique. The modeling techniques presented in this text are, in fact, used extensively in the business world and their use is increasing rapidly because of computer and information technology. Therefore, the chances of students using the modeling techniques that they learn from this text in a future job are very great indeed.

Even if these techniques are not used on the job, the logical approach to problem solving embodied in management science is valuable for all types of jobs in all types of organizations. Management science consists of more than just a collection of mathematical modeling techniques; it embodies a philosophy of approaching a problem in a logical manner, as does any science. Thus, this text not only teaches specific techniques but also provides a very useful method for

approaching problems.

My primary objective throughout all revisions of this text is readability. The modeling techniques presented in each chapter are explained with straightforward examples that avoid lengthy written explanations. These examples are organized in a logical step-by-step fashion that the student can subsequently apply to the Problems at the end of each chapter. I have tried to avoid complex mathematical notation and formulas wherever possible. These

various factors will, I hope,
help make the material more
interesting and less
intimidating to students.

Learning Features

This ninth edition of *Introduction to Management Science* includes many features that are designed to help sustain and accelerate the student's learning of the material. Some of these features remain from the previous editions while others are new to this edition. Several of the strictly mathematical topics like the simplex and transportation solution

methods are on the accompanying CD-ROM. This frees up text space for additional modeling examples in several of the chapters, allowing more emphasis on computer solutions with Excel spreadsheets, and added additional homework problems. In the following sections, we will summarize these and other learning features that appear in the text.

Text Organization

An important objective is to have a well-organized text that flows smoothly and follows a logical progression of topics, placing the different management science modeling techniques in their proper perspective. The first 10 chapters group together those chapters related to mathematical programming that can be solved using Excel spreadsheets, including linear, integer, nonlinear, and goal programming as well as network techniques.

Within these mathematical

programming chapters the traditional simplex procedure for solving linear programming problems mathematically is located on the CD-ROM that accompanies this text. It can still be covered by the student on the computer as part of linear programming or it can be excluded, without leaving a "hole" in the presentation of this topic. The integer programming mathematical branch and bound solution method is also on the CD-ROM. In [Chapter 6](#), on the transportation and assignment

problems, the strictly mathematical solution approaches, including the northwest corner, VAM, and steppingstone methods, are also on the accompanying CD-ROM. Since transportation and assignment problems are specific types of network problems, the two chapters that cover network flow models and project networks that can be solved with linear programming, as well as traditional model-specific solution techniques and software, follow [Chapter 6](#) on

transportation and assignment problems. In addition, in [Chapter 10](#), on nonlinear programming, the traditional mathematical solution techniques, including the substitution method and the method of Lagrange multipliers, are on the CD-ROM.

[Page xiv]

[Chapters 11](#) through [14](#) include topics generally thought of as being probabilistic, including probability and statistics, decision analysis, queuing, and

simulation. A module on Markov analysis is on the accompanying CD-ROM. Also, a module on game theory, is on the CD-ROM. Forecasting in [Chapters 15](#) and inventory in [Chapter 16](#) are both unique topics related to operations management.

New Topics and Sections in This Edition

In an effort to keep the book current and abreast of contemporary trends in

management science, and especially the increased emphasis on model development and solution with Excel spreadsheets, several chapters have been altered to include new sections. Specifically, [Chapter 8](#), on project management, has been completely rewritten to focus on activity-on-node networks and Microsoft Project. Also included are new sections Project Planning, The Project Team, Scope Statement, Work Breakdown Structure, Responsibility Assignment

Matrix, Project Scheduling, and Project Control.

Excel Spreadsheets

This new edition continues to emphasize Excel spreadsheet solutions of problems.

Spreadsheet solutions are demonstrated in all the chapters in the text (except for [Chapter 2](#), on linear programming modeling and graphical solution), for virtually every management science modeling technique presented.

These spreadsheet solutions are presented in optional subsections, allowing the instructor to decide whether to cover them. The text includes over 175 Excel spreadsheet screens, most of which include reference callout boxes that describe the solution steps within the spreadsheet. Files that include all the Excel spreadsheet model solutions for the examples in the text are included on the accompanying CD-ROM, and can be easily downloaded by the student to determine how the spreadsheet

was set up and the solution derived, and to use as templates to work homework problems. In addition, [Appendix B](#) at the end of the text provides a tutorial on how to set up and edit spreadsheets for problem solution. Following is an example of one of the Excel spreadsheet files (from [Chapter 3](#)) that is available on the CD-ROM accompanying the text.

[\[View full size image\]](#)

Microsoft Excel - Ex0803.1.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

Formulas: =C4D10-D4E11

1 The Beaver Creek Pottery Company

Products	Bowl	Mug	Resources			
Feet per unit	40	10	Usage	Constraint	Available	Left over
Resources						
Kiln (hours)	1	2	40	<=	40	0
Clay (lb/ton)	4	3	120	<=	120	0

Production

Bowls =	24
Mugs =	8
Profit =	1.50

maximize $Z = 40x_1 + 50x_2$

subject to

$$1x_1 + 2x_2 \leq 40$$

$$4x_1 + 3x_2 \leq 120$$

$$x_1, x_2 \geq 0$$

Spreadsheet "Add-Ins"

Several spreadsheet add-in packages are provided on the CD-ROM that is packaged with every copy of this text, as follows:

Excel QM

For some management science topics, the Excel formulas that are required for solution are lengthy and complex and, thus, are very tedious and time-consuming to type into a spreadsheet. In several of these instances in the book, including [Chapter 6](#) on transportation and assignment problems, [Chapter 12](#) on decision analysis, [Chapter 13](#) on queuing, [Chapter 15](#) on forecasting, and [Chapter 16](#) on inventory control, a

spreadsheet "add-in" called Excel QM is demonstrated. These add-ins provide a generic spreadsheet set-up with easy-to-use dialog boxes and all of the formulas already typed in for specific problem types. Unlike other "black box" software, these add-ins allow users to see the formulas used in each cell. The input, results, and the graphics are easily seen and can be easily changed, making this software ideal for classroom demonstrations and student explorations. Following is an

example of an Excel QM file (from [Chapter 13](#)) that is on the CD-ROM that accompanies the text.

[\[View full size image\]](#)

The screenshot shows an Excel spreadsheet titled "Exhibit13.12.xls". The spreadsheet is set up for a queueing model with multiple servers. It includes input parameters for arrival rate, service rate, and number of servers, and calculates various performance metrics.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Biggs Department Store Example With Multiple Servers																
2																	
3	Working Lines Mins																
4	The arrival RATE and service RATE both must be rates and use the same time unit. Given a line such as 10																
5	minutes, convert it to a rate such as 6 per hour																
6	Unit																
7	Arrive rate (λ)	10	Average server rate (μ)		0.0033												
8	Service rate (μ)	4	Average number of customers in the queue(L_q)		3.6112												
9	Number of servers(c)	3	Average number of customers in the system(L_s)		6.0112												
10			Average waiting time in the queue(W_q)		0.3611												
11			Average time in the system(W_s)		0.6011												
12			Probability (% of time) system is empty(P_0)		0.0449												
13																	

[Page xv]

Premium Solver for

Education

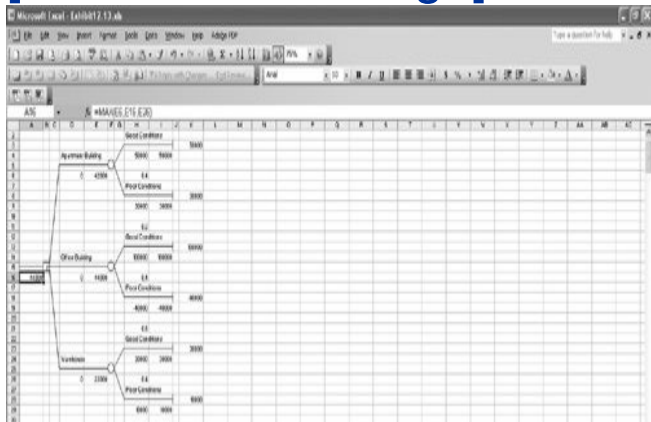
This is an upgraded version of the standard Solver that comes with Excel.

TreePlan

Another spreadsheet add-in program that is demonstrated in the text is *TreePlan*, a program that will set up a generic spreadsheet for the solution of decision-tree problems in [Chapter 12](#) on decision analysis. This is also

provided on the accompanying CD-ROM. Following is an example of one of the **TreePlan** files (from [Chapter 12](#)) that is on the text CD-ROM.

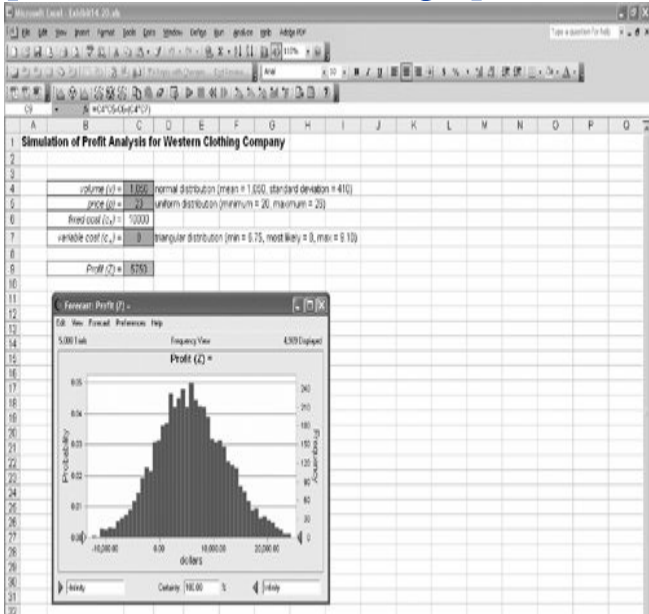
[\[View full size image\]](#)



Crystal Ball

Still another spreadsheet add-in program that is included on the accompanying CD-ROM and demonstrated in the book is *Crystal Ball*. Crystal Ball is demonstrated in [Chapter 14](#) on simulation and shows how to perform simulation analysis for certain types of risk analysis and forecasting problems. Following is an example of one of the **Crystal Ball** files (from [Chapter 14](#)) that is on the text CD-ROM.

[\[View full size image\]](#)



QM for Windows Software Package:

QM for Windows is the computer package that is included on the text CD-ROM and that many students and instructors will prefer to use with this text. This software is very user-friendly, requiring virtually no preliminary instruction except for the "help" screens that can be accessed directly from the program. It is demonstrated throughout the text in

conjunction with virtually every management science modeling technique, except simulation. The text includes 50 QM for Windows screens used to demonstrate example problems. Thus, for most topics problem solution is demonstrated via both Excel spreadsheets and QM for Windows. Files that include all the QM for Windows solutions for examples in the text are included on the accompanying CD-ROM. Following is an example of one of the QM for Windows files (from [Chapter 4](#))

that is on the CD-ROM.

[\[View full size image\]](#)

Original Problem w/answers								
Product Mix Example Solution								
	X1	X2	X3	X4		RHS	Dual	
Maximize	90	125	45	65			Max 90X1 +	
Processing time (hrs)	.1	.25	.08	.21	≤	72	233.3333	
Shipping capacity (boxes)	3	3	1	1	≤	1,200	22.2222	
Budget (\$)	36	48	25	35	≤	25,000	0	
Blank sweats (dozens)	1	1	0	0	≤	500	0	
Blank T's (dozens)	0	0	1	1	≤	500	4.1111	
Solution ->	175.5556	57.7778	500	0	Optimal Z ->	45,522.22	X3 + X4 ≤ 500	

Microsoft Project

As we indicated previously when talking about the new features in this edition, [Chapter 8](#) on Project Management focuses on the popular software package, *Microsoft Project*, which is available to users of this text. Following is an example of one of the *Microsoft Project* files (from [Chapter 8](#)) that is available on the text CD-ROM.

to offer students practice. This edition includes over 720 homework problems, 30 of which are new, and 52 end-of-chapter cases, 4 of which are new. In addition, four additional spreadsheet modeling cases are provided on this text's Web page, which can be accessed at <http://www.prenhall.com/taylor>

Management Science Applications Boxes

These boxes are located in

every chapter in the text. They describe how a company, organization, or agency uses the particular management science technique being presented and demonstrated in the chapter to compete in a global environment. There are 50 of these boxes, 16 of which are new, throughout the text and they encompass a broad range of business and public sector applications, both foreign and domestic.

Marginal Notes

Notes are included in the margins that serve the same basic function as notes that students themselves might write in the margin. They highlight certain topics to make it easier for the student to locate them, they summarize topics and important points, and they provide brief definitions of key terms and concepts.

Examples

The primary means of teaching

the various quantitative modeling techniques presented in this text is through examples. Thus, examples are liberally inserted throughout the text, primarily to demonstrate how problems are solved with the different quantitative techniques and to make them easier to understand. These examples are organized in a logical step-by-step solution approach that the student can subsequently apply to the homework problems.

Solved Example Problems

At the end of each chapter, just prior to the homework questions and problems, there is a section with solved examples to serve as a guide for doing the homework problems. These examples are solved in a detailed, step-by-step fashion.

Instructors' and Students' Supplements

For the Instructor:

- Excel Homework Solutions
In addition to the printed Instructor's Solutions Manual, almost every end-of-chapter homework and case problem in this text has a corresponding Excel solution file for the

instructor. This new edition includes 720 end-of-chapter homework problems and Excel solutions are provided for all but a few of them. Excel solutions are also provided for 50 of the 52 end-of-chapter case problems. These solution files can be accessed from the Instructor's Resource CD-ROM, as shown in the illustration below. These Excel files also include those homework and case problem solutions using

TreePlan (from [Chapter 12](#)) and those using *Crystal Ball* (from [Chapters 14](#)). In addition, *Microsoft Project* solution files are available for homework problems in [Chapter 8](#). These solution files are not available on the student CD-ROM that accompanies the text, but instructors can electronically post these solutions for their students to access or download directly to their computers.

- PowerPoint

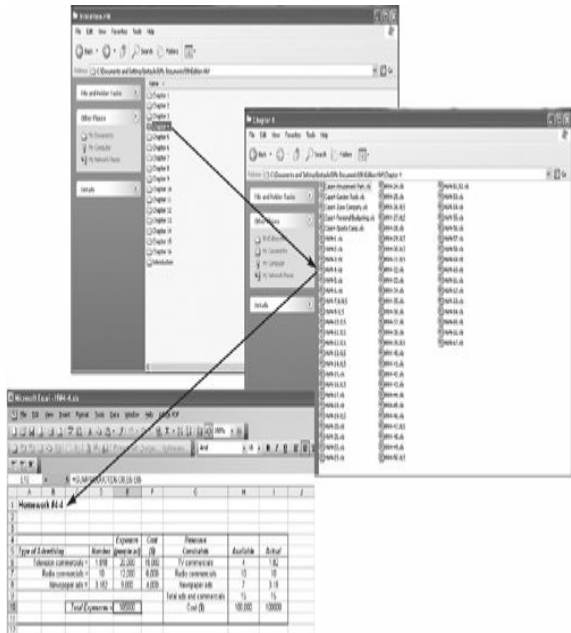
PresentationsPowerPoint presentations are available for every chapter to enhance lectures. They feature figures, tables, Excel, and main points from the text. They are available on the text Web site or on the Instructor's CD-ROM.

- Instructor's Solutions ManualThe instructor's Solutions Manual contains detailed solutions for all end-of-chapter exercises and cases. In addition to a printed solutions manual,

these solutions are provided electronically on the text's Web site and on a separate Instructor's CD-ROM in PDF format.

- Test Item FileThe test item file contains a variety of true/false, multiple choice, and problem solving questions for each chapter.

[\[View full size image\]](#)



[Page xviii]

- Instructor's CD-ROM This

separate CD-ROM for instructors only contains the following:

- All of the print supplements listed above, in electronic form.
- Electronic files for all of the example problem exhibits.
- Electronic files (with solutions) for almost all of the end-of-chapter homework problems

and cases. These files include solutions that use Excel, QM for Windows, Crystal Ball and TreePlan.

- All of the files and software programs on the students' CD-ROM.
- The TestGen software described below.
- Companion Web site This Web site, at www.prenhall.com/taylor contains all of the

supplements listed above (Instructor's Solutions Manual, PowerPoint slides, Test Item File) in electronic form and available for download.

TestGen Software

The print Test Item Files are designed for use with the TestGen test generating software. This computerized package allows instructors to custom design, save, and generate classroom tests. This software allows for greater

flexibility and ease of use. It provides many options for organizing and displaying tests, along with a search and sort feature.

For the Student:

- Student CD-ROMA CD-ROM is packaged with every copy of this book. This CD-ROM contains the following software packages:
Premium Solver for Education, Crystal Ball Professional
Textbook/Student Edition,

TreePlan and Excel QM.
Also on the CD-ROM are
Excel, Crystal Ball,
TreePlan, QM for Windows,
and Microsoft Project files
for the examples in the
text.

Acknowledgments

As with any large project, the revision of a textbook is not accomplished without the help of many people. The ninth edition of this book is no exception, and I would like to take this opportunity to thank those who have contributed to its preparation. First, I would like to thank my friend and colleague, Larry Moore, for his help in developing the organization and approach of

the original edition of this book and for his many suggestions during its revisions. We spent many hours discussing what an introductory text in management science should contain, and his ideas appear in these pages. Larry also served as a sounding board for many ideas regarding content, design, and preparation, and he read and edited many portions of the text, for which I am very grateful. I also thank the reviewers of this edition: Dr. B. S. Bal, Dewey Hemphill, David A. Larson, Sr., Christopher M.

Rump, Dothang Truong, Hulya Julie Yazici, and Ding Zhang.

I remain indebted to the reviewers of the previous editions: Nagraj Balakrishnan, Edward M. Barrow, Ali Behnezhad, Weldon J. Bowling, Rod Carlson, Petros Christofi, Yar M. Ebadi, Richard Ehrhardt, Warren W. Fisher, James Flynn, Wade Furgeson, Soumen Ghosh, James C. Goodwin Jr., Richard Gunther, Ann Hughes, Shivaji Khade, Shao-ju Lee, Robert L. Ludke, Peter A. Lyew, Robert D. Lynch, Dinesh Manocha, Mildred Massey,

Abdel-Aziz Mohamed, Thomas J. Nolan, Susan W. Palocsay, David W. Pentico, Cindy Randall, Roger Schoenfeldt, Charles H. Smith, Lisa Sokol, John Wang, and Barry Wray.

I am also very grateful to Tracy McCoy at Virginia Tech for her typing and editorial assistance. I would like to thank my production editor, Denise Culhane at Prentice Hall, for her valuable assistance and patience. I would also like to thank the text's accuracy checker, Annie Puciloski, for her diligence and thoroughness.

Finally, I would like to thank my editor, Alana Bradley at Prentice Hall, for her continual help and patience.

Chapter 1.

Management

Science

[Page 2]

Management science is the application of a scientific approach to solving management problems in order to help managers make better decisions. As implied by this definition, management science encompasses a number of mathematically oriented

techniques that have either been developed within the field of management science or been adapted from other disciplines, such as the natural sciences, mathematics, statistics, and engineering. This text provides an introduction to the techniques that make up management science and demonstrates their applications to management problems.

Management science is a recognized and established discipline in business. The applications of management science techniques are

widespread, and they have been frequently credited with increasing the efficiency and productivity of business firms. In various surveys of businesses, many indicate that they use management science techniques, and most rate the results to be very good. Management science (also referred to as *operations research*, *quantitative methods*, *quantitative analysis*, and *decision sciences*) is part of the fundamental curriculum of most programs in business.

Management science is a scientific approach to solving management problems.

As you proceed through the various management science models and techniques contained in this text, you should remember several things. First, most of the examples presented in this text are for business organizations

because businesses represent the main users of management science. However, management science techniques can be applied to solve problems in different types of organizations, including services, government, military, business and industry, and health care.

Management science can be used in a variety of organizations to solve many different types of problems.

Second, in this text all of the modeling techniques and solution methods are mathematically based. In some instances the manual, mathematical solution approach is shown because it helps to understand how the modeling techniques are applied to different problems. However, a computer solution is possible for each of the modeling techniques in this text, and in many cases the computer solution is emphasized. The

more detailed mathematical solution procedures for many of the modeling techniques are included as supplemental modules on the CD that accompanies this text.

Finally, as the various management science techniques are presented, keep in mind that management science is more than just a collection of techniques. Management science also involves the philosophy of approaching a problem in a logical manner (i.e., a scientific approach). The logical,

consistent, and systematic approach to problem solving can be as useful (and valuable) as the knowledge of the mechanics of the mathematical techniques themselves. This understanding is especially important for those readers who do not always see the immediate benefit of studying mathematically oriented disciplines such as management science.

Management science encompasses a logical approach to problem

solving.

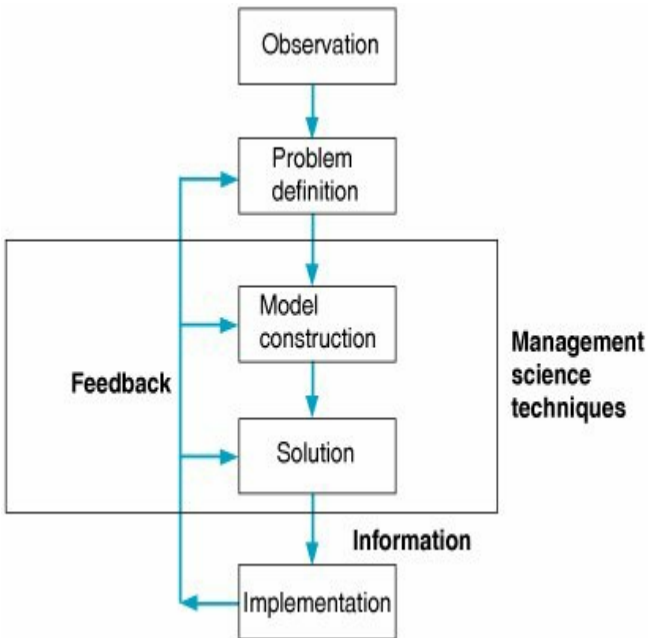
The Management Science Approach to Problem Solving

As indicated in the previous section, management science encompasses a logical, systematic approach to problem solving, which closely parallels what is known as the **scientific method** for attacking problems. This approach, as shown in **Figure 1.1**, follows a generally recognized and

ordered series of steps: (1) observation, (2) definition of the problem, (3) model construction, (4) model solution, and (5) implementation of solution results. We will analyze each of these steps individually.

Figure 1.1. The management science process

(This item is displayed on page 3 in the print version)



The steps of the

*scientific method are
(1) observation, (2)
problem definition,
(3) model
construction, (4)
model solution, and
(5) implementation.*

Observation

The first step in the management science process is the identification of a problem

that exists in the system (organization). The system must be continuously and closely observed so that problems can be identified as soon as they occur or are anticipated. Problems are not always the result of a crisis that must be reacted to but, instead, frequently involve an anticipatory or planning situation. The person who normally identifies a problem is the manager because the managers work in places where problems might occur. However, problems can often be

identified by a **management scientist**, a person skilled in the techniques of management science and trained to identify problems, who has been hired specifically to solve problems using management science techniques.

[Page 3]

*A **management scientist** is a person skilled in the application of management science techniques.*

Definition of the Problem

Once it has been determined that a problem exists, the problem must be clearly and concisely *defined*. Improperly defining a problem can easily result in no solution or an inappropriate solution.

Therefore, the limits of the problem and the degree to which it pervades other units of the organization must be included in the problem

definition. Because the existence of a problem implies that the objectives of the firm are not being met in some way, the goals (or objectives) of the organization must also be clearly defined. A stated objective helps to focus attention on what the problem actually is.

Model Construction

A management science **model** is an abstract representation of an existing problem situation.

It can be in the form of a graph or chart, but most frequently a management science model consists of a set of mathematical relationships. These mathematical relationships are made up of numbers and symbols.

*A **model** is an abstract mathematical representation of a problem situation.*

As an example, consider a business firm that sells a product. The product costs \$5 to produce and sells for \$20. A model that computes the total profit that will accrue from the items sold is

$$Z = \$20x - 5x$$

*A **variable** is a symbol used to represent an item that can take on any value.*

In this equation x represents the number of units of the product that are sold, and Z represents the total profit that results from the sale of the product. The *symbols* x and Z are *variables*. The term **variable** is used because no set numeric value has been specified for these items. The number of units sold, x , and the profit, Z , can be any amount (within limits); they can vary. These two variables can be further distinguished. Z

is a *dependent variable* because its value is dependent on the number of units sold; x is an *independent variable* because the number of units sold is *not* dependent on anything else (in this equation).

[Page 4]

Parameters are known, constant values that are often coefficients of variables in equations.

The numbers \$20 and \$5 in the equation are referred to as **parameters**. Parameters are constant values that are generally coefficients of the variables (symbols) in an equation. Parameters usually remain constant during the process of solving a specific problem. The parameter values are derived from **data** (i.e., pieces of information) from the problem environment. Sometimes the data are readily available and quite accurate.

For example, presumably the selling price of \$20 and product cost of \$5 could be obtained from the firm's accounting department and would be very accurate. However, sometimes data are not as readily available to the manager or firm, and the parameters must be either estimated or based on a combination of the available data and estimates. In such cases, the model is only as accurate as the data used in constructing the model.

Data are pieces of

*information from the
problem
environment.*

The equation as a whole is known as a **functional relationship** (also called *function and relationship*). The term is derived from the fact that profit, Z , is a *function* of the number of units sold, x , and the equation *relates* profit to units sold.

*A model is a
**functional
relationship** that
includes variables,
parameters, and
equations.*

Because only one functional relationship exists in this example, it is also the *model*. In this case the relationship is a model of the determination of profit for the firm. However, this model does not really

replicate a problem. Therefore, we will expand our example to create a problem situation.

Let us assume that the product is made from steel and that the business firm has 100 pounds of steel available. If it takes 4 pounds of steel to make each unit of the product, we can develop an additional mathematical relationship to represent steel usage:

$$4x = 100 \text{ lb. of steel}$$

This equation indicates that for every unit produced, 4 of the

available 100 pounds of steel will be used. Now our model consists of two relationships:

$$Z = \$20x - 5x$$

$$4x = 100$$

We say that the profit equation in this new model is an **objective function**, and the resource equation is a **constraint**. In other words, the objective of the firm is to achieve as much profit, Z , as possible, but the firm is constrained from achieving an infinite profit by the limited amount of steel available. To

signify this distinction between the two relationships in this model, we will add the following notations:

$$\text{maximize } Z = \$20x - 5x$$

subject to

$$4x = 100$$

This model now represents the manager's problem of determining the number of units to produce. You will recall that we defined the number of units to be produced as x . Thus, when we determine the value of x , it represents a potential (or recommended)

decision for the manager. Therefore, x is also known as a **decision variable**. The next step in the management science process is to solve the model to determine the value of the decision variable.

Model Solution

A management science technique usually applies to a specific model type.

Once models have been constructed in management science, they are solved using the management science techniques presented in this text. A management science solution technique usually applies to a specific type of model. Thus, the model type and solution method are both part of the management science technique. We are able to say that *a model is solved* because the model represents a problem. When we refer to

model solution, we also mean
problem solution.

[Page 5]

Time Out: For Pioneers in Management Science

Throughout this text TIME OUT boxes introduce you to the individuals who developed the various techniques that are described in the chapters. This will provide a historical perspective on the development of the field of management science. In this first instance we will briefly outline the development of management science.

Although a number of the mathematical techniques that make up management science date to the turn of the twentieth century or before, the field of management science itself can trace its beginnings to military operations research (OR) groups formed during World War II in

Great Britain circa 1939. These OR groups typically consisted of a team of about a dozen individuals from different fields of science, mathematics, and the military, brought together to find solutions to military-related problems. One of the most famous of these groups called "Blackett's circus" after its leader, Nobel laureate P. M. S. Blackett of the University of Manchester and a former naval officer included three physiologists, two mathematical physicists, one astrophysicist, one general physicist, two mathematicians, an Army officer, and a surveyor. Blackett's group and the other OR teams made significant contributions in improving Britain's early-warning radar system (which was instrumental in their victory in the Battle of Britain), aircraft gunnery, antisubmarine warfare, civilian

defense, convoy size determination, and bombing raids over Germany.

The successes achieved by the British OR groups were observed by two Americans working for the U.S. military, Dr. James B. Conant and Dr. Vannevar Bush, who recommended that OR teams be established in the U.S. branches of the military. Subsequently, both the Air Force and Navy created OR groups.

After World War II the contributions of the OR groups were considered so valuable that the Army, Air Force, and Navy set up various agencies to continue research of military problems. Two of the more famous agencies were the Navy's Operations Evaluation Group at MIT and Project RAND, established by the Air Force to study aerial warfare. Many of the individuals who developed operations

research and management science techniques did so while working at one of these agencies after World War II or as a result of their work there.

As the war ended and the mathematical models and techniques that were kept secret during the war began to be released, there was a natural inclination to test their applicability to business problems. At the same time, various consulting firms were established to apply these techniques to industrial and business problems, and courses in the use of quantitative techniques for business management began to surface in American universities. In the early 1950s the use of these quantitative techniques to solve management problems became known as management science, and it was popularized by a book of that name

by Stafford Beer of Great Britain.

For the example model developed in the previous section,

maximize $Z = \$20x - 5x$

subject to

$$4x = 100$$

the solution technique is simple algebra. Solving the constraint equation for x , we have

$$4x = 100$$

$$x = 100/4$$

$$x = 25 \text{ units}$$

Substituting the value of 25 for x into the profit function results in the total profit:

$$\begin{aligned} Z &= \$20x - 5x \\ &= 20(25) - 5(25) \\ &= \$375 \end{aligned}$$

Thus, if the manager decides to produce 25 units of the product and all 25 units sell, the business firm will receive \$375 in profit. Note, however, that the value of the decision variable does not constitute an actual decision; rather, it is *information* that serves as a recommendation or guideline,

helping the manager make a decision.

[Page 6]

Management Science

Application: Management Science at Taco Bell

Taco Bell, an international fast-food chain with annual sales of approximately \$4.6 billion, operates more than 6,500 locations worldwide. In the fast-food business the operating objective is, in general, to provide quality food, good service, and a clean environment. Although Taco Bell sees these three attributes as equally important, good service, as measured by its speed, has the greatest impact on revenues.

The 3-hour lunch period 11:00 A.M. to 2:00 P.M. accounts for 52% of Taco Bell's daily sales. Most fast-food restaurants have lines of waiting

customers during this period, and so speed of service determines sales capacity. If service time decreases, sales capacity increases, and vice versa. However, as speed of service increases, labor costs also increase. Because very few food items can be prepared in advance and inventoried, products must be prepared when they are ordered, making food preparation very labor intensive. Thus, speed of service depends on labor availability.

Taco Bell research studies showed that when customers are in line up to 5 minutes only, their perception of that waiting time is only a few minutes. However, after waiting time exceeds 5 minutes, customer perception of that waiting time increases exponentially. The longer the perceived waiting time, the more likely the customer is to leave the

restaurant without ordering. The company determined that a 3-minute average waiting time would result in only 2.5% of customers leaving. The company believed this was an acceptable level of attrition, and it established this waiting time as its service goal.



To achieve this goal Taco Bell developed a labor-management system based on an integrated set of management science models to

forecast customer traffic for every 15-minute interval during the day and to schedule employees accordingly to meet customer demand. This labor-management system includes a forecasting model to predict customer transactions; a simulation model to determine labor requirements based on these transactions; and an integer programming model to schedule employees and minimize payroll. From 1993 through 1997 the labor-management system using these models saved Taco Bell over \$53 million.

Source: J. Heuter and W. Swart, "An Integrated Labor-Management System for Taco Bell," *Interfaces* 28, no. 1 (January/February 1998): 7591.

Some management science techniques do not generate an answer or a recommended decision. Instead, they provide *descriptive results*: results that describe the system being modeled. For example, suppose the business firm in our example desires to know the average number of units sold each month during a year. The monthly *data* (i.e., sales) for the past year are as follows:

<i>Month</i>	<i>Sales</i>	<i>Month</i>	<i>Sales</i>
---------------------	---------------------	---------------------	---------------------

January	30	July	35
February	40	August	50
March	25	September	60
April	60	October	40
May	30	November	35
June	25	December	50
		<i>Total</i>	480 units

A management science solution can be either a recommended decision or information that helps a manager make a decision.

($480 \div 12$). This result is not a decision; it is information that describes what is happening in the system. The results of the management science techniques in this text are examples of the two types shown in this section: (1) solutions/decisions and (2) descriptive results.

Implementation

The final step in the management science process for problem solving described in

Figure 1.1 is implementation.

Implementation is the actual use of the model once it has been developed or the solution to the problem the model was developed to solve. This is a critical but often overlooked step in the process. It is not always a given that once a model is developed or a solution found, it is automatically used. Frequently the person responsible for putting the model or solution to use is not the same person who developed the model and, thus, the user may not fully

understand how the model works or exactly what it is supposed to do. Individuals are also sometimes hesitant to change the normal way they do things or to try new things. In this situation the model and solution may get pushed to the side or ignored altogether if they are not carefully explained and their benefit fully demonstrated. If the management science model and solution are not implemented, then the effort and resources used in their development have been

wasted.

-

Model Building: Break-Even Analysis

In the previous section we gave a brief, general description of how management science models are formulated and solved, using a simple algebraic example. In this section we will continue to explore the process of building and solving management science models, using **break-even analysis**, also called *profit analysis*. Break-even analysis is a good

topic to expand our discussion of model building and solution because it is straightforward, relatively familiar to most people, and not overly complex. In addition, it provides a convenient means to demonstrate the different ways management science models can be solved mathematically (by hand), graphically, and with a computer.

The purpose of break-even analysis is to determine the number of units of a product (i.e., the volume) to sell or produce that will equate total

revenue with total cost. The point where total revenue equals total cost is called the break-even point, and at this point profit is zero. The break-even point gives a manager a point of reference in determining how many units will be needed to ensure a profit.

Components of Break-Even Analysis

The three components of break-even analysis are

volume, cost, and profit.

Volume is the level of sales or production by a company. It can be expressed as the number of units (i.e., quantity) produced and sold, as the dollar volume of sales, or as a percentage of total capacity available.

Two type of costs are typically incurred in the production of a product: fixed costs and variable costs. **Fixed costs** are generally independent of the volume of units produced and sold. That is, fixed costs remain constant, regardless of how

many units of product are produced within a given range. Fixed costs can include such items as rent on plant and equipment, taxes, staff and management salaries, insurance, advertising, depreciation, heat and light, plant maintenance, and so on. Taken together, these items result in total fixed costs.

Fixed costs are independent of volume and remain constant.

Variable costs are determined on a per-unit basis. Thus, total variable costs depend on the number of units produced. Variable costs include such items as raw materials and resources, direct labor, packaging, material handling, and freight.

Variable costs
*depend on the
number of items
produced.*

Total variable costs are a function of the *volume* and the *variable cost per unit*. This relationship can be expressed mathematically as

$$\text{total variable cost} = v c_v$$

where c_v = variable cost per unit and v = volume (number of units) sold.

The **total cost** of an operation

is computed by summing total fixed cost and total variable cost, as follows:

total cost = total fixed cost +
total variable cost

or

$$TC = c_f + VC_v$$

where c_f = fixed cost.

Total cost (TC)
equals the fixed cost
(c_f) plus the variable
cost per unit (c_v)

*multiplied by volume
(v).*

As an example, consider Western Clothing Company, which produces denim jeans. The company incurs the following monthly costs to produce denim jeans:

fixed cost = c_f = \$10,000

variable cost = c_v = \$8 per pair

If we arbitrarily let the monthly sales volume, v , equal 400

pairs of denim jeans, the total cost is

$$TC = c_f + vc_v = \$10,000 + (400)(8) = \$13,200$$

The third component in our break-even model is **profit**. Profit is the difference between **total revenue** and total cost. Total revenue is the volume multiplied by the price per unit, total revenue = vp where p = price per unit.

Profit is the

*difference between
total revenue
(volume multiplied by
price) and total cost.*

For our clothing company example, if denim jeans sell for \$23 per pair and we sell 400 pairs per month, then the total monthly revenue is

$$\begin{aligned}\text{total revenue} &= vp = (400)(23) \\ &= \$9,200\end{aligned}$$

Now that we have developed relationships for total revenue and total cost, profit (Z) can be computed as follows:

total profit = total revenue – total cost

$$\begin{aligned} Z &= vp - (c_f + vc_v) \\ &= vp - c_f - vc_v \end{aligned}$$

Computing the Break-Even Point

For our clothing company example, we have determined total revenue and total cost to be \$9,200 and \$13,200, respectively. With these values,

there is no profit but, instead, a loss of \$4,000:

total profit = total revenue

total cost = \$9,200 13,200 =
\$4,000

We can verify this result by using our total profit formula,

$$Z = vp - c_f - vc_v$$

and the values $v = 400$, $p = \$23$, $c_f = \$10,000$, and $c_v = \$8$:

$$\begin{aligned} Z &= vp - c_f - vc_v \\ &= \$ (400)(23) - 10,000 - (400)(8) \\ &= \$9,200 - 10,000 - 3,200 \\ &= -\$4,000 \end{aligned}$$

[Page 9]

Obviously, the clothing company does not want to operate with a monthly loss of \$4,000 because doing so might eventually result in bankruptcy. If we assume that price is static because of market conditions and that fixed costs and the variable cost per unit are not subject to change, then the only part of our model that can

be varied is *volume*. Using the modeling terms we developed earlier in this chapter, price, fixed costs, and variable costs are parameters, whereas the volume, v , is a **decision variable**. In break-even analysis we want to compute the value of v that will result in zero profit.

*The **break-even point** is the volume (v) that equates total revenue with total cost where profit is zero.*

At the **break-even point**, where total revenue equals total cost, the profit, Z , equals zero. Thus, if we let profit, Z , equal zero in our total profit equation and solve for v , we can determine the break-even volume:

$$Z = vp - c_f - vc_v$$

$$0 = v(23) - 10,000 - v(8)$$

$$0 = 23v - 10,000 - 8v$$

$$15v = 10,000$$

$$v = 666.7 \text{ pairs of jeans}$$

In other words, if the company produces and sells 666.7 pairs

of jeans, the profit (and loss) will be zero and the company will *break even*. This gives the company a point of reference from which to determine how many pairs of jeans it needs to produce and sell in order to gain a profit (subject to any capacity limitations). For example, a sales volume of 800 pairs of denim jeans will result in the following monthly profit:

$$\begin{aligned} Z &= vp - c_f - vc_v \\ &= \$ (800)(23) - 10,000 - (800)(8) = \$2,000 \end{aligned}$$

In general, the break-even volume can be determined

using the following formula:

$$Z = vp - c_f - vc_v$$

$$0 = v(p - c_v) - c_f$$

$$v(p - c_v) = c_f$$

$$v = \frac{c_f}{p - c_v}$$

For our example,

$$v = \frac{c_f}{p - c_v}$$

$$= \frac{10,000}{23 - 8}$$

$$= 666.7 \text{ pairs of jeans}$$

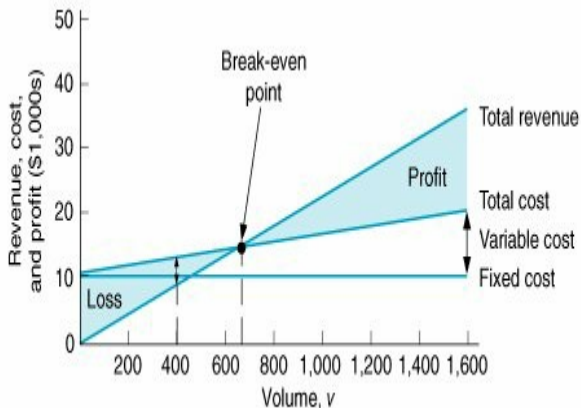
Graphical Solution

It is possible to represent many of the management science models in this text graphically and use these graphical models to solve problems. Graphical models also have the advantage of providing a "picture" of the model that can sometimes help us understand the modeling process better than the mathematics alone can. We can easily graph the break-even model for our Western Clothing Company example because the functions for total cost and total revenue are *linear*. That means we can

graph each relationship as a straight line on a set of coordinates, as shown in [Figure 1.2](#).

Figure 1.2. Break-even model

(This item is displayed on page 10 in the print version)



In [Figure 1.2](#), the fixed cost, c_f , has a constant value of \$10,000, regardless of the volume. The total cost line, TC ,

represents the sum of variable cost and fixed cost. The total cost line increases because variable cost increases as the volume increases. The total revenue line also increases as volume increases, but at a faster rate than total cost. The point where these two lines intersect indicates that total revenue equals total cost. The volume, v , that corresponds to this point is the *break-even volume*. The break-even volume in [Figure 1.2](#) is 666.7 pairs of denim jeans.

Sensitivity Analysis

We have now developed a general relationship for determining the break-even volume, which was the objective of our modeling process. This relationship enables us to see how the level of profit (and loss) is directly affected by changes in volume. However, when we developed this model, we assumed that our parameters, fixed and

variable costs and price, were constant. In reality such parameters are frequently uncertain and can rarely be assumed to be constant, and changes in any of the parameters can affect the model solution. The study of changes on a management science model is called **sensitivity analysis** that is, seeing how sensitive the model is to changes.

Sensitivity analysis can be performed on all management science models in one form or another. In fact, sometimes

companies develop models for the primary purpose of experimentation to see how the model will react to different changes the company is contemplating or that management might expect to occur in the future. As a demonstration of how sensitivity analysis works, we will look at the effects of some changes on our break-even model.

The first thing we will analyze is price. As an example, we will increase the price for denim jeans from \$23 to \$30. As

expected, this increases the total revenue, and it therefore reduces the break-even point from 666.7 pairs of jeans to 454.5 pairs of jeans:

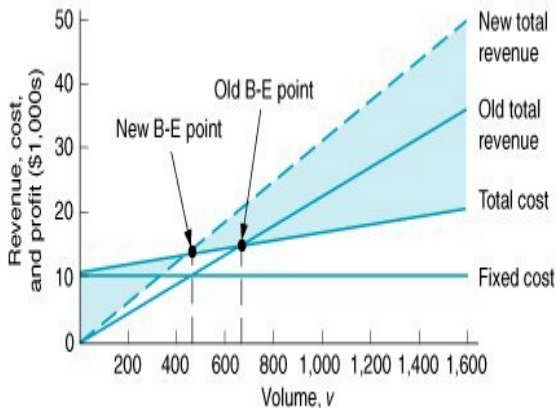
$$\begin{aligned}v &= \frac{c_f}{p - c_v} \\&= \frac{10,000}{30 - 8} = 454.5 \text{ pairs of denim jeans}\end{aligned}$$

In general, an increase in price lowers the break-even point, all other things held constant.

The effect of the price change on break-even volume is illustrated in [Figure 1.3](#).

Figure 1.3. Break-even model with an increase in price

(This item is displayed on page 11 in the print version)



Although a decision to increase price looks inviting from a strictly analytical point of view, it must be remembered that the lower break-even volume

and higher profit are *possible* but not guaranteed. A higher price can make it more difficult to sell the product. Thus, a change in price often must be accompanied by corresponding increases in costs, such as those for advertising, packaging, and possibly production (to enhance quality). However, even such direct changes as these may have little effect on product demand because price is often sensitive to numerous factors, such as the type of market, monopolistic elements, and

product differentiation.

[Page 11]

When we increased price, we mentioned the possibility of raising the quality of the product to offset a potential loss of sales due to the price increase. For example, suppose the stitching on the denim jeans is changed to make the jeans more attractive and stronger. This change results in an increase in variable costs of \$4 per pair of jeans, thus raising the variable cost per

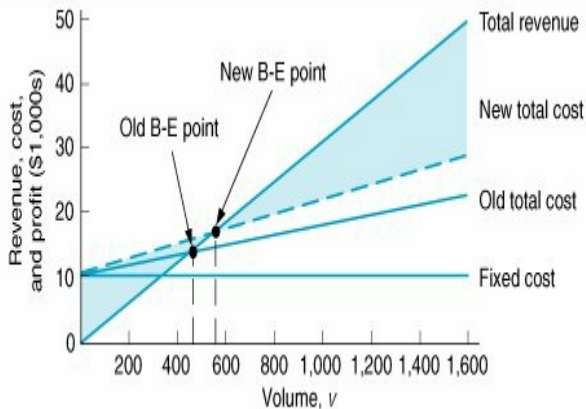
unit, c_v , to \$12 per pair. This change (in conjunction with our previous price change to \$30) results in a new break-even volume:

$$\begin{aligned} v &= \frac{c_f}{p - c_v} \\ &= \frac{10,000}{30 - 12} = 555.5 \text{ pairs of denim jeans} \end{aligned}$$

In general, an increase in variable costs will decrease the break-even point, all other things held constant.

This new break-even volume and the change in the total cost line that occurs as a result of the variable cost change are shown in [Figure 1.4](#).

Figure 1.4. Break-even model with an increase in variable cost



Next let's consider an increase in advertising expenditures to offset the potential loss in sales resulting from a price increase. An increase in advertising

expenditures is an addition to fixed costs. For example, if the clothing company increases its monthly advertising budget by \$3,000, then the total fixed cost, c_f , becomes \$13,000.

Using this fixed cost, as well as the increased variable cost per unit of \$12 and the increased price of \$30, we compute the break-even volume as follows:

$$\begin{aligned}
 v &= \frac{c_f}{p - c_v} \\
 &= \frac{13,000}{30 - 12} \\
 &= 722.2 \text{ pairs of denim jeans}
 \end{aligned}$$

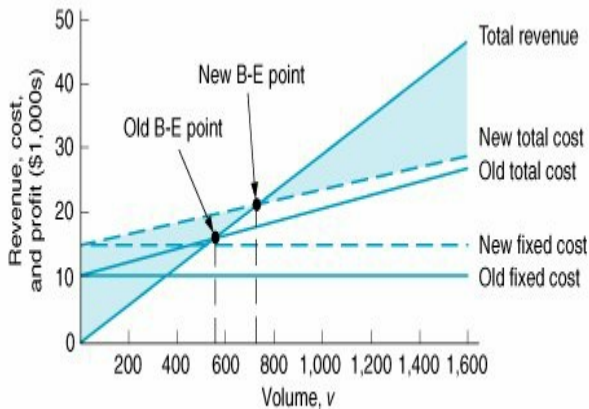
In general, an increase in fixed costs will increase the break-even point, all other things held constant.

This new break-even volume, representing changes in price,

fixed costs, and variable costs, is illustrated in [Figure 1.5](#). Notice that the break-even volume is now higher than the original volume of 666.7 pairs of jeans, as a result of the increased costs necessary to offset the potential loss in sales. This indicates the necessity to analyze the effect of a change in one of the break-even components on the whole break-even model. In other words, generally it is not sufficient to consider a change in one model component without considering the overall

effect.

Figure 1.5. Break-even model with a change in fixed cost



Computer Solution

Throughout the text we will demonstrate how to solve management science models on the computer by using Excel spreadsheets and QM for Windows, a general-purpose quantitative methods software package by Howard Weiss. QM for Windows has program modules to solve almost every type of management science problem you will encounter in this book. There are a number

of similar quantitative methods software packages available on the market, with similar characteristics and capabilities as QM for Windows. In most cases you simply input problem data (i.e., model parameters) into a model template, click on a solve button, and the solution appears in a Windows format. QM for Windows is included on the CD that accompanies this text.

Spreadsheets are not always easy to use, and you cannot conveniently solve every type of management science model

by using a spreadsheet. Most of the time you must not only input the model parameters but also set up the model mathematics, including formulas, as well as your own model template with headings to display your solution output. However, spreadsheets provide a powerful reporting tool in which you can present your model and results in any format you choose.

Spreadsheets such as Excel have become almost universally available to anyone who owns a computer. In

addition, spreadsheets have become very popular as a teaching tool because they tend to guide the student through a modeling procedure, and they can be interesting and fun to use. However, because spreadsheets are somewhat more difficult to set up and apply than is QM for Windows, we will spend more time explaining their use to solve various types of problems in this text.

One of the difficult aspects of using spreadsheets to solve management science problems is setting up a spreadsheet with some of the more complex models and formulas. For the most complex models in the text we will show how to use Excel QM, a supplemental spreadsheet macro that is included on the CD that accompanies this text. A *macro* is a template or an overlay that already has the model format with the necessary formulas set up on the spreadsheet so that the user only has to input the

model parameters. We will demonstrate Excel QM in six chapters, including this chapter, [Chapter 6](#) ("Transportation, Transshipment, and Assignment Problems"), [Chapter 12](#) ("Decision Analysis"), [Chapter 13](#) ("Queuing Analysis"), [Chapter 15](#) ("Forecasting"), and [Chapter 16](#) ("Inventory Management").

Later in this text we will also demonstrate two spreadsheet add-ins, TreePlan and Crystal Ball. TreePlan is a program for setting up and solving decision trees that we use in [Chapter 12](#)

("Decision Analysis"), whereas Crystal Ball is a simulation package that we use in [Chapter 14](#) ("Simulation"). Also, in [Chapter 8](#) ("Project Management") we will demonstrate Microsoft Project.

In this section we will demonstrate how to use Excel, Excel QM, and QM for Windows, using our break-even model example for Western Clothing Company.

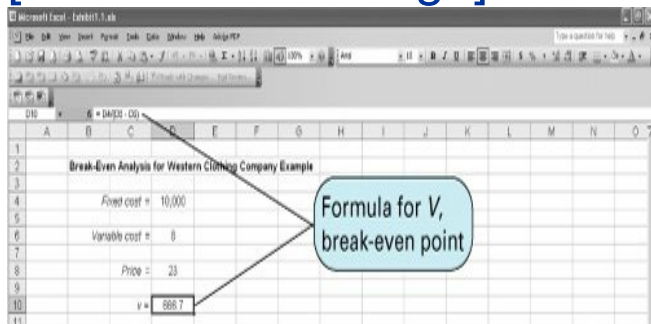
Excel Spreadsheets

To solve the break-even model using Excel, you must set up a spreadsheet with headings to identify your model parameters and variables and then input the appropriate mathematical formulas into the cells where you want to display your solution. [Exhibit 1.1](#) shows the spreadsheet for the Western Clothing Company example. Setting up the different headings to describe the parameters and the solution is not difficult, but it does require that you know your way around Excel a little. [Appendix B](#)

provides a brief tutorial titled "Setting Up and Editing a Spreadsheet" for solving management science problems.

Exhibit 1.1.

[\[View full size image\]](#)



Notice that cell D10 contains the break-even formula, which is displayed on the toolbar near the top of the screen. The fixed cost of \$10,000 is typed in cell D4, the variable cost of \$8 is in cell D6, and the price of \$23 is in cell D8.

As we present more complex models and problems in the chapters to come, the spreadsheets we will develop to solve these problems will become more involved and will

enable us to demonstrate different features of Excel and spreadsheet modeling.

[Page 14]

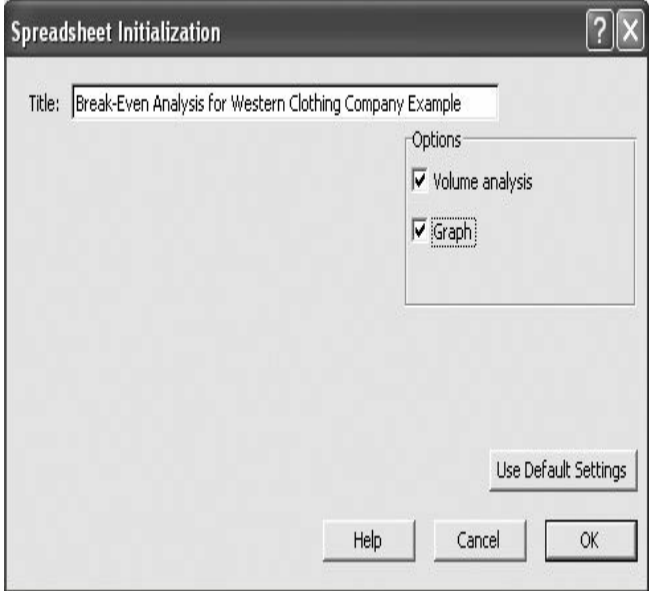
The Excel QM Macro for Spreadsheets

Excel QM is included on the CD that accompanies this text. You can install Excel QM onto your computer by following a brief series of steps displayed when the program is first accessed.

After Excel is started, Excel QM is normally accessed from the computer's program files, where it is usually loaded. When Excel QM is activated, "QM" will appear at the top of the spreadsheet (as indicated in [Exhibit 1.3](#)). Clicking on "QM" will pull down a menu of the topics in Excel QM, one of which is break-even analysis. Clicking on "Break-Even Analysis" will result in the window for spreadsheet initialization shown in [Exhibit 1.2](#). Every Excel QM macro listed on the menu will start

with a "Spreadsheet
Initialization" window similar to
this one.

Exhibit 1.2.



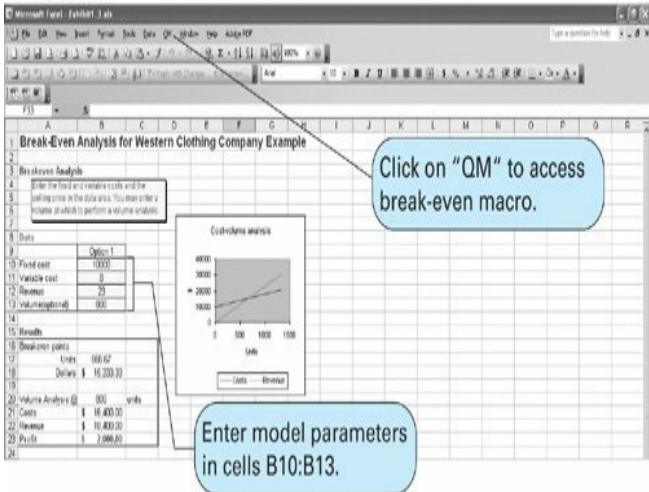
In the window in [Exhibit 1.2](#) you can enter a spreadsheet

title and choose under "Options" whether you also want volume analysis and a graph. Clicking on "OK" will result in the spreadsheet shown in [Exhibit 1.3](#). The first step is to input the values for the Western Clothing Company example in cells B10 to B13, as shown in [Exhibit 1.3](#). The spreadsheet shows the break-even volume in cell B17. However, notice that we have also chosen to perform some volume analysis by entering a hypothetical volume of 800 units in cell B13, which results

in the volume analysis in cells
B20 to B23.

Exhibit 1.3.

[\[View full size image\]](#)



[Page 15]

QM for Windows

You begin using QM for Windows by clicking on the "Module" button on the toolbar at the top of the main window that appears when you start the program. This will pull down a window with a list of all the model solution modules available in QM for Windows. Clicking on the "Break-even Analysis" module will access a new screen for typing in the problem title. Clicking again will access a screen with input cells for the model parameters that is, fixed cost, variable cost, and price (or

revenue). Next, clicking on the "Solve" button at the top of the screen will provide the solution to the Western Clothing Company example, as shown in [Exhibit 1.4](#).

Exhibit 1.4.

[\[View full size image\]](#)



You can also get the graphical model and solution for this problem by clicking on "Window" at the top of the solution screen and selecting the menu item for a graph of the problem. The break-even graph for the Western Clothing

example is shown in [Exhibit 1.5](#).

Exhibit 1.5.

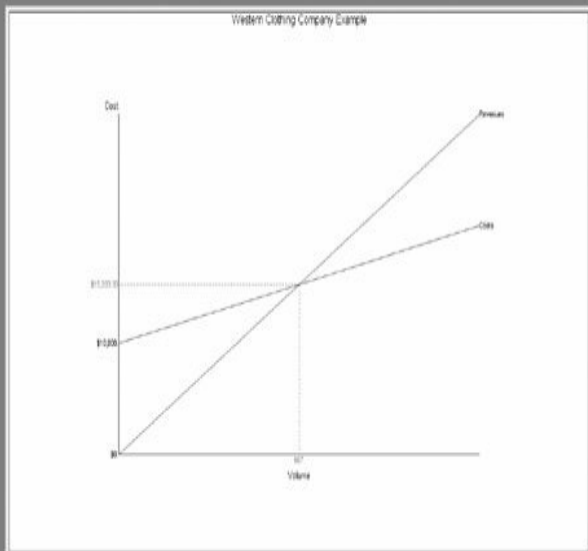
[\[View full size image\]](#)

File Edit View Format Help Tools Window Help

Standard toolbar icons: Undo, Redo, Copy, Paste, Find, Print, etc.

Address bar:

Status bar: Python 2.7.10 (tags/v2.7.10:6648ad33, Sep 11 2014) [AMD64] on win32



Management Science Modeling Techniques

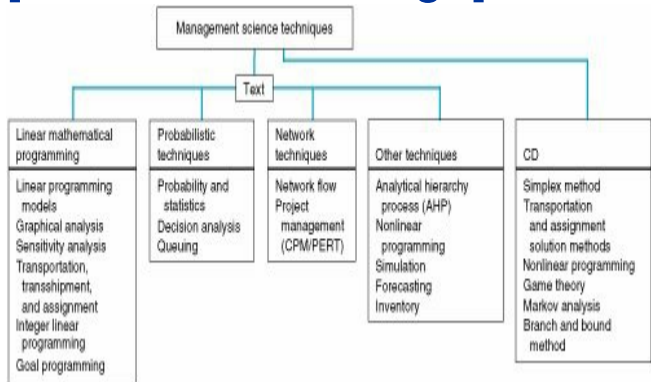
This text focuses primarily on two of the five steps of the management science process described in [Figure 1.1](#) model construction and solution.

These are the two steps that use the management science technique. In a textbook, it is difficult to show how an unstructured real-world problem is identified and defined because the problem

must be written out. However, once a problem statement has been given, we can show how a model is constructed and a solution is derived. The techniques presented in this text can be loosely classified into four categories, as shown in [Figure 1.6](#).

Figure 1.6.
**Classification of
management science
techniques**

[\[View full size image\]](#)



Linear Mathematical Programming Techniques

Chapters 2 through 6 and 9 present techniques that together make up **linear mathematical programming**. (The first example used to demonstrate model construction earlier in this chapter is a very rudimentary linear programming model.) The term *programming* used to identify this technique does not refer to computer programming but rather to a predetermined set of mathematical steps used to solve a problem. This particular class of techniques holds a predominant position in

this text because it includes some of the more frequently used and popular techniques in management science.

In general, linear programming models help managers determine solutions (i.e., make decisions) for problems that will achieve some objective in which there are restrictions, such as limited resources or a recipe or perhaps production guidelines. For example, you could actually develop a linear programming model to help determine a breakfast menu for yourself that would meet

dietary guidelines you may have set, such as number of calories, fat content, and vitamin level, while minimizing the cost of the breakfast.

Manufacturing companies develop linear programming models to help decide how many units of different products they should produce to maximize their profit (or minimize their cost), given scarce resources such as capital, labor, and facilities.

Six chapters in this text are devoted to this topic because there are several variations of

linear programming models that can be applied to specific types of problems. [Chapter 4](#) is devoted entirely to describing example linear programming models for several different types of problem scenarios. [Chapter 6](#), for example, focuses on one particular type of linear programming application for transportation, transshipment, and assignment problems. An example of a transportation problem is a manager trying to determine the lowest-cost routes to use to ship goods from several sources (such as

plants or warehouses) to several destinations (such as retail stores), given that each source may have limited goods available and each destination may have limited demand for the goods. Also, [Chapter 9](#) includes the topic of goal programming, which is a form of linear programming that addresses problems with more than one objective or goal.

[Page 17]

As mentioned previously in this chapter, some of the more

mathematical topics in the text are included as supplementary modules on the CD that accompanies the text. Among the linear programming topics included on the CD are modules on the simplex method; the transportation, transshipment, and assignment solution methods; and the branch and bound solution method for integer programming models. Also included on the CD are modules on nonlinear programming, game theory, and Markov analysis.

Probabilistic Techniques

Probabilistic techniques are presented in [Chapters 11](#) through [13](#). These techniques are distinguished from mathematical programming techniques in that the results are probabilistic. Mathematical programming techniques assume that all parameters in the models are known with *certainty*. Therefore, the solution results are assumed to be known with certainty, with no probability that other solutions might exist. A

technique that assumes certainty in its solution is referred to as **deterministic**. In contrast, the results from a probabilistic technique *do* contain uncertainty, with some possibility that alternative solutions might exist. In the model solution presented earlier in this chapter, the result of the first example ($x = 25$ units to produce) is deterministic, whereas the result of the second example (estimating an average of 40 units sold each month) is probabilistic.

An example of a probabilistic technique is decision analysis, the subject of [Chapter 12](#). In decision analysis it is shown how to select among several different decision alternatives, given uncertain (i.e., probabilistic) future conditions. For example, a developer may want to decide whether to build a shopping mall, build an office complex, build condominiums, or not build anything at all, given future economic conditions that might be good, fair, or poor, each with a probability of occurrence.

[Chapter 13](#), on queuing analysis, presents probabilistic techniques for analyzing waiting lines that might occur, for example, at the grocery store, at a bank, or at a movie. The results of waiting line analysis are statistical averages showing, among other things, the average number of customers in line waiting to be served or the average time a customer might have to wait for service.

Network Techniques

Networks, the topic of [Chapters 7](#) and [8](#), consist of models that are represented as diagrams rather than as strictly mathematical relationships. As such, these models offer a pictorial representation of the system under analysis. These models represent either probabilistic or deterministic systems.

For example, in shortest route problems, one of the topics in [Chapter 7](#) ("Network Flow Models"), a network diagram can be drawn to help a manager determine the

shortest route among a number of different routes from a source to a destination. For example, you could use this technique to determine the shortest or quickest car route from St. Louis to Daytona Beach for a spring break vacation. In [Chapter 8](#) ("Project Management"), a network is drawn that shows the relationships of all the tasks and activities for a project, such as building a house or developing a new computer system. This type of network can help a manager plan the

best way to accomplish each of the tasks in the project so that it will take the shortest amount of time possible. You could use this type of technique to plan for a concert or an intramural volleyball tournament on your campus.

[Page 18]

Other Techniques

Some topics in the text are not easily categorized; they may overlap between several

categories, or they may be unique. The analytical hierarchy process (AHP) in [Chapter 9](#) is such a topic that is not easily classified. It is a mathematical technique for helping the decision maker choose between several alternative decisions, given more than one objective; however, it is not a form of linear programming, as is goal programming, the shared topic in [Chapter 9](#), on multicriteria decision making. The structure of the mathematical models for nonlinear programming

problems in [Chapter 10](#) is similar to the linear programming problems in [Chapters 2](#) through [6](#); however, the mathematical equations and functions in nonlinear programming can be nonlinear instead of linear, thus requiring the use of calculus to solve them. Simulation, the subject of [Chapter 14](#), is probably the single most unique topic in the text. It has the capability to solve probabilistic and deterministic problems and is often the technique of last resort when no other

management science technique will work. In simulation a mathematical model is constructed (typically using a computer) that replicates a real-world system under analysis, and then that simulation model is used to solve problems in the "simulated" real-world system. For example, with simulation you could build a model to simulate the traffic patterns of vehicles at a busy intersection to determine how to set the traffic light signals.

Forecasting, the subject of

[Chapter 15](#), and inventory management, in [Chapter 16](#), are topics traditionally considered to be part of the field of operations management. However, because they are both important business functions that also rely heavily on quantitative models for their analysis, they are typically considered important topics in the study of management science as well. Both topics also include probabilistic as well as deterministic aspects. In [Chapter 15](#) we will look at

several different quantitative models that help managers predict what the future demand for products and services will look like. In general, historical sales and demand data are used to build a mathematical function or formula that can be used to estimate product demand in the future. In [Chapter 16](#) we will look at several different quantitative models that help organizations determine how much inventory to keep on hand in order to minimize inventory costs, which can be significant.

Business Usage of Management Science Techniques

Not all management science techniques are equally useful or equally used by business firms and other organizations. Some techniques are used quite frequently by business practitioners and managers; others are used less often. The most frequently used techniques are linear and

integer programming, simulation, network analysis (including critical path method/project evaluation and review technique [CPM/PERT]), inventory control, decision analysis, and queuing theory, as well as probability and statistics. An attempt has been made in this text to provide a comprehensive treatment of all the topics generally considered within the field of management science, regardless of how frequently they are used. Although some topics may have limited direct applicability, their

study can reveal informative and unique means of approaching a problem and can often enhance one's understanding of the decision-making process.

[Page 19]

Management Science

Application: Management Science at FedEx

In 1973 Frederick W. Smith started Federal Express Corporation to provide overnight delivery of small, high-value items, such as pharmaceuticals, aerospace components, and computer parts. The company began operation on March 12, 1973, in 11 cities in the south with a fleet of 22 twin-engine executive jets, each with a payload of about 3 tons. That first day the company delivered only 6 packages, and three days later FedEx discontinued air delivery service because of a lack of business. At that point FedEx took stock of its situation and began using

management science techniques and models to help solve its problems and redefine its operation. Since that time, management science has been an integral part of the company's spectacular success. Initially FedEx used quantitative techniques to analyze its original choice of cities, and a month after it shut down service, a rejuvenated FedEx began servicing a new 26-city system based on quantitative analysis of markets and cities. FedEx is now an \$8 billion plus corporation and the world's largest express transportation company, delivering more than 2 million items to over 200 countries each day. Its transportation system includes more than 500 aircraft and 35,000 vehicles. The management science applications the company has conducted over the years include

simulation models to develop economical flight schedules and determine resource requirements at its hub terminals; planning models to evaluate alternative routes and evaluate expansion plans; forecasting models to forecast aircraft maintenance requirements and develop maintenance schedules and to forecast when new terminal hubs would be needed to increase capacity for an expanding business; Markov models to plan for pilot requirements in the future and to show pilots their expected career path with the company; queuing analysis to develop and analyze the company's computerized call centers for customer orders and dispatching; and integer linear programming to help the company develop its SuperHub system instead of a less economical series of smaller hubs. All

these techniques, plus others, are presented in this text. From its beginning, management science applications and models have been used in many of FedEx's crucial business-shaping decisions.



Source: R. O. Maxon, J. L. McKenney, W. Carlson, and D. Copeland, "Absolutely, Positively Operations Research: The Federal Express Story," *Interfaces* 27, no. 2 (March/April 1997): 1736.

The variety and breadth of management science applications and of the potential for applying management science, not only in business and industry but also in government, health care, and service organizations, are extensive. Areas of application include project planning, capital budgeting, production planning, inventory analysis, scheduling, marketing planning, quality control, plant

location, maintenance policy, personnel management, and product demand forecasting, among others. In this text the applicability of management science to a variety of problem areas is demonstrated via individual chapter examples and the problems that accompany each chapter.

A small portion of the thousands of applications of management science that occur each year are recorded in various academic and professional journals. Frequently, these journal

articles are as complex as the applications themselves and are very difficult to read. However, one particular journal, *Interfaces*, is devoted specifically to the application of management science and is written not just for college professors but for businesspeople, practitioners, and students as well. *Interfaces* is published by INFORMS (Institute for Operations Research and Management Sciences), an international professional organization whose members include college

professors, businesspeople, scientists, students, and a variety of professional people interested in the practice and application of management science and operations research.

[Page 20]

Interfaces regularly publishes articles that report on the application of management science to a wide variety of problems. The chapters that follow present examples of applications of management

science from *Interfaces* and other professional journals. These examples, as presented here, do not detail the actual models and the model components. Instead, they briefly indicate the type of problem the company or organization faced, the objective of the solution approach developed to solve the problem, and the benefits derived from the model or technique (i.e., what was accomplished). The interested reader who desires more detailed information about

these and other management science applications is encouraged to go to the library and peruse *Interfaces* and the many other journals that contain articles on the application of management science.

Management Science Models in Decision Support Systems

Historically management science models have been applied to the solution of specific types of problems; for example, a waiting line model is used to analyze a specific waiting line system at a store or bank. However, the evolution of computer and information technology has enabled the

development of expansive computer systems that combine several management science models and solution techniques in order to address more complex, interrelated organizational problems. A **decision support system (DSS)** is a computer-based system that helps decision makers address complex problems that cut across different parts of an organization and operations.

A DSS is normally *interactive*, combining various databases and different management

science models and solution techniques with a user interface that enables the decision maker to ask questions and receive answers. In its simplest form any computer-based software program that helps a decision maker make a decision can be referred to as a DSS. For example, an Excel spreadsheet like the one shown for break-even analysis in [Exhibit 1.1](#) or the QM for Windows model shown in [Exhibit 1.4](#) can realistically be called a DSS. Alternatively enterprisewide DSSs can

encompass many different types of models and large data warehouses, and they can serve many decision makers in an organization. They can provide decision makers with interrelated information and analyses about almost anything in a company.

[Figure 1.7](#) illustrates the basic structure of a DSS with a database component, a modeling component, and a user interface with the decision maker. As noted earlier, each of these components can be small and singular, with one

analytical model linked to a database, or they can be very large and complex, linking many models and large databases. A DSS can be primarily a data-oriented system, or it can be a model-oriented system. A new type of DSS, called an *online analytical processing* system, or *OLAP*, focuses on the use of analytical techniques such as management science models and statistics for decision making. A desktop DSS for a single user can be a spreadsheet program such as

Excel to develop specific solutions to individual problems. [Exhibit 1.1](#) includes all the components of a DSS cost, volume, and price data, a break-even model, and the opportunity for the user to manipulate the data and see the results (i.e., a user interface). Expert Choice is another example of a desktop DSS that uses the analytical hierarchy process (AHP) described in [Chapter 9](#) to structure complex problems by establishing decision criteria, developing priorities, and

ranking decision alternatives.

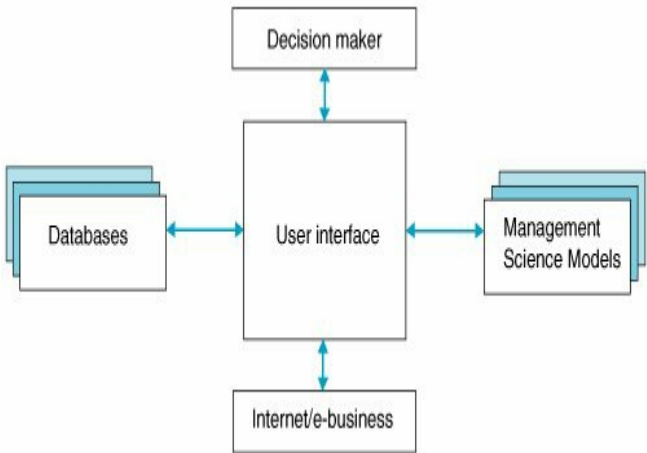
On the other end of the DSS spectrum, an enterprise resource planning (ERP) system is software that can connect the components and functions of an entire company. It can transform data, such as individual daily sales, directly into information that supports immediate decisions in other parts of the company, such as ordering, manufacturing, inventory, and distribution. A large-scale DSS such as an ERP system in a company might include a forecasting model

([Chapter 15](#)) to analyze sales data and help determine future product demand; an inventory model ([Chapter 16](#)) to determine how much inventory to keep on hand; a linear programming model ([Chapters 25](#)) to determine how much material to order and product to produce, and when to produce it; a transportation model ([Chapter 6](#)) to determine the most cost-effective method of distributing a product to customers; and a network flow model ([Chapter 7](#)) to determine the best delivery routes. All

these different management science models and the data necessary to support them can be linked in a single enterprisewide DSS that can provide many decisions to many different decision makers.

[Page 21]

Figure 1.7. A decision support system



In addition to helping managers answer specific questions and make decisions, a DSS may be most useful in answering what-if? questions and performing

sensitivity analysis. In other words, a DSS provides a computer-based laboratory to perform experiments. By linking various management science models together with different databases, a user can change a parameter in one model related to one company function and see what the effect will be in a model related to a different operation in the company. For example, by changing the data in a forecasting model, a manager could see the impact of a hypothetical change in product

demand on the production schedule, as determined by a linear programming model.

Advances in information and computer technology have provided the opportunity to apply management science models to a broad array of complex organizational problems by linking different models to databases in a DSS. These advances have also made the application of management science models more readily available to individual users in the form of desktop DSSs that can help

managers make better decisions relative to their day-to-day operations. In the future it will undoubtedly become even easier to apply management science to the solution of problems with the development of newer software, and management science will become even more important and pervasive as an aid to decision makers as managers are linked within companies with sophisticated computer systems and between companies via the Internet.

Many companies now interface

with new types of DSS over the Internet. In e-business applications companies can link to other business units around the world through computer systems called *intranets*, with other companies through systems called *extranets*, and over the Internet. For example, electronic data interchange (EDI) and point-of-sale data (through bar codes) can provide companies with instantaneous records of business transactions and sales at retail stores that are immediately entered into a

company's DSS to update inventory and production scheduling, using management science models. Internet transportation exchanges enable companies to arrange cost-effective transportation of their products at Web sites that match shipping loads with available trucks at the lowest cost and fastest delivery speed, using sophisticated management science models.

Management Science

Application: A Decision

Support System for

Aluminum Can Production at Coors

Valley Metal Container (VMC), a joint venture between Coors Brewing Company and American National Can, operates the largest single facility for aluminum can production in the world in Golden, Colorado. The plant, encompassing six production lines, manufactures over 4 billion cans per year for seven Coors beer labels produced at Coors breweries in Colorado, Virginia, and Tennessee. The breweries' weekly production schedules are unpredictable, and if sufficient cans are not available,

brewery fill lines can be shut down, at a cost of \$65 per minute. In order to cope with variable brewery demand, VMC builds up inventories of cans for all seven Coors labels during winter and spring, when beer demand is lower, in order to meet higher demand in the summer. Each production line at the can production plant produces multiple labels, and when a line is switched from one label to another, a label change occurs. Finished cans go into either short-term inventory (in trailers), from which they are shipped to a brewery within a day, or two types of longer-term inventory, in which cans are stored on pallets for future delivery.

VMC developed a DSS to determine the weekly production schedule for cans that would meet brewery demand while minimizing the number

of costly label changes and associated inventory costs. The DSS includes an Excel-based user interface for data entry. The Excel spreadsheets are linked to a linear programming model that develops the optimal production schedule that minimizes the costs associated with label changes while meeting demand. The spreadsheet format allows data to be entered and adjusted easily on one worksheet, and the production schedule is developed on another worksheet. The production schedule is then reported on another specially formatted spreadsheet, which the user can view, print, and edit. Costs savings with this DSS average about \$3,000 per week and annual savings of over \$160,000.



Source: E. Katok and D. Ott, "Using Mixed-Integer Programming to Reduce Label Changes in the Coors Aluminum Can Plant," *Interfaces* 30, no. 2 (March/April 2000): 112.

Summary

In the chapters that follow, the model construction and solutions that constitute each management science technique are presented in detail and illustrated with examples. In fact, the primary method of presenting the techniques is through examples. Thus, the text offers you a broad spectrum of knowledge of the mechanics of management science techniques and the

types of problems to which these techniques are applied. However, the ultimate test of a management scientist or a manager who uses management science techniques is the ability to transfer textbook knowledge to the business world. In such instances there is an *art* to the application of management science, but it is an art predicated on practical experience and sound textbook knowledge. Providing the first of these necessities is beyond the scope of textbooks;

providing the second is the objective of this text.

*Management science
is an art.*

References

Ackoff, Russell L., and Sasieni, Maurice W. *Fundamentals of Operations Research*. New York: John Wiley & Sons, 1968.

Beer, Stafford. *Management Sciences: The Business Use of Operations Research*. New York: Doubleday, 1967.

Churchman, C. W., Ackoff, R. L., and Arnoff, E. L. *Introduction to Operations Research*. New York: John Wiley & Sons, 1957.

Fabrycky, W. J., and Torgersen, P. E. *Operations Economy: Industrial Applications of*

Operations Research. Upper Saddle River, NJ: Prentice Hall, 1966.

Hillier, F. S., and Lieberman, G. J. *Operations Research*, 4th ed. San Francisco: Holden-Day, 1987.

Taha, Hamdy A. *Operations Research, An Introduction*, 4th ed. New York: Macmillan, 1987.

Teichroew, P. *An Introduction to Management Science*. New York: John Wiley & Sons, 1964.

Wagner, Harvey M. *Principles of*

Management Science. Upper Saddle River, NJ: Prentice Hall, 1975.

. *Principles of Operations Research*. 2d ed. Upper Saddle River, NJ: Prentice Hall, 1975.

Problems

The Willow Furniture Company produces tables. The fixed monthly cost of production is \$10,000 and the variable cost per table is \$65. Tables sell for \$180 apiece.

- 1.** For a monthly volume of 300 tables, determine the total cost, total revenue, and profit.
- 2.** Determine the monthly break-even point for the Willow Furniture Company.

The Retread Tire Company recaps tires. The annual cost of the recapping operation is \$60,000. The variable cost of recapping a tire is \$9. The company charges \$25 to recap a tire.

- 2.**
- 1.** For an annual volume of 12,000 tires, determine the total cost, total revenue, and profit.
 - 2.** Determine the annual break-even volume for the Retread Tire Company operation.

The Rolling Creek Textile Mill produces denim. The fixed monthly cost is \$21,000, and the variable cost per yard of denim is \$0.45. The mill sells each yard of denim for \$1.30.

- 3.**
- 1.** For a monthly volume of 18,000 yards of denim, determine the total cost, total revenue, and profit.
 - 2.** Determine the annual break-even volume for the Rolling Creek Textile Mill.

Evergreen Fertilizer Company produces fertilizer. The company's fixed monthly cost is \$16,000, and its variable cost per pound of fertilizer is \$0.10. The company sells 800,000 pounds of fertilizer per month.

4. Evergreen sells the fertilizer for \$0.40 per unit. Determine the monthly break-even volume for the company.

5. Graphically illustrate the break-even volume for the Retread Tire Company determined in [Problem 4](#).

6. Graphically illustrate the break-even volume for the Evergreen Fertilizer Company determined in [Problem 4](#).

7. Andy Mendoza makes handcrafted dolls and sells at craft fairs. He is considering making the dolls to sell in stores. He estimates an initial investment for plant and equipment of \$25,000, whereas labor, material, packing and shipping will be about \$10 per doll. If the dolls are sold for \$30 each, what sales volume is required for Andy to break even?

- If the maximum operating capacity of the Tire Company, as described in [Problem 7](#), is 100,000 tires annually, determine the break-even volume as a percentage of that capacity.
- 8.**

[Page 24]

- If the maximum operating capacity of the Creek Textile Mill described in [Problem 8](#) is 100,000 yards of denim per month, determine the break-even volume as a percentage of capacity.
- 9.**

- If the maximum operating capacity of the Fertilizer Company described in [Problem 9](#) is 120,000 pounds of fertilizer per month, determine the break-even volume as a percentage of capacity.
- 10.**

If the Retread Tire Company in [Problem 10](#) has its pricing for recapping a tire from \$25 to \$30, determine the break-even volume as a percentage of capacity.

11. what effect will the change have on the volume?

12. If Evergreen Fertilizer Company in [Prob](#) changes the price of its fertilizer from \$ pound to \$0.60 per pound, what effect change have on the break-even volume?

13. If Evergreen Fertilizer Company change production process to add a weed killer fertilizer in order to increase sales, the per pound will increase from \$0.15 to \$ effect will this change have on the break-even volume computed in [Problem 12](#)?

14. If Evergreen Fertilizer Company increase advertising expenditures by \$14,000 per effect will the increase have on the break-even volume computed in [Problem 13](#)?

- Pastureland Dairy makes cheese, which it sells at local supermarkets. The fixed monthly production is \$4,000, and the variable cost per pound of cheese is \$0.21. The cheese currently sells for \$0.75 per pound; however, the dairy is considering raising the price to \$0.95 per pound. The dairy currently produces and sells 9,000 pounds of cheese per month, but if it raises its price to \$0.95 per pound, sales will decrease to 5,700 pounds per month. Should the dairy raise the price?
- 15.**

- For the doll-manufacturing enterprise discussed in [Problem 7](#), Andy Mendoza has determined that \$10,000 worth of advertising will increase sales volume by 400 dolls. Should he spend this amount for advertising?
- 16.**

Andy Mendoza in [Problem 7](#) is concerned that demand for his dolls will not exceed the break-even point. He believes he can reduce his initial

17.

investment by purchasing used sewing and fewer machines. This will reduce his investment from \$25,000 to \$17,000. will also require his employees to work and perform more operations by hand, increasing variable cost from \$10 to \$1 Will these changes reduce his break-even

The General Store at State University is bookstore located near the dormitories academic supplies, toiletries, sweatshirt shirts, magazines, packaged food items: canned soft drinks and fruit drinks. The the store has noticed that several pizza services near campus make frequent d manager is therefore considering selling the store. She could buy premade frozen and heat them in an oven. The cost of and freezer would be \$27,000. The frozen cost \$3.75 each to buy from a distributor prepare (including labor and a box). To competitive with the local delivery serv

18.

manager believes she should sell the pizza for \$8.95 apiece. The manager needs to write a proposal for the university's director of services.

- 1.** Determine how many pizzas would be sold to break even.
- 2.** If The General Store sells 20 pizzas, how many days would it take to break even?
- 3.** The manager of the store anticipates that once the local pizza delivery service loses business, they will react by raising prices. If after a month (30 days) the manager has to lower the price of a pizza to \$7.95 to keep demand at 20 pizzas as she expects, what will the new break-even point be, and how long will it take to break even?

Kim Davis has decided to purchase a coffee machine for the university's director of services.

but she is unsure about which rate plan. The "regular" plan charges a fixed fee of \$30 per month for 1,000 minutes of airtime plus \$0.20 per minute for any time over 1,000 minutes. The "executive" plan charges a fixed fee of \$50 per month for 1,200 minutes of airtime plus \$0.15 per minute over 1,200 minutes.

19.

[Page 25]

- 1.** If Kim expects to use the phone for 1,500 minutes per month, which plan should she choose?
- 2.** At what level of use would Kim be indifferent between the two plans?

Annie McCoy, a student at Tech, plans to open a hot dog stand inside Tech's football stadium for all home games. There are seven home games scheduled for the upcoming season. She has offered to the Tech athletic department a vendor's fee of \$3,000 for the season. Her stand and equipment will cost \$1,000. She expects to sell 100 hot dogs per game at \$2.00 each. Variable costs are \$0.50 per hot dog. Fixed costs are \$1,000 for the season. She expects to have a net profit of \$1,500.

equipment will cost her \$4,500 for the estimates that each hot dog she sells v \$0.35. She has talked to friends at oth universities who sell hot dogs at games their information and the athletic depar forecast that each game will sell out, st

20. anticipates that she will sell approximat hot dogs during each game.

- 1.** What price should she charge for order to break even?
- 2.** What factors might occur during t that would alter the volume sold & break-even price Annie might char
- 3.** What price would you suggest the charge for a hot dog to provide h reasonable profit while remaining with other food vendors?

Molly Dymond and Kathleen Taylor are the possibility of teaching swimming to

the summer. A local swim club opens its pool from 11:00 a.m. to 1:00 p.m. each day, so it is available to rent for the entire morning. The cost of renting the pool for the 10-week period for which Molly and Kathleen need it is \$1,700. The pool would also charge Molly and Kathleen an admission, towel service, and lifeguarding fee of \$7 per pupil, and Molly and Kathleen estimate an additional \$5 cost per pupil for the pool's several assistants. Molly and Kathleen plan to charge \$75 per student for the 10-week summer class.

21.

- 1.** How many pupils do Molly and Kathleen need to enroll in their class to break even?
- 2.** If Molly and Kathleen want to make a profit of \$5,000 for the summer, how many pupils do they need to enroll?
- 3.** Molly and Kathleen estimate that they will not be able to enroll more than 60 pupils. If they enroll this many pupils, how much more do they need to charge per pupil in order to realize their profit goal of \$5,000?

The College of Business at Tech is planning an online MBA program. The initial start-up costs for computing equipment, facilities, course development, and staff recruitment and development is \$350,000. The college will charge tuition of \$18,000 per student per year. However, the university administration will reimburse the college \$12,000 per student for the first year for administrative costs. For subsequent years, the college will receive its share of the tuition payments.

22.

- 1.** How many students does the college need to enroll in the first year to break even?
- 2.** If the college can enroll 75 students in the first year, how much profit will it make?
- 3.** The college believes it can increase tuition to \$24,000, but doing so would reduce enrollment to 35. Should the college be doing this?

23.

The Star Youth Soccer Club helps to support boys' and girls' teams financially, primarily the payment of coaches. The club puts on a tournament each fall to help pay its expenses. The cost of putting on the tournament is \$8,000, mainly for development, printing, and tournament brochures. The tournament costs \$400 per team. For every team that enters, it costs the club about \$75 to pay referees. With a three-game minimum each team is guaranteed. If the club needs to clear \$60,000 from the tournament, how many teams should it invite?

In the example used to demonstrate marginal cost construction in this chapter (p. 4), a firm produces a product, x , for \$20 that costs \$5 to make. It uses 100 pounds of steel to make the product. It takes 4 pounds of steel to make each unit of the model that was constructed is

maximize $Z = 15x$
subject to
 $4x = 100$

24.

[Page 26]

Now suppose that there is a second product that has a profit of \$10 and requires 2 pounds of steel to make, such that the model becomes

maximize $Z = 15x + 10y$
subject to
 $4x + 2y = 100$

Can you determine a solution to this new model that will achieve the objective? Explain.

Consider a model in which two products are produced. There are 100 pounds of material and 80 hours of labor available. It requires 4 pounds of material and 1 hour of labor to produce a unit of x , and 4 pounds of material and 2 hours of labor to produce a unit of y .

25.

labor to produce a unit of y . The profit per unit, and the profit for y is \$50 per unit. To want to know how many units of x and y to produce to maximize profit, the model

$$\text{maximize } Z = 30x + 50y$$

subject to

$$2x + 4y = 100$$

$$x + 5y = 80$$

Determine the solution to this problem and state your answer.

26.

The Easy Drive Car Rental Agency needs 300 cars in its Nashville operation and 300 cars in Jacksonville, and it currently has 400 cars in both Atlanta and Birmingham. It costs \$30 to move a car from Atlanta to Nashville, \$40 to move a car from Atlanta to Jacksonville, \$50 to move a car from Birmingham to Nashville, and \$60 to move a car from Birmingham to Jacksonville. The agency wants to determine how many

be transported from the agencies in Atlanta to the agencies in Birmingham to the agencies in Nashville to the agencies in Jacksonville in order to meet demand while minimizing the transport costs. Develop a mathematical model for this problem and solve it to determine a solution.

Ed Norris has developed a Web site for his textbook business at State University. In order to advertise he needs to forecast the number of visits he expects in the future. For the past 5 months he has had the following number of visits:

Month	1	2	3	4	5
-------	---	---	---	---	---

27.

Site visits	6,300	10,200	14,700	18,500	25,000
-------------	-------	--------	--------	--------	--------

Determine a forecast for Ed to use for
explain the logic used to develop your f

When Marie McCoy wakes up on Saturday morning, she remembers that she had the PTA she would make some cakes and homemade bread for its bake sale that day. However, she does not have time to go to the store and get ingredients, and she has only 3 hours time to bake things in her oven. Because different breads require different baking temperatures, she cannot bake them simultaneously, and she has only 3 hours available to bake. A cake requires 2 cups of flour, and a loaf of bread requires 8 cups of flour. A cake requires 45 minutes to bake, and a loaf of bread requires 30 minutes to bake. The PTA will sell a cake for \$10 and a loaf of bread for \$6. Marie wants to decide how many cakes and loaves of bread she should make. Identify all possible solutions to this problem (i.e.,

- 28.** flour, and a loaf of bread requires 8 cup
20 cups of flour. A cake requires 45 min
bake, and a loaf of bread requires 30 m
PTA will sell a cake for \$10 and a loaf o
\$6. Marie wants to decide how many c
loaves of bread she should make. Ident
possible solutions to this problem (i.e.,

combinations of cakes and loaves of bread (if you have the time and flour to bake) and sell one.

Case Problem

THE CLEAN CLOTHES CORNER LAUNDRY

When Molly Lai purchased the Clean Clothes Corner Laundry, she thought that because it was in a good location near several high-income neighborhoods, she would automatically generate good business if she improved the laundry's physical appearance. Thus, she initially invested a lot of her cash

reserves in remodeling the exterior and interior of the laundry. However, she just about broke even in the year following her acquisition of the laundry, which she didn't feel was a sufficient return, given how hard she had worked. Molly didn't realize that the dry-cleaning business is very competitive and that success is based more on price and quality service, including quickness of service, than on the laundry's appearance.

In order to improve her service, Molly is considering

purchasing new dry-cleaning equipment, including a pressing machine that could substantially increase the speed at which she can dry-clean clothes and improve their appearance. The new machinery costs \$16,200 installed and can clean 40 clothes items per hour (or 320 items per day). Molly estimates her variable costs to be \$0.25 per item dry-cleaned, which will not change if she purchases the new equipment. Her current fixed costs are \$1,700 per month. She charges

customers \$1.10 per clothing item.

- 1.** What is Molly's current monthly volume?
- 2.** If Molly purchases the new equipment, how many additional items will she have to dry-clean each month to break even?
- 3.** Molly estimates that with the new equipment she can increase her volume to 4,300 items per month. What monthly profit would she realize with that level

of business during the next 3 years? After 3 years?

4. Molly believes that if she doesn't buy the new equipment but lowers her price to \$0.99 per item, she will increase her business volume. If she lowers her price, what will her new break-even volume be? If her price reduction results in a monthly volume of 3,800 items, what will her monthly profit be?
5. Molly estimates that if she purchases the new

equipment and lowers her price to \$0.99 per item, her volume will increase to about 4,700 units per month. Based on the local market, that is the largest volume she can realistically expect. What should Molly do?

Case Problem

THE OCOBEE RIVER RAFTING COMPANY

Vicki Smith, Penny Miller, and Darryl Davis are students at State University. In the summer they often go rafting with other students down the Ocobee River in the nearby Blue Ridge Mountain foothills. The river has a number of minor rapids but is not generally dangerous. The students' rafts basically consist

of large rubber tubes, sometimes joined together with ski rope. They have noticed that a number of students who come to the river don't have rubber rafts and often ask to borrow theirs, which can be very annoying. In discussing this nuisance, it occurred to Vicki, Penny, and Darryl that the problem might provide an opportunity to make some extra money. They considered starting a new enterprise, the Ocobee River Rafting Company, to sell rubber rafts at the river. They determined that their

initial investment would be about \$3,000 to rent a small parcel of land next to the river on which to make and sell the rafts; to purchase a tent to operate out of; and to buy some small equipment such as air pumps and a rope cutter. They estimated that the labor and material cost per raft will be about \$12, including the purchase and shipping costs for the rubber tubes and rope. They plan to sell the rafts for \$20 apiece, which they think is about the maximum price students will pay for a

preassembled raft.

Soon after they determined these cost estimates, the newly formed company learned about another rafting company in North Carolina that was doing essentially what they planned to do. Vicki got in touch with one of the operators of that company, and he told her the company would be willing to supply rafts to the Ocobee River Rafting Company for an initial fixed fee of \$9,000 plus \$8 per raft, including shipping. (The Ocobee River Rafting Company would still have to

rent the parcel of riverside land and tent for \$1,000.) The rafts would already be inflated and assembled. This alternative appealed to Vicki, Penny, and Darryl because it would reduce the amount of time they would have to work pumping up the tubes and putting the rafts together, and it would increase time for their schoolwork.

Although the students prefer the alternative of purchasing the rafts from the North Carolina company, they are concerned about the large initial cost and worried about

whether they will lose money. Of course, Vicki, Penny, and Darryl realize that their profit, if any, will be determined by how many rafts they sell. As such, they believe that they first need to determine how many rafts they must sell with each alternative in order to make a profit and which alternative would be best given different levels of demand. Furthermore, Penny has conducted a brief sample survey of people at the river and estimates that demand for rafts for the summer will be

around 1,000 rafts.

Perform an analysis for the Ocobee River Rafting Company to determine which alternative would be best for different levels of demand. Indicate which alternative should be selected if demand is approximately 1,000 rafts and how much profit the company would make.

Case Problem

CONSTRUCTING A DOWNTOWN PARKING LOT IN DRAPER

The town of Draper, with a population of 20,000, sits adjacent to State University, which has an enrollment of 27,000 students. Downtown Draper merchants have long complained about the lack of parking available to their customers. This is one primary reason for the steady migration

of downtown businesses to a mall several miles outside town. The local chamber of commerce has finally convinced the town council to consider the construction of a new multilevel indoor parking facility downtown. Kelly Mattingly, the town's public works director, has developed plans for a facility that would cost \$4.5 million to construct. To pay for the project, the town would sell municipal bonds with a duration of 30 years at 8% interest. Kelly also estimates that five employees would be

required to operate the lot on a daily basis, at a total annual cost of \$140,000. It is estimated that each car that enters the lot would park for an average of 2.5 hours and pay an average fee of \$3.20. Further, it is estimated that each car that parks in the lot would (on average) cost the town \$0.60 in annual maintenance for cleaning and repairs to the facility. Most of the downtown businesses (which includes a number of restaurants) are open 7 days per week.

- 1.** Using break-even analysis, determine the number of cars that would have to park in the lot on an annual basis to pay off the project in the 30-year time frame.
- 2.** From the results in (A), determine the approximate number of cars that would have to park in the lot on a daily basis. Does this seem to be a reasonable number to achieve, given the size of the town and college population?

Chapter 2. Linear Programming: Model Formulation and Graphical Solution

[Page 30]

Many major decisions faced by a manager of a business focus on the best way to achieve the objectives of the firm, subject to the restrictions placed on the manager by the operating

environment. These restrictions can take the form of limited resources, such as time, labor, energy, material, or money; or they can be in the form of restrictive guidelines, such as a recipe for making cereal or engineering specifications. One of the most frequent objectives of business firms is to gain the most profit possible or, in other words, to *maximize* profit. The objective of individual organizational units within a firm (such as a production or packaging department) is often to *minimize* cost. When a

manager attempts to solve a general type of problem by seeking an objective that is subject to restrictions, the management science technique called **linear programming** is frequently used.

Objectives of a business frequently are to maximize profit or minimize cost.

Linear programming is a model that consists of linear relationships representing a firm's decision(s), given an objective and resource constraints.

There are three steps in applying the linear programming technique. First, the problem must be identified as being solvable by linear

programming. Second, the unstructured problem must be formulated as a mathematical model. Third, the model must be solved by using established mathematical techniques. The linear programming technique derives its name from the fact that the functional relationships in the mathematical model are *linear*, and the solution technique consists of predetermined mathematical steps that is, a *program*. In this chapter we will concern ourselves with the formulation of the

mathematical model that represents the problem and then with solving this model by using a graph.

Model Formulation

A linear programming model consists of certain common components and characteristics. The model components include decision variables, an objective function, and model constraints, which consist of decision variables and parameters. **Decision variables** are mathematical symbols that represent levels of activity by the firm. For

example, an electrical manufacturing firm desires to produce x_1 radios, x_2 toasters, and x_3 clocks, where x_1 , x_2 , and x_3 are symbols representing unknown variable quantities of each item. The final values of x_1 , x_2 , and x_3 , as determined by the firm, constitute a *decision* (e.g., the equation $x_1 = 100$ radios is a decision by the firm to produce 100 radios).

Decision variables
are mathematical

*symbols that
represent levels of
activity.*

The **objective function** is a linear mathematical relationship that describes the objective of the firm in terms of the decision variables. The objective function always consists of either *maximizing* or *minimizing* some value (e.g., maximize the profit or minimize the cost of producing

radios).

*The **objective function** is a linear relationship that reflects the objective of an operation.*

The *model constraints* are also linear relationships of the decision variables; they represent the restrictions placed on the firm by the operating environment. The

restrictions can be in the form of limited resources or restrictive guidelines. For example, only 40 hours of labor may be available to produce radios during production. The actual numeric values in the objective function and the constraints, such as the 40 hours of available labor, are **parameters**.

*A **constraint** is a linear relationship that represents a restriction on decision making.*

Parameters are
*numerical values that
are included in the
objective functions
and constraints.*

The next section presents an example of how a linear programming model is formulated. Although this example is simplified, it is realistic and represents the

type of problem to which linear programming can be applied. In the example, the model components are distinctly identified and described. By carefully studying this example, you can become familiar with the process of formulating linear programming models.

A Maximization Model Example

Beaver Creek Pottery Company is a small crafts operation run by a Native American tribal council. The company employs skilled artisans to produce clay bowls and mugs with authentic Native American designs and colors. The two primary resources used by the company are special pottery clay and skilled labor. Given these limited resources, the company

desires to know how many bowls and mugs to produce each day in order to maximize profit. This is generally referred to as a *product mix* problem type. This scenario is illustrated in [Figure 2.1](#).

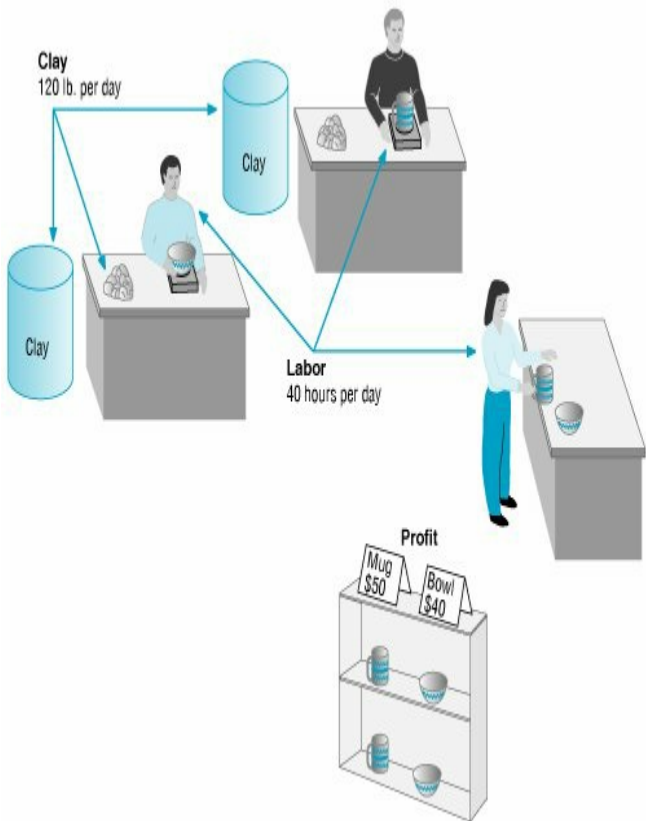
[Page 31]

Time Out: For George B. Dantzig

Linear programming, as it is known today, was conceived in 1947 by George B. Dantzig while he was the head of the Air Force Statistical Control's Combat Analysis Branch at the Pentagon. The military referred to its plans for training, supplying, and deploying combat units as "programs." When Dantzig analyzed Air Force planning problems, he realized that they could be formulated as a system of linear inequalitieshence his original name for the technique, "programming in a linear structure," which was later shortened to "linear programming."

Figure 2.1. Beaver Creek Pottery Company

[\[View full size image\]](#)

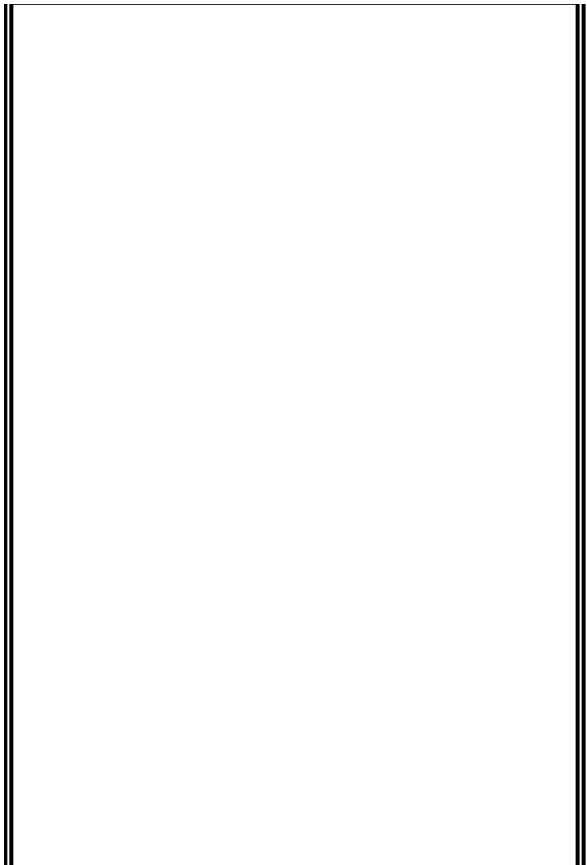


The two products have the following resource requirements for production and profit per item produced (i.e., the model parameters):

Resource Requirements

Product	LABOR (HR./UNIT)	CLAY (LB./UNIT)	PROFIT (\$/UNIT)
Bowl	1	4	40
Mug	2	3	50

There are 40 hours of labor and 120 pounds of clay available each day for production. We will formulate this problem as a linear programming model by defining each component of the model separately and then combining the components into a single model. The steps in this formulation process are summarized as follows:



Summary of LP Model

Formulation Steps

Define the decision variables

- Step 1.** How many bowls and mugs to produce

Define the objective function

- Step 2.** Maximize profit

Define the constraints

- Step 3.** The resources (clay and labor) available

A linear programming model consists of decision variables, an objective function, and constraints.

Decision Variables

The decision confronting management in this problem is

how many bowls and mugs to produce. The two decision variables represent the number of bowls and mugs to be produced on a daily basis. The quantities to be produced can be represented symbolically as

x_1 = number of bowls to produce

x_2 = number of mugs to produce

The Objective Function

The objective of the company is to maximize total profit. The company's profit is the sum of the individual profits gained from each bowl and mug. Profit derived from bowls is determined by multiplying the unit profit of each bowl, \$40, by the number of bowls produced, x_1 . Likewise, profit derived from mugs is derived from the unit profit of a mug, \$50, multiplied by the number of mugs produced, x_2 . Thus, total profit, which we will define symbolically as Z , can be expressed mathematically as

$\$40x_1 + \$50x_2$. By placing the term *maximize* in front of the profit function, we express the objective of the firm to maximize total profit:

$$\text{maximize } Z = \$40x_1 + 50x_2$$

where

Z = total profit per day

$\$40x_1$ = profit from bowls

$\$50x_2$ = profit from mugs

Model Constraints

In this problem two resources are used for production labor and clay both of which are limited. Production of bowls and mugs requires both labor and clay. For each bowl produced, 1 hour of labor is required.

Therefore, the labor used for the production of bowls is $1x_1$ hours. Similarly, each mug requires 2 hours of labor; thus, the labor used to produce mugs

every day is $2x_2$ hours. The total labor used by the company is the sum of the individual amounts of labor used for each product:

$$1x_1 + 2x_2$$

[Page 33]

However, the amount of labor represented by $1x_1 + 2x_2$ is limited to 40 hours per day; thus, the complete labor constraint is

$$1x_1 + 2x_2 \leq 40 \text{ hr.}$$

The "less than or equal to" ($\leq$) inequality is employed instead of an equality ($=$) because the 40 hours of labor is a maximum limitation that *can be used*, not an amount that *must be used*. This constraint allows the company some flexibility; the company is not restricted to using exactly 40 hours but can use whatever amount is necessary to maximize profit, up to and including 40 hours. This means that it is possible to have idle, or excess, capacity (i.e., some of the 40 hours may not be used).

The constraint for clay is formulated in the same way as the labor constraint. Because each bowl requires 4 pounds of clay, the amount of clay used daily for the production of bowls is $4x_1$ pounds; and because each mug requires 3 pounds of clay, the amount of clay used daily for mugs is $3x_2$. Given that the amount of clay available for production each day is 120 pounds, the material constraint can be formulated as

$$4x_1 + 3x_2 \leq 120 \text{ lb.}$$

A final restriction is that the number of bowls and mugs produced must be either zero or a positive value because it is impossible to produce negative items. These restrictions are referred to as **nonnegativity constraints** and are expressed mathematically as

Nonnegativity constraints restrict the decision variables to zero or positive values.

$$x_1 \geq 0, x_2 \geq 0$$

The complete linear programming model for this problem can now be summarized as follows:

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

$$1x_1 + 2x_2 \leq 40$$

$$4x_1 + 3x_2 \leq 120$$

$$x_1, x_2 \geq 0$$

The solution of this model will result in numeric values for x_1 and x_2 that will maximize total

profit, Z . As *one possible* solution, consider $x_1 = 5$ bowls and $x_2 = 10$ mugs. First, we will substitute this hypothetical solution into each of the constraints in order to make sure that the solution does not require more resources than the constraints show are available:

$$\begin{aligned} 1(5) + 2(10) &\leq 40 \\ 25 &\leq 40 \end{aligned}$$

and

$$\begin{aligned} 4(5) + 3(10) &\leq 120 \\ 50 &\leq 120 \end{aligned}$$

Because neither of the constraints is violated by this hypothetical solution, we say the solution is **feasible** (i.e., it is possible). Substituting these solution values in the objective function gives $Z = 40(5) + 50(10) = \$700$. However, for the time being, we do not have any way of knowing whether \$700 is the *maximum* profit.

A feasible solution does not violate any of the constraints.

Now consider a solution of $x_1 = 10$ bowls and $x_2 = 20$ mugs.
This solution results in a profit of

$$\begin{aligned} Z &= \$40(10) + 50(20) \\ &= 400 + 1,000 \\ &= \$1,400 \end{aligned}$$

[Page 34]

Although this is certainly a better solution in terms of profit, it is **infeasible** (i.e., not possible) because it violates the resource constraint for labor:

$$1(10) + 2(20) \leq 40$$
$$50 \not\leq 40$$

An ***infeasible problem*** violates at least one of the constraints.

The solution to this problem must maximize profit without violating the constraints. The solution that achieves this objective is $x_1 = 24$ bowls and $x_2 = 8$ mugs, with a

corresponding profit of \$1,360. The determination of this solution is shown using the graphical solution approach in the following section.

Graphical Solutions of Linear Programming Models

Following the formulation of a mathematical model, the next stage in the application of linear programming to a decision-making problem is to find the solution of the model. A common solution approach is to solve algebraically the set of mathematical relationships that form the model either manually

or using a computer program, thus determining the values for the decision variables.

However, because the relationships are *linear*, some models and solutions can be illustrated *graphically*.

Graphical solutions are limited to linear programming problems with only two decision variables.

The graphical method is realistically limited to models with only two decision variables, which can be represented on a graph of two dimensions. Models with three decision variables can be graphed in three dimensions, but the process is quite cumbersome, and models of four or more decision variables cannot be graphed at all.

Although the graphical method is limited as a solution approach, it is very useful at this point in our presentation of linear programming in that it

gives a picture of how a solution is derived. Graphs can provide a clearer understanding of how the computer and mathematical solution approaches presented in subsequent chapters work and, thus, a better understanding of the solutions.

The graphical method provides a picture of how a solution is obtained for a linear programming problem.

Graphical Solution of a Maximization Model

The product mix model will be used to demonstrate the graphical interpretation of a linear programming problem. Recall that the problem describes Beaver Creek Pottery Company's attempt to decide how many bowls and mugs to produce daily, given limited amounts of labor and clay. The complete linear programming

model was formulated as

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$

$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$

$$x_1, x_2 \geq 0$$

where

x_1 = number of bowls produced

x_2 = number of mugs produced

[Figure 2.2](#) is a set of coordinates for the decision

variables x_1 and x_2 , on which the graph of our model will be drawn. Note that only the positive quadrant is drawn (i.e., the quadrant where x_1 and x_2 will always be positive) because of the nonnegativity constraints, $x_1 \geq 0$ and $x_2 \geq 0$.

Management Science

Application: Operational Cost Control at Kellogg

Kellogg is the world's largest cereal producer and a leading producer of convenience foods, with worldwide sales in 1999 of almost \$7 billion. The company started with a single product, Kellogg's Corn Flakes, in 1906 and over the years developed a product line of other cereals, including Rice Krispies and Corn Pops, and convenience foods such as Pop-Tarts and Nutri-Grain cereal bars. Kellogg operates 5 plants in the United States and Canada and 7 distribution centers, and it contracts with 15 co-packers to produce or pack some of the Kellogg products. Kellogg must coordinate the

production, packaging, inventory, and distribution of roughly 80 cereal products alone at these various facilities.

For more than a decade Kellogg has been using a large-scale linear programming model called the Kellogg Planning System (KPS) to plan its weekly production, inventory, and distribution decisions. The model decision variables include the amount of each product produced in a production process at each plant, the units of product packaged, the amount of inventory held, and the shipments of products to other plants and distribution centers. Model constraints include production processing time, packaging capacity, balancing constraints that make sure that all products produced are also packaged during the week, inventory balancing constraints, and inventory

safety stock requirements. The model objective is cost minimization. Kellogg has also developed a tactical version of this basic operational linear programming model for long-range planning for 12 to 24 months into the future. The KPS model is credited with saving Kellogg \$4.5 million in reduced production, inventory, and distribution costs in 1995, and it is estimated that KPS has saved Kellogg many more millions of dollars since the mid-1990s. The tactical version of KPS recently helped the company consolidate production capacity with estimated projected savings of almost \$40 million.



Source: G. Brown, J. Keegan, B. Vigus, and K. Wood, "The Kellogg Company Optimizes Production, Inventory, and Distribution," *Interfaces* 31, no. 6 (November/December 2001): 115.

Figure 2.2.

Coordinates for graphical analysis

x_2

60

50

40

30

20

10

0

10

20

30

40

50

60

 x_1

The first step in drawing the graph of the model is to plot the constraints on the graph. This is done by treating both constraints as equations (or straight lines) and plotting each line on the graph. Let's consider the labor constraint line first:

$$x_1 + 2x_2 = 40$$

Constraint lines are plotted as equations.

A simple procedure for plotting this line is to determine two points that are on the line and then draw a straight line through the points. One point can be found by letting $x_1 = 0$ and solving for x_2 :

$$\begin{aligned}(0) + 2x_2 &= 40 \\ x_2 &= 20\end{aligned}$$

Thus, one point is at the coordinates $x_1 = 0$ and $x_2 = 20$. A second point can be found by letting $x_2 = 0$ and

solving for x_1 :

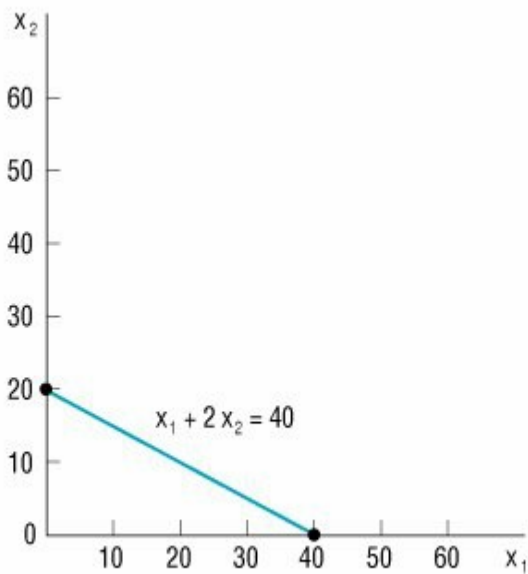
$$x_1 + 2(0) = 40$$

$$x_1 = 40$$

Now we have a second point, $x_1 = 40$, $x_2 = 0$. The line on the graph representing this equation is drawn by connecting these two points, as shown in [Figure 2.3](#). However, this is only the graph of the constraint *line* and does not reflect the entire constraint, which also includes the values that are less than or equal to ($\leq$) this line. The *area* representing the entire

constraint is shown in [Figure 2.4](#).

Figure 2.3. Graph of the labor constraint line



To test the correctness of the constraint area, we check any two points one inside the constraint area and one outside. For example, check point A in [Figure 2.4](#), which is at the intersection of $x_1 = 10$ and $x_2 = 10$. Substituting these values into the following labor constraint,

$$10 + 2(10) \leq 40$$
$$30 \leq 40 \text{ hr.}$$

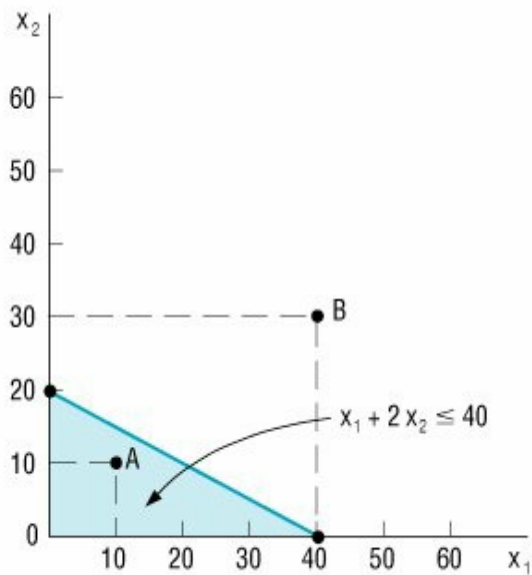
[Page 37]

shows that point A is indeed within the constraint area, as

these values for x_1 and x_2 yield a quantity that does not exceed the limit of 40 hours. Next, we check point B at $x_1 = 40$ and $x_2 = 30$:

$$40 + 2(30) \leq 40$$
$$100 \not\leq 40 \text{ hr.}$$

Figure 2.4. The labor constraint area



Point B is obviously outside the constraint area because the values for x_1 and x_2 yield a quantity (100) that exceeds the limit of 40 hours.

We draw the line for the clay constraint the same way as the one for the labor constraint by finding two points on the constraint line and connecting them with a straight line. First, let $x_1 = 0$ and solve for x_2 :

$$\begin{aligned}4(0) + 3x_2 &= 120 \\x_2 &= 40\end{aligned}$$

Performing this operation

results in a point, $x_1 = 0, x_2 = 40$. Next, we let $x_2 = 0$ and then solve for x_1 :

$$4x_1 + 3(0) = 120$$

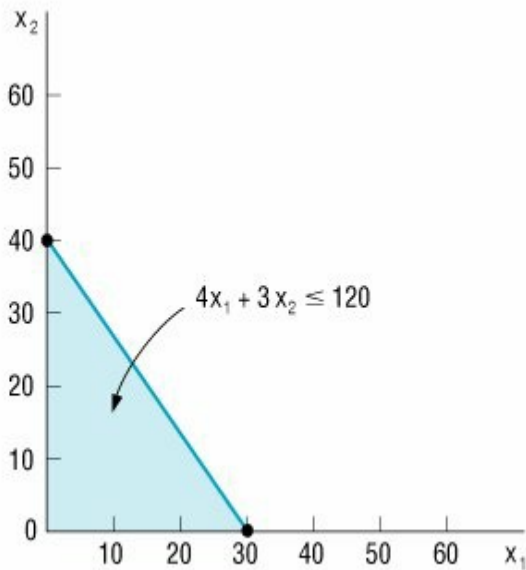
$$x_1 = 30$$

This operation yields a second point, $x_1 = 30, x_2 = 0$. Plotting these points on the graph and connecting them with a line gives the constraint line and area for clay, as shown in [Figure 2.5](#).

Figure 2.5. The

constraint area for clay

**(This item is displayed on page 38
in the print version)**

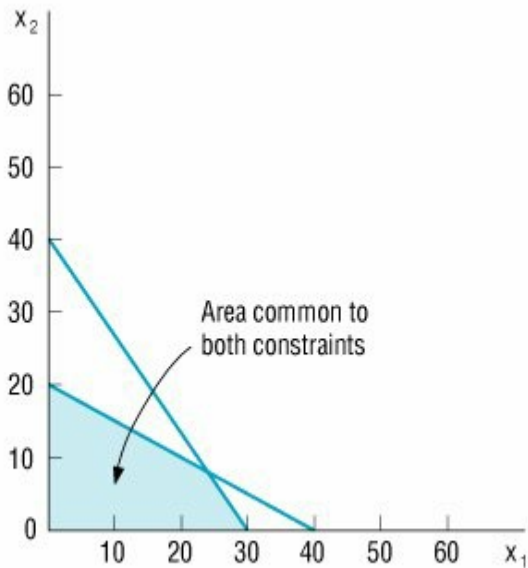


Combining the two individual graphs for both labor and clay ([Figures 2.4](#) and [2.5](#)) produces a graph of the model constraints, as shown in [Figure 2.6](#). The shaded area in [Figure 2.6](#) is the area that is common to both model constraints. Therefore, this is the only area on the graph that contains points (i.e., values for x_1 and x_2) that will satisfy both constraints simultaneously. For example, consider the points R , S , and T in [Figure 2.7](#). Point R satisfies both constraints; thus, we say it is a *feasible* solution

point. Point S satisfies the clay constraint ($4x_1 + 3x_2 \leq 120$) but exceeds the labor constraint; thus, it is infeasible. Point T satisfies neither constraint; thus, it is also infeasible.

[Page 38]

Figure 2.6. Graph of both model constraints



The shaded area in [Figure 2.7](#) is referred to as the *feasible solution area* because all the points in this area satisfy both constraints. Some point within this feasible solution area will result in *maximum profit* for Beaver Creek Pottery Company. The next step in the graphical solution approach is to locate this point.

The feasible solution area is an area on the graph that is bounded by the constraint equations.

The Optimal Solution Point

The second step in the graphical solution method is to locate the point in the feasible solution area that will result in the greatest total profit. To begin the solution analysis, we first plot the objective function line for an *arbitrarily* selected level of profit. For example, if we say profit, Z , is \$800, the objective function is

$$\$800 = 40x_1 + 50x_2$$

Plotting this line just as we plotted the constraint lines results in the graph shown in [Figure 2.8](#). Every point on this line is in the feasible solution area and will result in a profit of \$800 (i.e., every combination of x_1 and x_2 on this line will give a Z value of \$800). However, let us see whether an even greater profit will still provide a feasible solution. For example, consider profits of \$1,200 and \$1,600, as shown in [Figure 2.9](#).

**Figure 2.7. The
feasible solution area
constraints**

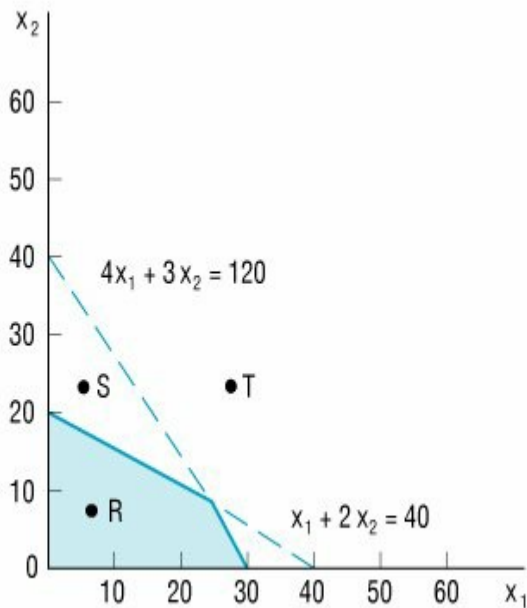
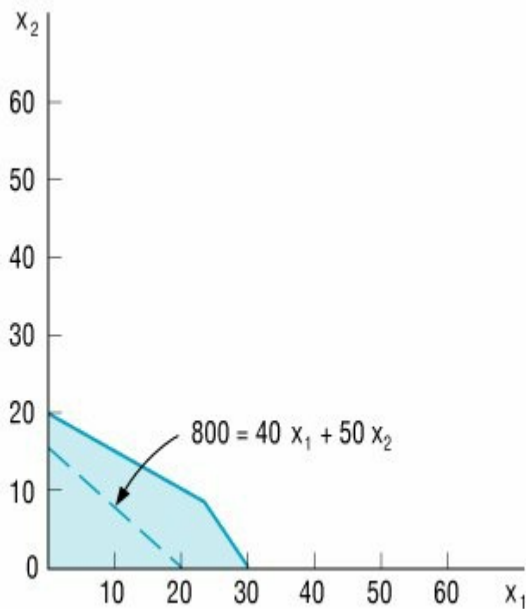


Figure 2.8. Objective function line for $Z =$ \$800



A portion of the objective function line for a profit of \$1,200 is outside the feasible solution area, but part of the line remains within the feasible area. Therefore, this profit line indicates that there are feasible solution points that give a profit greater than \$800. Now let us increase profit again, to \$1,600. This profit line, also shown in [Figure 2.9](#), is completely outside the feasible solution area. The fact that no points on this line are feasible indicates that a profit of \$1,600 is not possible.

Because a profit of \$1,600 is too great for the constraint limitations, as shown in [Figure 2.9](#), the question of the maximum profit value remains. We can see from [Figure 2.9](#) that profit increases as the objective function line moves away from the origin (i.e., the point $x_1 = 0, x_2 = 0$). Given this characteristic, the maximum profit will be attained at the point where the objective function line is farthest from the origin *and* is still touching a point in the feasible solution area. This

point is shown as point B in [Figure 2.10](#).

[Page 40]

Figure 2.9.
Alternative objective
function lines for
profits, Z , of \$800,
\$1,200, and \$1,600

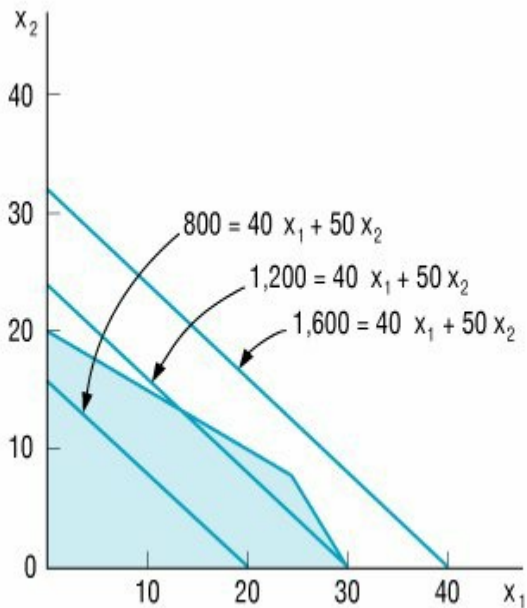
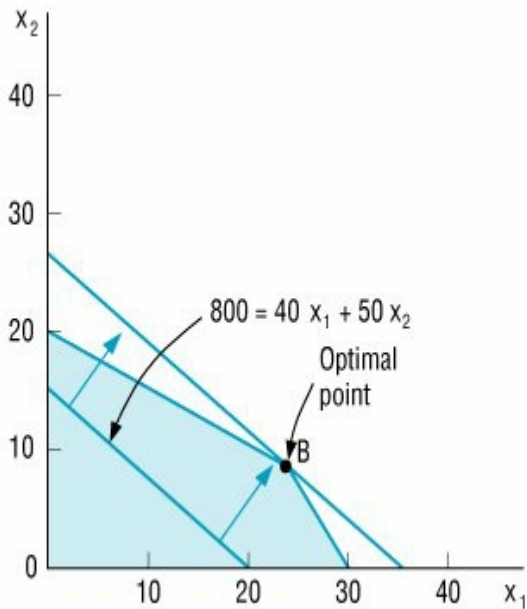


Figure 2.10.
Identification of
optimal solution
point



To find point B , we place a straightedge parallel to the objective function line $\$800 = 40x_1 + 50x_2$ in [Figure 2.10](#) and move it outward from the origin as far as we can without losing contact with the feasible solution area. Point B is referred to as the [**optimal \(i.e., best\) solution.**](#)

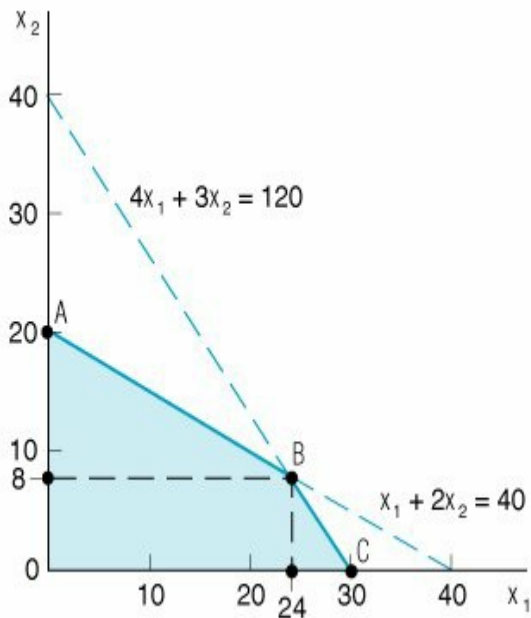
The ***optimal solution*** is the best feasible solution.

The Solution Values

The third step in the graphical solution approach is to solve for the values of x_1 and x_2 once the optimal solution point has been found. It is possible to determine the x_1 and x_2 coordinates of point B in [Figure 2.10](#) directly from the graph, as shown in [Figure 2.11](#). The graphical coordinates corresponding to point B in [Figure 2.11](#) are $x_1 = 24$ and $x_2 = 8$. This is the optimal solution for the decision variables in the

problem. However, unless an absolutely accurate graph is drawn, it is frequently difficult to determine the correct solution directly from the graph. A more exact approach is to determine the solution values mathematically once the optimal point on the graph has been determined. The mathematical approach for determining the solution is described in the following pages. First, however, we will consider a few characteristics of the solution.

Figure 2.11. Optimal solution coordinates



In [Figure 2.10](#), as the objective function was increased, the last point it touched in the feasible solution area was on the boundary of the feasible solution area. The solution point is always on this boundary because the boundary contains the points farthest from the origin (i.e., the points corresponding to the greatest profit). This characteristic of linear programming problems reduces the number of possible solution points considerably, from all points in the solution area to

just those points on the boundary. However, the number of possible solution points is reduced even more by another characteristic of linear programming problems.

The optimal solution point is the last point the objective function touches as it leaves the feasible solution area.

The solution point will be on the boundary of the feasible solution area and at one of the *corners* of the boundary where two constraint lines intersect. (The graphical axes, you will recall, are also constraints because $x_1 \geq 0$ and $x_2 \geq 0$.)

These corners (points *A*, *B*, and *C* in [Figure 2.11](#)) are protrusions, or *extremes*, in the feasible solution area; they are called **extreme points**. It has been proven mathematically that the optimal solution in a linear programming model will always occur at an extreme

point. Therefore, in our sample problem the possible solution points are limited to the three extreme points, A , B , and C . The optimal extreme point is the extreme point the objective function touches last as it leaves the feasible solution area, as shown in [Figure 2.10](#).

Extreme points are corner points on the boundary of the feasible solution area.

From the graph shown in [Figure 2.10](#), we know that the optimal solution point is B . Because point B is formed by the intersection of two constraint lines, as shown in [Figure 2.11](#), these two lines are *equal* at point B . Thus, the values of x_1 and x_2 at that intersection can be found by solving the two equations *simultaneously*.

*Constraint equations
are solved
simultaneously at the
optimal extreme*

*point to determine
the variable solution
values.*

First, we convert both
equations to functions of x_1 :

$$x_1 + 2x_2 = 40$$

$$x_1 = 40 - 2x_2$$

and

$$4x_1 + 3x_2 = 120$$

$$4x_1 = 120 - 3x_2$$

$$x_1 = 30 - (3x_2/4)$$

Now we let x_1 in the first equation equal x_1 in the second equation,

$$40 - 2x_2 = 30 - (3x_2/4)$$

and solve for x_2 :

$$5x_2/4 = 10$$

$$x_2 = 8$$

Substituting $x_2 = 8$ into either one of the original equations gives a value for x_1 :

$$\begin{aligned}x_1 &= 40 - 2x_2 \\x_1 &= 40 - 2(8) \\&= 24\end{aligned}$$

Thus, the optimal solution at point B in [Figure 2.11](#) is $x_1 = 24$ and $x_2 = 8$. Substituting these values into the objective function gives the maximum profit,

$$\begin{aligned}Z &= \$40x_1 + 50x_2 \\Z &= \$40(24) + 50(8) \\&= \$1,360\end{aligned}$$

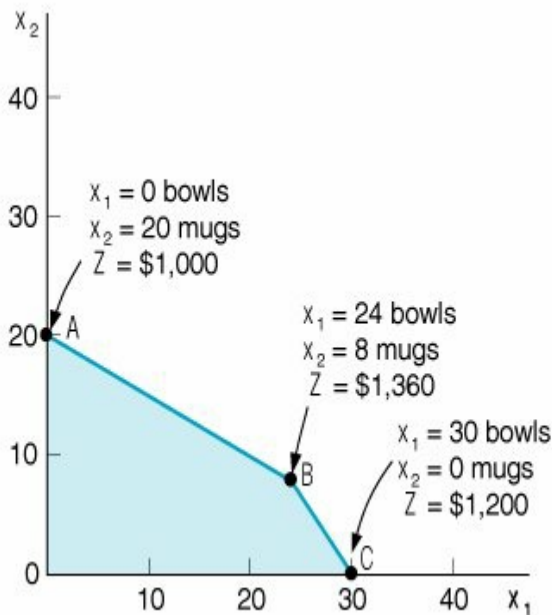
In terms of the original problem, the solution indicates that if the pottery company produces 24 bowls and 8 mugs,

it will receive \$1,360, the maximum daily profit possible (given the resource constraints).

Given that the optimal solution will be at one of the extreme corner points, A , B , or C , we can also find the solution by testing each of the three points to see which results in the greatest profit, rather than by graphing the objective function and seeing which point it last touches as it moves out of the feasible solution area. [Figure 2.12](#) shows the solution values for all three points, A , B , and C ,

and the amount of profit, Z , at each point.

Figure 2.12. Solutions at all corner points



As indicated in the discussion of [Figure 2.10](#), point B is the optimal solution point because it is the last point the objective function touches before it leaves the solution area. In other words, the objective function determines which extreme point is optimal. This is because the objective function designates the profit that will accrue from each combination of x_1 and x_2 values at the extreme points. If the objective function had had different coefficients (i.e., different x_1 and x_2 profit

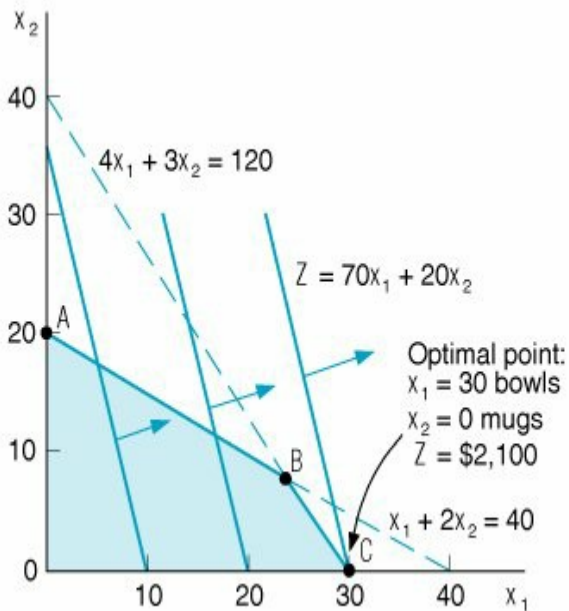
values), one of the extreme points other than B might have been optimal.

[Page 43]

Let's assume for a moment that the profit for a bowl is \$70 instead of \$40, and the profit for a mug is \$20 instead of \$50. These values result in a new objective function, $Z = \$70x_1 + 20x_2$. If the model constraints for labor or clay are not changed, the feasible solution area remains the same, as shown in [Figure 2.13](#).

However, the location of the objective function in [Figure 2.13](#) is different from that of the original objective function in [Figure 2.10](#). The reason for this change is that the new profit coefficients give the linear objective function a new *slope*.

Figure 2.13. The optimal solution with
 $Z = 70x_1 + 20x_2$



The slope can be determined by transforming the objective function into the general equation for a straight line, $y = a + bx$, where y is the dependent variable, a is the y intercept, b is the slope, and x is the independent variable. For our sample objective function, x_2 is the dependent variable corresponding to y (i.e., it is on the vertical axis), and x_1 is the independent variable. Thus, the objective function can be transformed into the general equation of a line as follows:

$$Z = 70x_1 + 20x_2$$

$$20x_2 = Z - 70x_1$$

$$x_2 = \frac{Z}{20} - \frac{7}{2}x_1$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ y & a & b \end{array}$$

*The **slope** is
computed as the
"rise" over the "run."*

This transformation identifies the slope of the new objective function as 7/2 (the minus sign indicates that the line slopes

downward). In contrast, the slope of the original objective function was $4/5$.

If we move this new objective function out through the feasible solution area, the last extreme point it touches is point C. Simultaneously solving the constraint lines at point C results in the following solution:

$$x_1 = 30$$

$$4x_1 + 3x_2 = 120$$

and

$$x_2 = 40 - (4x_1/3)$$

$$x_2 = 40 - 4(30)/3$$

$$x_2 = 0$$

Thus, the optimal solution at point C in [Figure 2.13](#) is $x_1 = 30$ bowls, $x_2 = 0$ mugs, and $Z = \$2,100$ profit. Altering the objective function coefficients results in a new solution.

[Page 44]

This brief example of the effects of altering the objective function highlights two useful points. First, the optimal extreme point is determined by

the objective function, and an extreme point on one axis of the graph is as likely to be the optimal solution as is an extreme point on a different axis. Second, the solution is sensitive to the values of the coefficients in the objective function. If the objective function coefficients are changed, as in our example, the solution may change. Likewise, if the constraint coefficients are changed, the solution space and solution points may change also. This information can be of

consequence to the decision maker trying to determine how much of a product to produce.

Sensitivity analysis the use of linear programming to evaluate the effects of changes in model parameters is discussed in Chapter 3.

Sensitivity analysis
*is used to analyze
changes in model
parameters.*

It should be noted that some problems do not have a single extreme point solution. For example, when the objective function line parallels one of the constraint lines, an entire line segment is bounded by two adjacent corner points that are optimal; there is no single extreme point on the objective function line. In this situation there are **multiple optimal solutions**. This and other irregular types of solution outcomes in linear programming are discussed at the end of this chapter.

Multiple optimal solutions can occur when the objective function is parallel to a constraint line.

Slack Variables

Once the optimal solution was found at point B in [Figure 2.12](#), simultaneous equations were solved to determine the values of x_1 and x_2 . Recall that the

solution occurs at an extreme point where constraint equation lines intersect with each other or with the axis. Thus, the model constraints are considered as *equations* (=) rather than $\leq$ or $\geq$ inequalities.

There is a standard procedure for transforming $\leq$ inequality constraints into equations. This transformation is achieved by adding a new variable, called a **slack variable**, to each constraint. For the pottery company example, the model constraints are

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$

$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$

A slack variable is added to a $\leq$ constraint to convert it to an equation (=).

The addition of a unique slack variable, s_1 , to the labor constraint and s_2 to the constraint for clay results in the following equations:

$$x_1 + 2x_2 + s_1 = 40 \text{ hr. of labor}$$

$$4x_1 + 3x_2 + s_2 = 120 \text{ lb. of clay}$$

***A slack variable
represents unused
resources.***

The slack variables in these equations, s_1 and s_2 , will take on any value necessary to make the left-hand side of the equation equal to the right-

hand side. For example,
consider a hypothetical solution
of $x_1 = 5$ and $x_2 = 10$.

Substituting these values into
the foregoing equations yields

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \text{ hr. of labor} \\5 + 2(10) + s_1 &= 40 \text{ hr. of labor} \\s_1 &= 15 \text{ hr. of labor}\end{aligned}$$

and

$$\begin{aligned}4x_1 + 3x_2 + s_2 &= 120 \text{ lb. of clay} \\4(5) + 3(10) + s_2 &= 120 \text{ lb. of clay} \\s_2 &= 70 \text{ lb. of clay}\end{aligned}$$

In this example, $x_1 = 5$ bowls
and $x_2 = 10$ mugs represent a
solution that does not make

use of the total available amount of labor and clay. In the labor constraint, 5 bowls and 10 mugs require only 25 hours of labor. This leaves 15 hours that are not used. Thus, s_1 represents the amount of *unused* labor, or slack.

[Page 45]

In the clay constraint, 5 bowls and 10 mugs require only 50 pounds of clay. This leaves 70 pounds of clay unused. Thus, s_2 represents the amount of *unused* clay. In general, slack

variables represent the amount of *unused resources*.

The ultimate instance of unused resources occurs at the origin, where $x_1 = 0$ and $x_2 = 0$. Substituting these values into the equations yields

$$x_1 + 2x_2 + s_1 = 40$$

$$0 + 2(0) + s_1 = 40$$

$$s_1 = 40 \text{ hr. of labor}$$

and

$$4x_1 + 3x_2 + s_2 = 120$$

$$4(0) + 3(0) + s_2 = 120$$

$$s_2 = 120 \text{ lb. of clay}$$

Because no production takes place at the origin, all the resources are unused; thus, the slack variables equal the total available amounts of each resource: $s_1 = 40$ hours of labor and $s_2 = 120$ pounds of clay.

What is the effect of these new slack variables on the objective function? The objective function for our example represents the profit gained from the production of bowls and mugs,

$$Z = \$40x_1 + \$50x_2$$

The coefficient \$40 is the contribution to profit of each bowl; \$50 is the contribution to profit of each mug. What, then, do the slack variables s_1 and s_2 contribute? They contribute *nothing* to profit because they represent unused resources. Profit is made only after the resources are put to use in making bowls and mugs. Using slack variables, we can write the objective function as

A slack variable

*contributes nothing
to the objective
function value.*

$$\text{maximize } Z = \$40x_1 + \$50x_2 + 0s_1 + 0s_2$$

As in the case of decision variables (x_1 and x_2), slack variables can have only nonnegative values because negative resources are not possible. Therefore, for this model formulation, x_1 , x_2 , s_1 ,

and $s_2 \geq 0$.

The complete linear programming model can be written in what is referred to as *standard form* with slack variables as follows:

maximize $Z = \$40x_1 + \$50x_2 + 0s_1 + 0s_2$

subject to

$$x_1 + 2x_2 + s_1 = 40$$

$$4x_1 + 3x_2 + s_2 = 120$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The solution values, including the slack at each solution point, are summarized as follows:

Solution Summary with

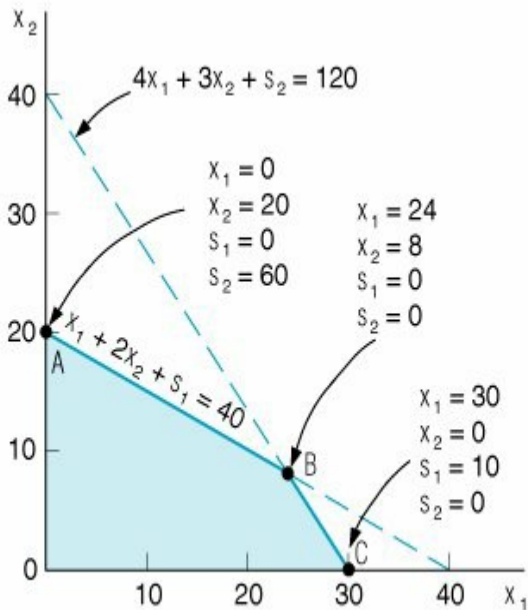
Slack

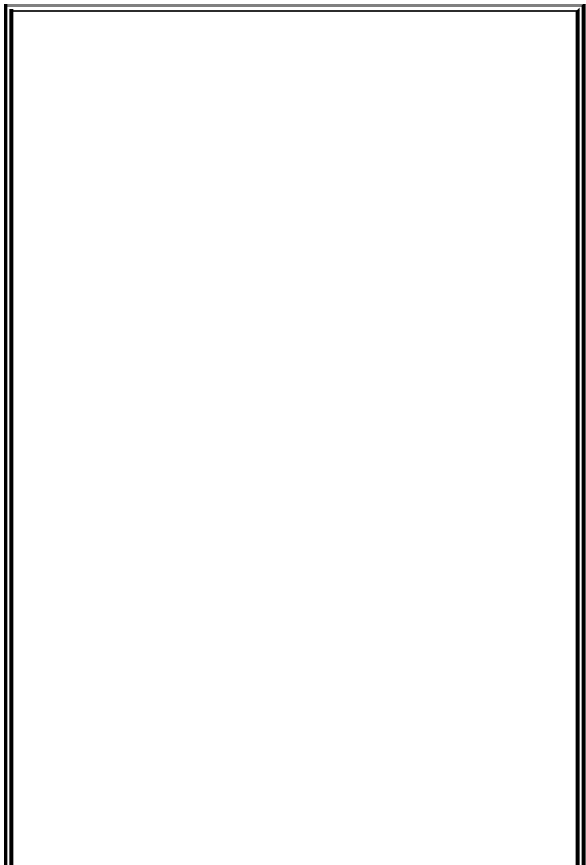
Point	Solution Values	Z	Slack
A	$x_1 = 0$ bowls, $x_2 = 20$ mugs	\$1,000	$s_1 = 0$ hr.; $s_2 = 60$ lb.
B	$x_1 = 24$ bowls, $x_2 = 8$ mugs	\$1,360	$s_1 = 0$ hr.; $s_2 = 0$ lb.
C	$x_1 = 30$ bowls, $x_2 = 0$ mugs	\$1,200	$s_1 = 10$ hr.; $s_2 = 0$ lb.

[Figure 2.14](#) shows the graphical solution of this example, with slack variables included at each solution point.

[Page 46]

Figure 2.14. Solutions at points *A*, *B*, and *C* with slack





Management Science

Application: Estimating Food Nutrient Values at Minnesota's Nutrition Coordinating Center

The Nutrition Coordinating Center (NCC) at the University of Minnesota maintains a food composition database used by institutions around the world. This database is used to calculate dietary material intake; to plan menus; to explore the relationships between diet and disease; to meet regulatory requirements; and to monitor the effects of education, intervention, and regulation. These calculations require an enormous amount of nutrient value data for different food

products.

Every year, more than 12,000 new brand-name food products are introduced into the marketplace. However, it's not feasible to perform a chemical analysis on all foods for multiple nutrients, and so the NCC *estimates* thousands of nutrient values each year. The NCC has used a time-consuming trial-and-error approach to estimating these nutrient values. These estimates are a composite of the foods already in the database. The NCC developed a linear programming model that determines ingredient amounts in new food products. These estimated ingredient amounts are subsequently used to calculate nutrient amounts for a food product. The nutritionist normally uses the linear programming model to derive an initial estimate of ingredient amounts

and then fine-tunes these amounts. The model has reduced the average time it takes to estimate a product formula from 8.3 minutes (using the trial-and-error approach) to 2.1 minutes of labor time and effort.



Source: B. J. Westrich, M. A. Altmann, and S. J. Potthoff, "Minnesota's Nutrition Coordinating Center Uses Mathematical Optimization to Estimate Food Nutrient Values," *Interfaces* 28, no. 5

Summary of the Graphical Solution Steps

The steps for solving a graphical linear programming model are summarized here:

1. Plot the model constraints as equations on the graph; then, considering the inequalities of the

constraints, indicate the feasible solution area.

[Page 47]

Plot the objective function; then move this line out

- 2.** from the origin to locate the optimal solution point.

Solve simultaneous equations at the solution

- 3.** point to find the optimal solution values.

Or

Solve simultaneous equations at each corner

- 2.** point to find the solution values at each point.

Substitute these values into the objective function to

- 3.** find the set of values that results in the maximum Z value.

A Minimization Model Example

As mentioned at the beginning of this chapter, there are two types of linear programming problems: maximization problems (like the Beaver Creek Pottery Company example) and minimization problems. A minimization problem is formulated the same basic way as a maximization problem, except for a few minor differences. The

following sample problem will demonstrate the formulation of a minimization model.

A farmer is preparing to plant a crop in the spring and needs to fertilize a field. There are two brands of fertilizer to choose from, Super-gro and Crop-quick. Each brand yields a specific amount of nitrogen and phosphate per bag, as follows:

Chemical Contribution

Brand	NITROGEN (LB./BAG)	PHOSPHATE (LB./BAG)
--------------	-------------------------------	--------------------------------

Super-gro	2	4
Crop-quick	4	3

The farmer's field requires at least 16 pounds of nitrogen and 24 pounds of phosphate. Super-gro costs \$6 per bag, and Crop-quick costs \$3. The farmer wants to know how many bags of each brand to purchase in order to minimize the total cost of fertilizing. This

scenario is illustrated in [Figure 2.15](#).

The steps in the linear programming model formulation process are summarized as follows:



Summary of LP Model

Formulation Steps

Define the decision variables

Step 1. How many bags of Super-gro and Crop-quick to buy

Define the objective function

Step 2. Minimize cost

Define the constraints

Step 3. The field requirements for nitrogen and phosphate

Decision Variables

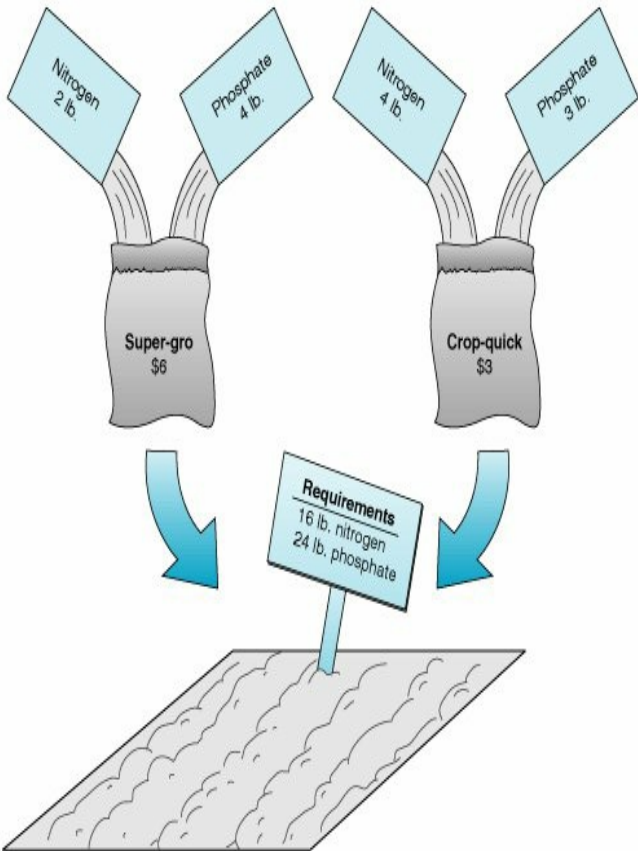
This problem contains two decision variables, representing the number of bags of each brand of fertilizer to purchase:

x_1 = bags of Super-gro

x_2 = bags of Crop-quick

Figure 2.15. Fertilizing farmer's field

[\[View full size image\]](#)



The Objective Function

The farmer's objective is to minimize the total cost of fertilizing. The total cost is the sum of the individual costs of each type of fertilizer purchased. The objective function that represents total cost is expressed as

$$\text{minimize } Z = \$6x_1 + 3x_2$$

where

$\$6x_1$ = cost of bags of Super-gro

$\$3x_2$ = cost of bags of Crop-quick

Model Constraints

The requirements for nitrogen and phosphate represent the constraints of the model. Each bag of fertilizer contributes a number of pounds of nitrogen and phosphate to the field. The

constraint for nitrogen is

$$2x_1 + 4x_2 \geq 16 \text{ lb.}$$

where

$2x_1$ = the nitrogen contribution (lb.) per bag of Super-gro

$4x_2$ = the nitrogen contribution (lb.) per bag of Crop-quick

Rather than a $\leq$ (less than or equal to) inequality, as used in the Beaver Creek Pottery Company model, this constraint requires a $\geq$ (greater than or equal to) inequality. This is

because the nitrogen content for the field is a minimum requirement specifying that at least 16 pounds of nitrogen be deposited on the farmer's field. If a minimum cost solution results in more than 16 pounds of nitrogen on the field, that is acceptable; however, the amount cannot be less than 16 pounds.

[Page 49]

The constraint for phosphate is constructed like the constraint for nitrogen:

$$4x_1 + 3x_2 \geq 24 \text{ lb.}$$

With this example we have shown two of the three types of linear programming model constraints, $\leq$ and $\geq$. The third type is an exact equality, $=$. This type specifies that a constraint requirement must be exact. For example, if the farmer had said that the phosphate requirement for the field was exactly 24 pounds, the constraint would have been

*The three types of
linear programming*

*constraints are $\leq$, $=$,
and $\geq$.*

$$4x_1 + 3x_2 = 24 \text{ lb.}$$

As in our maximization model, there are also nonnegativity constraints in this problem to indicate that negative bags of fertilizer cannot be purchased:

$$x_1, x_2 \geq 0$$

The complete model

formulation for this minimization problem is

$$\text{minimize } Z = \$6x_1 + 3x_2$$

subject to

$$2x_1 + 4x_2 \geq 16 \text{ lb. of nitrogen}$$

$$4x_1 + 3x_2 \geq 24 \text{ lb. of phosphate}$$

$$x_1, x_2 \geq 0$$

Graphical Solution of a Minimization Model

We follow the same basic steps in the graphical solution of a minimization model as in a maximization model. The fertilizer example will be used

to demonstrate the graphical solution of a minimization model.

The first step is to graph the equations of the two model constraints, as shown in [Figure 2.16](#). Next, the feasible solution area is chosen, to reflect the $\geq$ inequalities in the constraints, as shown in [Figure 2.17](#).

Figure 2.16. Constraint lines for fertilizer model

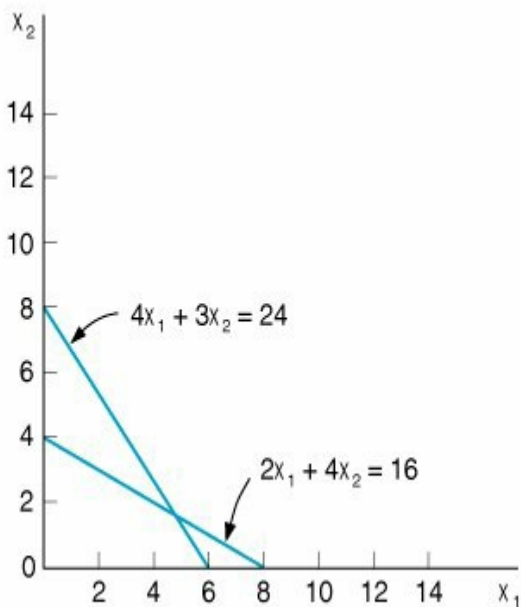
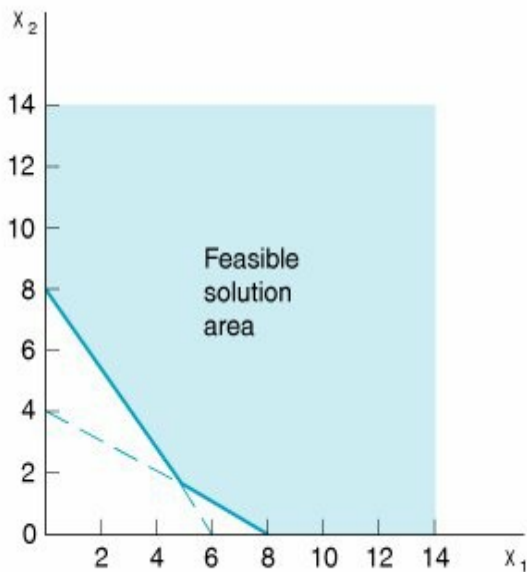


Figure 2.17. Feasible solution area



After the feasible solution area has been determined, the second step in the graphical solution approach is to locate the optimal point. Recall that in a maximization problem, the optimal solution is on the boundary of the feasible solution area that contains the point(s) farthest from the origin. The optimal solution point in a minimization problem is also on the boundary of the feasible solution area; however, the boundary contains the point(s) *closest* to the origin (zero being the lowest cost

possible).

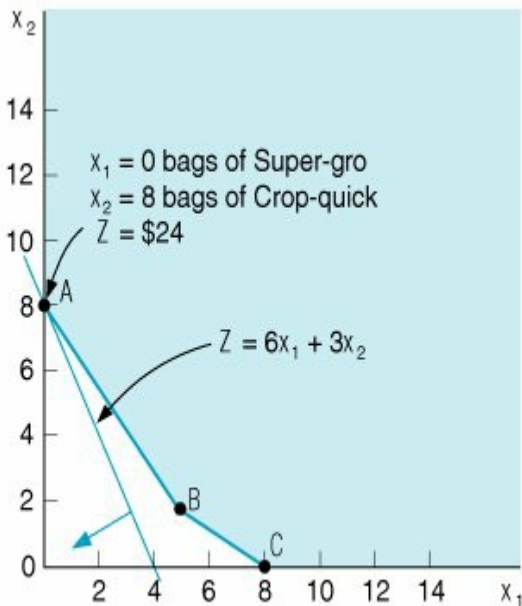
The optimal solution of a minimization problem is at the extreme point closest to the origin.

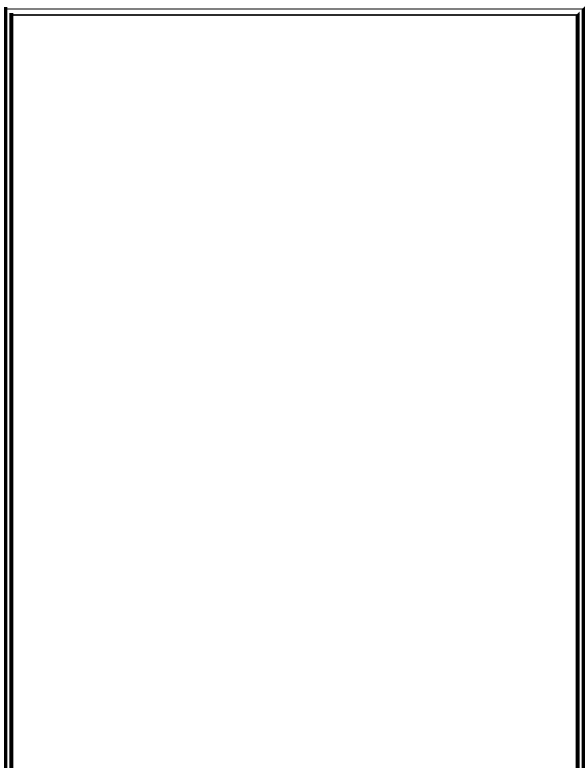
As in a maximization problem, the optimal solution is located at one of the extreme points of the boundary. In this case the corner points represent extremities in the boundary of

the feasible solution area that are *closest* to the origin. [Figure 2.18](#) shows the three corner points A , B , and C and the objective function line.

As the objective function edges *toward* the origin, the last point it touches in the feasible solution area is A . In other words, point A is the closest the objective function can get to the origin without encompassing infeasible points. Thus, it corresponds to the lowest cost that can be attained.

Figure 2.18. The optimal solution point





Management Science

Application: Determining Optimal Fertilizer Mixes at Soquimich (South America)

Soquimich, a Chilean fertilizer manufacturer, is the leading producer and distributor of specialty fertilizers in the world, with revenues of almost US\$0.5 billion in more than 80 countries. Soquimich produces four main specialty fertilizers and more than 200 fertilizer blends, depending on the needs of its customers. Farmers want the company to quickly recommend optimal fertilizer blends that will provide the appropriate quantity of ingredients for their particular crop at the lowest possible cost. A farmer will provide a

company sales representative with information about previous crop yields and his or her target yields and then company representatives will visit the farm to obtain soil samples, which are analyzed in the company labs. A report is generated, which indicates the soil requirements for nutrients, including nitrogen, phosphorus, potassium, boron, magnesium, sulfur, and zinc. Given these soil requirements, company experts determine an optimal fertilizer blend, using a linear programming model that includes constraints for the nutrient quantities required by the soil (for a particular crop) and an objective function that minimizes production costs.

Previously the company determined fertilizer blend recommendations by using a time-consuming manual procedure conducted by experts. The

linear programming model enables the company to provide accurate, quick, low-cost (discounted) estimates to its customers, which has helped the company gain new customers and increase its market share.



Source: A. M. Angel, L. A. Taladriz, and R. Weber. "Soquimich Uses a System Based on Mixed-Integer Linear Programming and Expert Systems to Improve Customer

The final step in the graphical solution approach is to solve for the values of x_1 and x_2 at point A. Because point A is on the x_2 axis, $x_1 = 0$; thus,

$$\begin{aligned}4(0) + 3x_2 &= 24 \\x_2 &= 8\end{aligned}$$

Given that the optimal solution is $x_1 = 0$, $x_2 = 8$, the minimum cost, Z , is

$$\begin{aligned}Z &= \$6x_1 + \$3x_2 \\Z &= 6(0) + 3(8) \\&= \$24\end{aligned}$$

This means the farmer should not purchase any Super-gro but, instead, should purchase eight bags of Crop-quick, at a total cost of \$24.

Surplus Variables

Greater than or equal to constraints cannot be converted to equations by adding slack variables, as with $\leq$ constraints. Recall our

fertilizer model, formulated as

minimize $Z = \$6x_1 + \$3x_2$

subject to

$2x_1 + 4x_2 \geq 16$ lb. of nitrogen

$4x_1 + 3x_2 \geq 24$ lb. of phosphate

$x_1, x_2 \geq 0$

[Page 52]

where

x_1 = bags of Super-gro
fertilizer

x_2 = bags of Crop-quick
fertilizer

Z = farmer's total cost (\$) of purchasing fertilizer

Because this problem has $\geq$ constraints as opposed to the $\leq$ constraints of the Beaver Creek Pottery Company maximization example, the constraints are converted to equations a little differently.

A surplus variable is subtracted from a $\geq$ constraint to convert it to an equation (=).

Instead of adding a slack variable with a $\geq$ constraint, we subtract a **surplus variable**.

Whereas a slack variable is added and reflects unused resources, a surplus variable is subtracted and reflects the excess above a minimum resource requirement level. Like a slack variable, a surplus variable is represented symbolically by s_1 and must be nonnegative.

For the nitrogen constraint, the

subtraction of a surplus variable gives

*A **surplus variable** represents an excess above a constraint requirement level.*

$$2x_1 + 4x_2 - s_1 = 16$$

The surplus variable s_1 transforms the nitrogen constraint into an equation.

As an example, consider the hypothetical solution

$$x_1 = 0$$

$$x_2 = 10$$

Substituting these values into the previous equation yields

$$2(0) + 4(10) - s_1 = 16$$

$$-s_1 = 16 - 40$$

$$s_1 = 24 \text{ lb. of nitrogen}$$

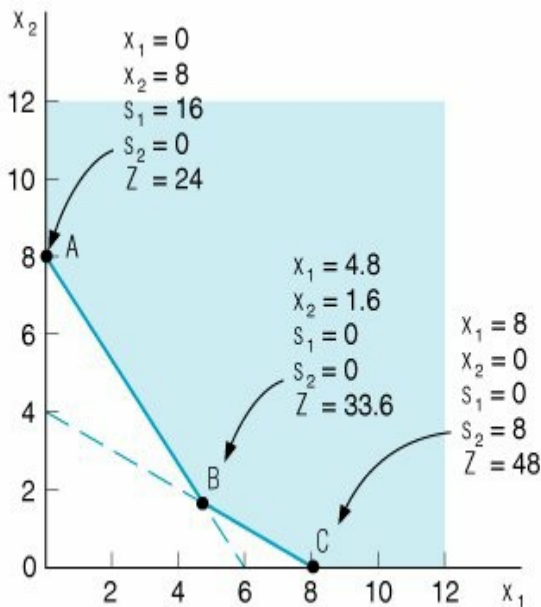
In this equation s_1 can be interpreted as the *extra* amount of nitrogen above the minimum requirement of 16

pounds that would be obtained by purchasing 10 bags of Crop-quick fertilizer.

In a similar manner, the constraint for phosphate is converted to an equation by subtracting a surplus variable, s_2 :

$$4x_1 + 3x_2 - s_2 = 24$$

Figure 2.19. Graph of the fertilizer example



As is the case with slack variables, surplus variables contribute nothing to the overall cost of a model. For example, putting additional nitrogen or phosphate on the field will not affect the farmer's cost; the only thing affecting cost is the number of bags of fertilizer purchased. As such, the standard form of this linear programming model is summarized as

minimize $Z = \$6x_1 + 3x_2 + 0s_1 + 0s_2$
subject to

$$2x_1 + 4x_2 - s_1 = 16$$

$$4x_1 + 3x_2 - s_2 = 24$$

$$x_1, x_2, s_1, s_2 \geq 0$$

[Figure 2.19](#) shows the graphical solutions for our example, with surplus variables included at each solution point.

Irregular Types of Linear Programming Problems

The basic forms of typical maximization and minimization problems have been shown in this chapter. However, there are several special types of atypical linear programming problems. Although these special cases do not occur frequently, they will be described so that you can recognize them when they arise. These special types include problems with more

than one optimal solution, infeasible problems, and problems with unbounded solutions.

For some linear programming models, the general rules do not always apply.

Multiple Optimal Solutions

Consider the Beaver Creek Pottery Company example, with the objective function changed from $Z = 40x_1 + 50x_2$ to $Z = 40x_1 + 30x_2$:

$$\begin{aligned} &\text{maximize } Z = 40x_1 + 30x_2 \\ &\text{subject to} \\ &\quad x_1 + 2x_2 \leq 40 \text{ hr. of labor} \\ &\quad 4x_1 + 3x_2 \leq 120 \text{ lb. of clay} \\ &\quad x_1, x_2 \geq 0 \end{aligned}$$

where

x_1 = bowls produced

x_2 = mugs produced

The graph of this model is

shown in [Figure 2.20](#). The slight change in the objective function makes it now *parallel* to the constraint line, $4x_1 + 3x_2 = 120$. Both lines now have the same slope of $4/3$.

Therefore, as the objective function edge moves outward from the origin, it touches the whole line segment BC rather than a single extreme corner point before it leaves the feasible solution area. This means that every point along this line segment is optimal (i.e., each point results in the same profit of $Z = \$1,200$). The

endpoints of this line segment, B and C , are typically referred to as the **alternate optimal solutions**. It is understood that these points represent the endpoints of a range of optimal solutions.

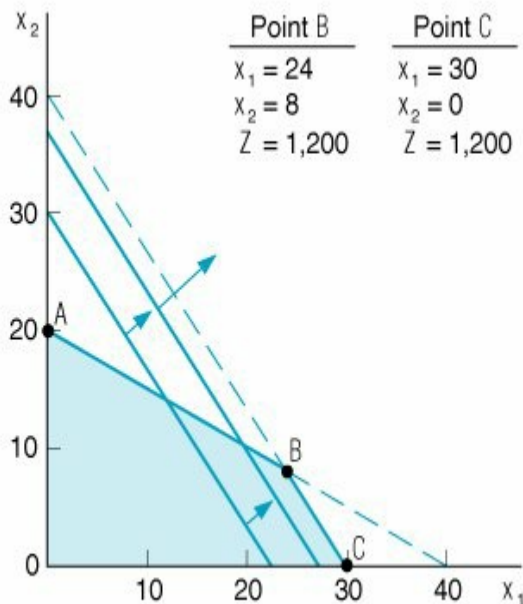
Alternate optimal solutions are at the endpoints of the constraint line segment that the objective function parallels.

Multiple optimal solutions provide greater flexibility to the decision maker.

The pottery company, therefore, has several options in deciding on the number of bowls and mugs to produce. Multiple optimal solutions can benefit the decision maker because the number of decision

options is enlarged. The multiple optimal solutions (along the line segment BC in [Figure 2.20](#)) allow the decision maker greater flexibility. For example, in the case of Beaver Creek Pottery Company, it may be easier to sell bowls than mugs; thus, the solution at point C , where only bowls are produced, would be more desirable than the solution at point B , where a mix of bowls and mugs is produced.

**Figure 2.20. Graph of
the Beaver Creek
Pottery Company
example with
multiple optimal
solutions**



An Infeasible Problem

In some cases a linear programming problem has no feasible solution area; thus, there is no solution to the problem. An example of an infeasible problem is formulated next and depicted graphically in [Figure 2.21](#):

$$\text{maximize } Z = 5x_1 + 3x_2$$

subject to

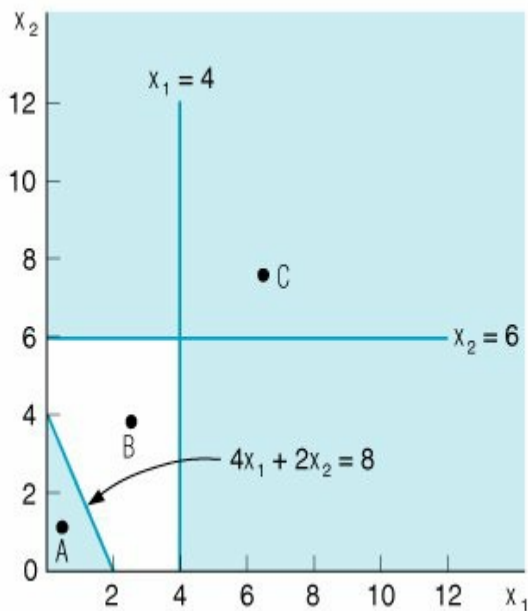
$$4x_1 + 2x_2 \leq 8$$

$$x_1 \geq 4$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Figure 2.21. Graph of an infeasible problem



An **infeasible problem** has no feasible solution area; every possible solution point violates one or more constraints.

[Page 55]

Point A in [Figure 2.21](#) satisfies only the constraint $4x_1 + 2x_2 \leq 8$, whereas point C satisfies only the constraints $x_1 \geq 4$ and

$x_2 \geq 6$. Point B satisfies none of the constraints. The three constraints do not overlap to form a feasible solution area. Because no point satisfies all three constraints simultaneously, there is no solution to the problem. Infeasible problems do not typically occur, but when they do, they are usually a result of errors in defining the problem or in formulating the linear programming model.

An Unbounded Problem

In some problems the feasible solution area formed by the model constraints is not closed. In these cases it is possible for the objective function to increase indefinitely without ever reaching a maximum value because it never reaches the boundary of the feasible solution area.

In an unbounded problem the objective function can increase indefinitely without reaching a maximum value.

An example of this type of problem is formulated next and shown graphically in [Figure 2.22](#):

$$\text{maximize } Z = 4x_1 + 2x_2$$

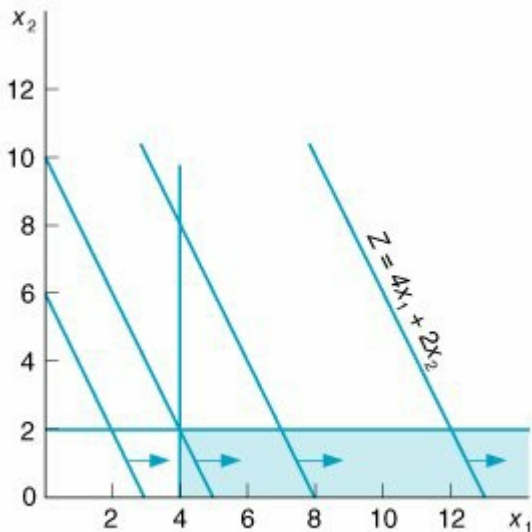
subject to

$$x_1 \geq 4$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Figure 2.22. An unbounded problem



In [Figure 2.22](#) the objective function is shown to increase

without bound; thus, a solution is never reached.

The solution space is not completely closed in.

Unlimited profits are not possible in the real world; an unbounded solution, like an infeasible solution, typically reflects an error in defining the problem or in formulating the model.

Characteristics of Linear Programming Problems

Now that we have had the opportunity to construct several linear programming models, let's review the characteristics that identify a linear programming problem.

The components of a linear programming model are an objective function, decision variables,

and constraints.

A linear programming problem requires a choice between alternative courses of action (i.e., a decision). The decision is represented in the model by decision variables. A typical choice task for a business firm is deciding how much of several different products to produce, as in the Beaver Creek Pottery Company example presented earlier in this chapter.

Identifying the choice task and defining the decision variables is usually the first step in the formulation process because it is quite difficult to construct the objective function and constraints without first identifying the decision variables.

[Page 56]

The problem encompasses an objective that the decision maker wants to achieve. The two most frequently encountered objectives for a

business are maximizing profit and minimizing cost.

A third characteristic of a linear programming problem is that restrictions exist, making unlimited achievement of the objective function impossible. In a business firm these restrictions often take the form of limited resources, such as labor or material; however, the sample models in this chapter exhibit a variety of problem restrictions. These restrictions, as well as the objective, must be definable by mathematical functional relationships that

are linear. Defining these relationships is typically the most difficult part of the formulation process.

Properties of Linear Programming Models

In addition to encompassing only linear relationships, a linear programming model also has several other implicit properties, which have been exhibited consistently throughout the examples in this chapter. The term *linear*

not only means that the functions in the models are graphed as a straight line; it also means that the relationships exhibit **proportionality**. In other words, the rate of change, or slope, of the function is constant; therefore, changes of a given size in the value of a decision variable will result in exactly the same relative changes in the functional value.

Proportionality means the slope of a

*constraint or
objective function
line is constant.*

Linear programming also requires that the objective function terms and the constraint terms be **additive**. For example, in the Beaver Creek Pottery Company model, the total profit (Z) must equal the sum of profits earned from making bowls ($\$40x_1$) and mugs ($\$50x_2$). Also, the total

resources used must equal the sum of the resources used for each activity in a constraint (e.g., labor).

The terms in the objective function or constraints are additive.

Another property of linear programming models is that the solution values (of the decision variables) cannot be

restricted to integer values; the decision variables can take on any fractional value. Thus, the variables are said to be *continuous* or **divisible**, as opposed to *integer* or *discrete*. For example, although decision variables representing bowls or mugs or airplanes or automobiles should realistically have integer (whole number) solutions, the solution methods for linear programming will not necessarily provide such solutions. This is a property that will be discussed further as solution methods are presented

in subsequent chapters.

*The values of
decision variables are
continuous or
divisible.*

The final property of linear programming models is that the values of all the model parameters are assumed to be constant and known with **certainty**. In real situations, however, model parameters are

frequently uncertain because they reflect the future as well as the present, and future conditions are rarely known with certainty.

All model parameters are assumed to be known with certainty.

To summarize, a linear programming model has the following general properties: linearity, proportionality,

additivity, divisibility, and certainty. As various linear programming solution methods are presented throughout this book, these properties will become more obvious, and their impact on problem solution will be discussed in greater detail.

Summary

The two example problems in this chapter were formulated as linear programming models in order to demonstrate the modeling process. These problems were similar in that they concerned achieving some objective subject to a set of restrictions or requirements. Linear programming models exhibit certain common characteristics:

- An objective function to be maximized or minimized
- A set of constraints
- Decision variables for measuring the level of activity
- Linearity among all constraint relationships and the objective function

The graphical approach to the solution of linear programming problems is not a very efficient means of solving problems. For

one thing, drawing accurate graphs is tedious. Moreover, the graphical approach is limited to models with only two decision variables. However, the analysis of the graphical approach provides valuable insight into linear programming problems and their solutions.

In the graphical approach, once the feasible solution area and the optimal solution point have been determined from the graph, simultaneous equations are solved to determine the values of x_1 and x_2 at the

solution point. In [Chapter 3](#) we will show how linear programming solutions can be obtained using computer programs.

References

Baumol, W. J. *Economic Theory and Operations Analysis*, 4th ed. Upper Saddle River, NJ: Prentice Hall, 1977.

Charnes, A., and Cooper, W. W. *Management Models and Industrial Applications of Linear Programming*. New York: John Wiley & Sons, 1961.

Dantzig, G. B. *Linear Programming and Extensions*. Princeton, NJ: Princeton University Press, 1963.

Gass, S. *Linear Programming*, 4th ed. New York: McGraw-Hill,

1975.

Hadley, G. *Linear Programming*. Reading, MA: Addison-Wesley, 1962.

Hillier, F. S., and Lieberman, G. J. *Introduction to Operations Research*, 4th ed. San Francisco: Holden-Day, 1986.

Llewellyn, R. W. *Linear Programming*. New York: Holt, Rinehart and Winston, 1964.

Taha, H. A. *Operations Research, an Introduction*, 4th ed. New York: Macmillan, 1987.

Wagner, H. M. *Principles of Operations Research*, 2nd ed.
Upper Saddle River, NJ:
Prentice Hall, 1975.

Example Problem Solutions

As a prelude to the problems, this section presents example solutions to two linear programming problems.

Problem Statement

Moore's Meatpacking Company produces a hot dog mixture in 1,000-pound batches. The mixture contains two

ingredients chicken and beef.
The cost per pound of each of
these ingredients is as follows:

Ingredient	Cost/lb.
Chicken	\$3
Beef	\$5

Each batch has the following
recipe requirements:

- 1.** At least 500 pounds of chicken
- 2.** At least 200 pounds of beef

The ratio of chicken to beef must be at least 2 to 1. The company wants to know the optimal mixture of ingredients that will minimize cost.

Formulate a linear programming model for this problem.

Solution

Identify Decision Variables

Recall that the problem should not be "swallowed whole."

Step 1. Identify each part of the model separately, starting with the decision variables:

x_1 = lb. of chicken

x_2 = lb. of beef

Formulate the

Objective Function

Step 2.

minimize $Z = \$3x_1 + \$5x_2$

where

Z = cost per 1,000-lb batch

$\$3x_1$ = cost of chicken

$\$5x_2$ = cost of beef

Establish Model Constraints

The constraints of this problem are embodied in the recipe restrictions and (not to be overlooked) the fact that each batch must

consist of 1,000
pounds of mixture:

$$x_1 + x_2 = 1,000 \text{ lb.}$$

$$x_1 \geq 500 \text{ lb. of chicken}$$

$$x_2 \geq 200 \text{ lb. of beef}$$

Step 3. $x_1/x_2 \geq 2/1$ or $x_1 - 2x_2 \geq 0$

and

$$x_1, x_2 \geq 0$$

The Model

minimize $Z = \$3x_1 + \$5x_2$

subject to

$$x_1 + x_2 = 1,000$$

$$x_1 \geq 500$$

$$x_2 \geq 200$$

$$x_1 - 2x_2 \geq 0$$

$$x_1, x_2 \geq 0$$

Problem Statement

Solve the following linear programming model graphically:

maximize $Z = 4x_1 + 5x_2$

subject to

$$x_1 + 2x_2 \leq 10$$

$$6x_1 + 6x_2 \leq 36$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

[Page 59]

Solution

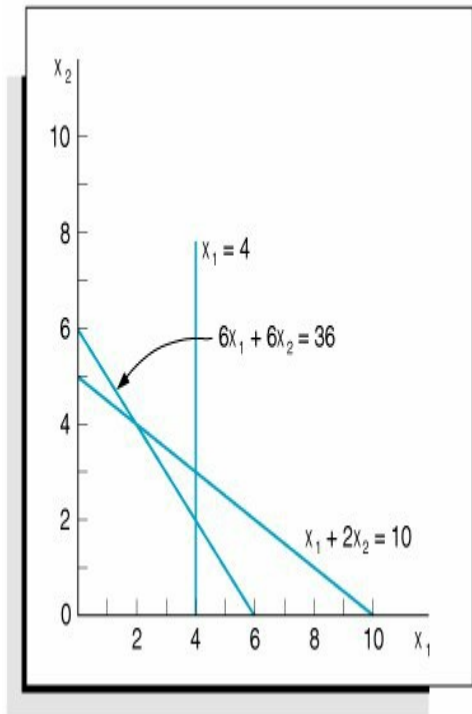
Plot the Constraint Lines as Equations

A simple method for plotting constraint lines is to set one of

the constraint variables equal to zero and solve for the other variable to establish a point on one of the axes. The three constraint lines are graphed in the following figure:

The constraint equations

Step 1.



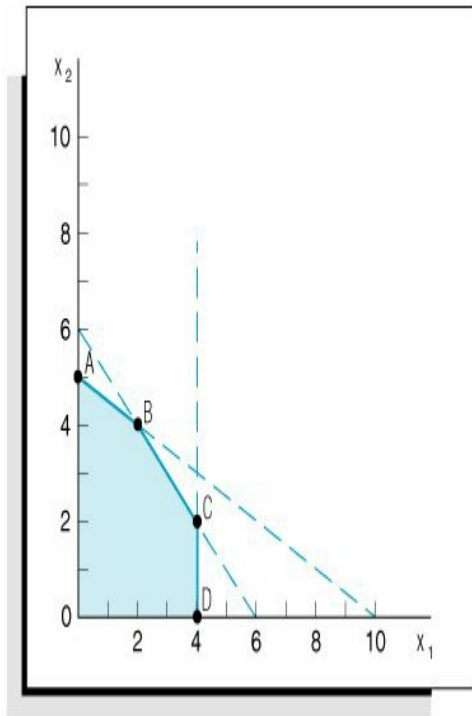
Determine the

Feasible Solution Area

The feasible solution area is determined by identifying the space that jointly satisfies the $\leq$ conditions of all three constraints, as shown in the following figure:

Step 2.

The feasible solution space and extreme points



Determine the

Solution Points

The solution at point A can be determined by noting that the constraint line intersects the x_2 axis at 5; thus, $x_2 = 5$, $x_1 = 0$, and $Z = 25$. The solution at point D on the other axis can be determined similarly; the constraint intersects the axis at $x_1 = 4$, $x_2 = 0$, and $Z = 16$.

[Page 60]

The values at points B and C must be found by solving simultaneous equations. Note that point B is formed by the intersection of the lines $x_1 + 2x_2 = 10$ and $6x_1 + 6x_2 = 36$. First, convert both of these equations to functions of x_1 :

$$x_1 + 2x_2 = 10$$

$$x_1 = 10 - 2x_2$$

and

Step 3.

$$6x_1 + 6x_2 = 36$$

$$6x_1 = 36 - 6x_2$$

$$x_1 = 6 - x_2$$

Now set the equations equal and solve for x_2 :

$$10 - 2x_2 = 6 - x_2$$

$$-x_2 = -4$$

$$x_2 = 4$$

Substituting $x_2 = 4$
into either of the two

equations gives a value for x_1 :

$$x_1 = 6 x_2$$

$$x_1 = 6 (4)$$

$$x_1 = 2$$

Thus, at point B , $x_1 = 2$, $x_2 = 4$, and $Z = 28$.

At point C , $x_1 = 4$.

Substituting $x_1 = 4$ into the equation $x_1 =$

6 x_2 gives a value for x_2 :

$$4 = 6 x_2$$

$$x_2 = 2$$

Thus, $x_1 = 4$, $x_2 = 2$,
and $Z = 26$.

Determine the Optimal Solution

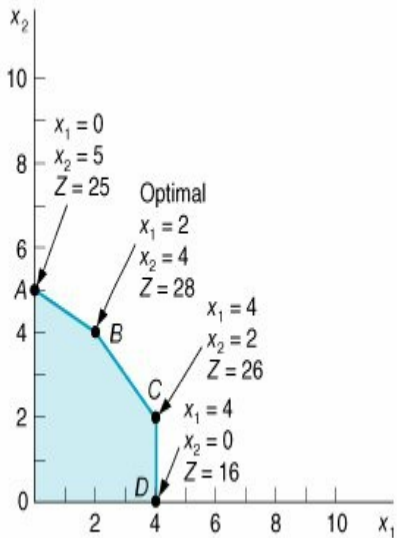
The optimal solution is
at point B , where $x_1 =$

2, $x_2 = 4$, and $Z = 28$.

The optimal solution and solutions at the other extreme points are summarized in the following figure:

Optimal solution point

Step 4.



Problems

In [Problem 28](#) in [Chapter 1](#), when Marie McCoy wakes up Saturday morning, she remembers that she promised the PTA she would make some cakes and/or homemade bread for its bake sale that afternoon. However, she does not have time to go to the store to get ingredients, and she has only a short time to bake things in her oven. Because cakes and breads require different baking temperatures, she cannot bake them simultaneously, and she has only 3 hours available to bake. A [1.](#) cake requires 3 cups of flour, and a

loaf of bread requires 8 cups; Marie has 20 cups of flour. A cake requires 45 minutes to bake, and a loaf of bread requires 30 minutes. The PTA will sell a cake for \$10 and a loaf of bread for \$6. Marie wants to decide how many cakes and loaves of bread she should make.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

A company produces two products that are processed on two assembly lines. Assembly line 1 has 100 available hours, and assembly line 2 has 42 available hours. Each product requires 10 hours of processing time on line 1, while on

- line 2 product 1 requires 7 hours and product 2 requires 3 hours.
2. The profit for product 1 is \$6 per unit, and the profit for product 2 is \$4 per unit.
 1. Formulate a linear programming model for this problem.
 2. Solve this model by using graphical analysis.

The Munchies Cereal Company makes a cereal from several ingredients. Two of the ingredients, oats and rice, provide vitamins A and B. The company wants to know how many ounces of oats and rice it should include in each box of cereal to meet the minimum requirements of 48 milligrams of vitamin A and 12

milligrams of vitamin B while minimizing cost. An ounce of oats contributes 8 milligrams of vitamin A and 1 milligram of vitamin B, whereas an ounce of rice contributes 6 milligrams of A and 2 milligrams of B. An ounce of oats costs \$0.05, and an ounce of rice costs \$0.03.

1. Formulate a linear programming model for this problem.
2. Solve this model by using graphical analysis.

What would be the effect on the optimal solution in [Problem 3](#) if the cost of rice increased from \$0.03 per ounce to \$0.06 per ounce?

- 4.

The Kalo Fertilizer Company makes a fertilizer using two chemicals that provide nitrogen, phosphate, and potassium. A pound of ingredient 1 contributes 10 ounces of nitrogen and 6 ounces of phosphate, while a pound of ingredient 2 contributes 2 ounces of nitrogen, 6 ounces of phosphate, and 1 ounce of potassium. Ingredient 1 costs \$3 per pound, and ingredient 2 costs \$5 per pound. The company wants to know how many pounds of

- 5.** each chemical ingredient to put into a bag of fertilizer to meet the minimum requirements of 20 ounces of nitrogen, 36 ounces of phosphate, and 2 ounces of potassium while minimizing cost.

- 1.** Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

The Pinewood Furniture Company produces chairs and tables from two resources labor and wood. The company has 80 hours of labor and 36 pounds of wood available each day. Demand for chairs is limited to 6 per day. Each chair requires 8 hours of labor and 2 pounds of wood, whereas a table requires 10 hours of labor and 6 pounds of wood. The profit derived from each chair is \$400 and from each table, \$100. The company wants to determine the number of chairs and tables to produce each day in order to maximize profit.

1. Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

[Page 62]

7. In [Problem 6](#), how much labor and wood will be unused if the optimal numbers of chairs and tables are produced?

8. In [Problem 6](#), explain the effect on the optimal solution of changing the profit on a table from \$100 to \$500.

The Crumb and Custard Bakery makes coffee cakes and Danish pastries in large pans. The main

ingredients are flour and sugar. There are 25 pounds of flour and 16 pounds of sugar available, and the demand for coffee cakes is 5. Five pounds of flour and 2 pounds of sugar are required to make a pan of coffee cakes, and 5 pounds of flour and 4 pounds of sugar are required to make a pan of Danish.

9. A pan of coffee cakes has a profit of \$1, and a pan of Danish has a profit of \$5. Determine the number of pans of cakes and Danish to produce each day so that profit will be maximized.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

- 10.** In [Problem 9](#), how much flour and sugar will be left unused if the optimal numbers of cakes and Danish are baked?

Solve the following linear programming model graphically:

$$\text{maximize } Z = 3x_1 + 6x_2$$

subject to

11. $3x_1 + 2x_2 \leq 18$

$$x_1 + x_2 \geq 5$$

$$x_1 \leq 4$$

$$x_1, x_2 \geq 0$$

The Elixir Drug Company produces a drug from two ingredients. Each ingredient contains the same three antibiotics, in different proportions. One gram of ingredient 1

contributes 3 units, and 1 gram of ingredient 2 contributes 1 unit of antibiotic 1; the drug requires 6 units. At least 4 units of antibiotic 2 are required, and the ingredients each contribute 1 unit per gram. At least 12 units of antibiotic 3 are required; a gram of ingredient 1 contributes 2 units, and a gram of ingredient 2 contributes 6 units.

- 12.** The cost for a gram of ingredient 1 is \$80, and the cost for a gram of ingredient 2 is \$50. The company wants to formulate a linear programming model to determine the number of grams of each ingredient that must go into the drug in order to meet the antibiotic requirements at the minimum cost.

- 1.** Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

A jewelry store makes necklaces and bracelets from gold and platinum. The store has 18 ounces of gold and 20 ounces of platinum. Each necklace requires 3 ounces of gold and 2 ounces of platinum, whereas each bracelet requires 2 ounces of gold and 4 ounces of platinum. The demand for bracelets is no more than four. A necklace earns \$300 in profit and a bracelet, \$400. The store wants to determine the number of necklaces and bracelets to make in order to maximize profit.

13.

1. Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

In [Problem 13](#), explain the effect on the optimal solution of increasing the profit on a bracelet

14. from \$400 to \$600. What will be the effect of changing the platinum requirement for a necklace from 2 ounces to 3 ounces?

[Page 63]

In [Problem 13](#):

1. The maximum demand for bracelets is four. If the store produces the optimal number of bracelets and necklaces, will the maximum demand for bracelets be met? If not, by

15.

how much will it be missed?

2. What profit for a necklace would result in no bracelets being produced, and what would be the optimal solution for this profit?

A clothier makes coats and slacks. The two resources required are wool cloth and labor. The clothier has 150 square yards of wool and 200 hours of labor available. Each coat requires 3 square yards of wool and 10 hours of labor, whereas each pair of slacks requires 5 square yards of wool and 4 hours of labor. The profit for a coat is \$50, and the profit for

16. slacks is \$40. The clothier wants to determine the number of coats and pairs of slacks to make so that profit will be maximized.

1. Formulate a linear programming model for this problem.
2. Solve this model by using graphical analysis.

In [Problem 16](#), what would be the effect on the optimal solution if the available labor were increased from 200 to 240 hours?

17.

Solve the following linear programming model graphically:

18.

maximize $Z = 1.5x_1 + x_2$

subject to

$$x_1 \leq 4$$

$$x_2 \leq 6$$

$$x_1 + x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

19. Transform the model in [Problem 18](#) into standard form and indicate the value of the slack variables at each corner point solution.

Solve the following linear programming model graphically:

20.

maximize $Z = 5x_1 + 8x_2$

subject to

$$3x_1 + 5x_2 \leq 50$$

$$2x_1 + 4x_2 \leq 40$$

$$x_1 \leq 8$$

$$x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

- 21.** Transform the model in [Problem 20](#) into standard form and indicate the value of the slack variables at each corner point solution.

Solve the following linear programming model graphically:

22.

maximize $Z = 6.5x_1 + 10x_2$

subject to

$$2x_1 + 4x_2 \leq 40$$

$$x_1 + x_2 \leq 15$$

$$x_1 \geq 8$$

$$x_1, x_2 \geq 0$$

[Page 64]

In [Problem 22](#), if the constraint $x_1 \geq 8$ is changed to $x_1 \leq 8$, what

23. effect does this have on the feasible solution space and the optimal solution?

Universal Claims Processors processes insurance claims for large national insurance companies. Most claim processing

is done by a large pool of computer operators, some of whom are permanent and some of whom are temporary. A permanent operator can process 16 claims per day, whereas a temporary operator can process 12 per day, and on average the company processes at least 450 claims each day. The company has 40 computer workstations. A permanent operator generates about 0.5 claims with errors each

- 24.** day, whereas a temporary operator averages about 1.4 defective claims per day. The company wants to limit claims with errors to 25 per day. A permanent operator is paid \$64 per day, and a temporary operator is paid \$42 per day. The company wants to determine the number of permanent and temporary

operators to hire in order to minimize costs.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 24](#), explain the effect on the optimal solution of changing the daily pay for a permanent claims processor from \$64 to \$54.

25. Explain the effect of changing the daily pay for a temporary claims processor from \$42 to \$36.

In [Problem 24](#), what would be the effect on the optimal solution if Universal Claims Processors

26. decided not to try to limit the

number of defective claims each day?

In [Problem 24](#), explain the effect on the optimal solution if the

- 27.** minimum number of claims the firm processes each day increased from 450 to at least 650.

Solve the following linear programming model graphically:

$$\text{minimize } Z = 8x_1 + 6x_2$$

- 28.** subject to

$$4x_1 + 2x_2 \geq 20$$

$$-6x_1 + 4x_2 \leq 12$$

$$x_1 + x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming model graphically:

$$\text{minimize } Z = 3x_1 + 6x_2$$

subject to

29.

$$3x_1 + 2x_2 \leq 18$$

$$x_1 + x_2 \geq 5$$

$$x_1 \leq 4$$

$$x_2 \leq 7$$

$$x_2/x_1 \leq 7/8$$

$$x_1, x_2 \geq 0$$

In [Problem 29](#), what would be the effect on the solution if the

30. constraint $x_2 \leq 7$ were changed to $x_2 \geq 7$?

Solve the following linear programming model graphically:

minimize $Z = 5x_1 + x_2$

31. subject to

$$3x_1 + 4x_2 = 24$$

$$x_1 \leq 6$$

$$x_1 + 3x_2 \leq 12$$

$$x_1, x_2 \geq 0$$

[Page 65]

Solve the following linear programming model graphically:

maximize $Z = 3x_1 + 2x_2$

subject to

32. $2x_1 + 4x_2 \leq 22$

$$-x_1 + 4x_2 \leq 10$$

$$4x_1 - 2x_2 \leq 14$$

$$x_1 - 3x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming model graphically:

$$\text{minimize } Z = 8x_1 + 2x_2$$

subject to

33. $2x_1 - 6x_2 \leq 12$

$$5x_1 + 4x_2 \geq 40$$

$$x_1 + 2x_2 \geq 12$$

$$x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Gillian's Restaurant has an ice-cream counter where it sells two main products, ice cream and frozen yogurt, each in a variety of flavors. The restaurant makes one order for ice cream and yogurt each week, and the store has enough freezer space for 115

gallons total of both products. A gallon of frozen yogurt costs \$0.75 and a gallon of ice cream costs \$0.93, and the restaurant budgets \$90 each week for these products. The manager estimates that each week the restaurant sells at least twice as much ice cream as frozen yogurt. Profit per gallon of ice cream is \$4.15, and profit per gallon of yogurt is \$3.60.

34.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 34](#), how much additional profit would the restaurant realize each week if it

35.

increased its freezer capacity to accommodate 20 extra gallons total of ice cream and yogurt?

Copperfield Mining Company owns two mines, each of which produces three grades of ore high, medium, and low. The company has a contract to supply a smelting company with at least 12 tons of high-grade ore, 8 tons of medium-grade ore, and 24 tons of low-grade ore. Each mine produces a certain amount of each type of ore during each hour that it operates. Mine 1 produces 6 tons of high-grade ore, 2 tons of medium-grade ore, and 4 tons of low-grade ore per hour. Mine 2 produces 2, 2, and 12 tons, respectively, of high-, medium-, and low-grade ore per hour. It costs Copperfield \$200 per

hour to mine each ton of ore from mine 1, and it costs \$160 per hour to mine each ton of ore from mine 2. The company wants to determine the number of hours it needs to operate each mine so that its contractual obligations can be met at the lowest cost.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

A canning company produces two sizes of cans regular and large. The cans are produced in 10,000-can lots. The cans are processed through a stamping operation and a coating operation. The company has 30 days available for both

stamping and coating. A lot of regular-size cans requires 2 days to stamp and 4 days to coat, whereas a lot of large cans requires 4 days to stamp and 2 days to coat. A lot of regular-size cans earns \$800 profit, and a lot of large-size cans earns \$900 profit. In order to fulfill its

- 37.** obligations under a shipping contract, the company must produce at least nine lots. The company wants to determine the number of lots to produce of each size can (x_1 and x_2) in order to maximize profit.

[Page 66]

- 1.** Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

A manufacturing firm produces two products. Each product must undergo an assembly process and a finishing process. It is then transferred to the warehouse, which has space for only a limited number of items. The firm has 80 hours available for assembly and 112 hours for finishing, and it can store a maximum of 10 units in the warehouse. Each unit of product 1 has a profit of \$30 and requires 4 hours to assemble and 14 hours to finish. Each unit of product 2 has a profit of \$70 and requires 10 hours to assemble and 8 hours to finish. The firm wants to determine the quantity of each product to produce in order to

maximize profit.

1. Formulate a linear programming model for this problem.
2. Solve this model by using graphical analysis.

Assume that the objective function in [Problem 38](#) has been changed from $Z = 30x_1 + 70x_2$ to $Z =$

- 39.** $90x_1 + 70x_2$. Determine the slope of each objective function and discuss what effect these slopes have on the optimal solution.

The Valley Wine Company produces two kinds of wine Valley Nectar and Valley Red. The wines are produced from 64 tons of

grapes the company has acquired this season. A 1,000-gallon batch of Nectar requires 4 tons of grapes, and a batch of Red requires 8 tons. However, production is limited by the availability of only 50 cubic yards of storage space for aging and 120 hours of processing time. A batch of each type of wine requires 5 yd.³ of storage space. The processing time for a batch of Nectar is 15 hours, and the processing time for a batch of Red is 8 hours. Demand for each type of wine is limited to seven batches. The profit for a batch of Nectar is \$9,000, and the profit for a batch of Red is \$12,000. The company wants to determine the number of 1,000-gallon batches of Nectar (x_1) and Red (x_2) to produce in order to maximize profit.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 40](#):

- 1.** How much processing time will be left unused at the optimal solution?

41.

- 2.** What would be the effect on the optimal solution of increasing the available storage space from 50 to 60 yd.³?

Kroeger supermarket sells its own brand of canned peas as well as

several national brands. The store makes a profit of \$0.28 per can for its own peas and a profit of \$0.19 for any of the national brands. The store has 6 square feet of shelf space available for canned peas, and each can of peas takes up 9 square inches of that space. Point-of-sale records show that each week the store never sells more than one-half as many cans of its own brand as it does of the national brands. The store wants to know how many cans of its own brand of peas and how many cans of the national brands to stock each week on the allocated shelf space in order to maximize profit.

- 1.** Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

In [Problem 42](#), if Kroeger discounts the price of its own brand of peas, the store will sell at least 1.5 times as much of the national brands as **43.** its own brand, but its profit margin on its own brand will be reduced to \$0.23 per can. What effect would the discount have on the optimal solution?

Shirtstop makes T-shirts with logos and sells them in its chain of retail stores. It contracts with two different plants one in Puerto Rico and one in The Bahamas. The shirts from the plant in Puerto Rico cost \$0.46 a piece, and 9% of them are defective and can't be

sold. The shirts from The Bahamas cost only \$0.35 each, but they have an 18% defective rate.

- 44.** Shirtstop needs 3,500 shirts. To retain its relationship with the two plants, it wants to order at least 1,000 shirts from each. It would also like at least 88% of the shirts it receives to be salable.

[Page 67]

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 44](#):

- 1.** Suppose Shirtstop decided it

wanted to minimize the defective shirts while keeping costs below \$2,000.

45.

Reformulate the problem with these changes and solve graphically.

- 2.** How many fewer defective items were achieved with the model in (a) than with the model in [Problem 44](#)?

Angela and Bob Ray keep a large garden in which they grow cabbage, tomatoes, and onions to make two kinds of relish: chow-chow and tomato. The chow-chow is made primarily of cabbage, whereas the tomato relish has more tomatoes than chow-chow. Both relishes include onions, and negligible amounts of bell peppers and spices. A jar of chow-chow

contains 8 ounces of cabbage, 3 ounces of tomatoes, and 3 ounces of onions, whereas a jar of tomato relish contains 6 ounces of tomatoes, 6 ounces of cabbage, and 2 ounces of onions. The Rays grow 120 pounds of cabbage, 90 pounds of tomatoes, and 45 pounds of onions each summer. The Rays can produce no more than 24 dozen jars of relish. They make \$2.25 in profit from a jar of chow-chow and \$1.95 in profit from a jar of tomato relish. The Rays want to know how many jars of each kind of relish to produce to generate the most profit.

1. Formulate a linear programming model for this problem.
2. Solve this model graphically.

In [Problem 46](#), the Rays have checked their sales records for the past five years and have found **47.** that they sell at least 50% more chow-chow than tomato relish. How will this additional information affect their model and solution?

A California grower has a 50-acre farm on which to plant strawberries and tomatoes. The grower has available 300 hours of labor per week and 800 tons of fertilizer, and he has contracted for shipping space for a maximum of 26 acres' worth of strawberries and 37 acres' worth of tomatoes. An acre of strawberries requires 10 hours of labor and 8 tons of fertilizer, whereas an acre of tomatoes requires 3 hours of labor

48. and 20 tons of fertilizer. The profit from an acre of strawberries is \$400, and the profit from an acre of tomatoes is \$300. The farmer wants to know the number of acres of strawberries and tomatoes to plant to maximize profit.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 48](#), if the amount of fertilizer required for each acre of strawberries were determined to be 20 tons instead of 8 tons, what would be the effect on the optimal solution?

49.

The admissions office at Tech wants to determine how many in-state and how many out-of-state students to accept for next fall's entering freshman class. Tuition for an in-state student is \$7,600 per year, whereas out-of-state tuition is \$22,500 per year. A total of 12,800 in-state and 8,100 out-of-state freshmen have applied for next fall, and Tech does not want to accept more than 3,500 students. However, because Tech is a state institution, the state

- 50.** mandates that it can accept no more than 40% out-of-state students. From past experience the admissions office knows that 12% of in-state students and 24% of out-of-state students will drop out during their first year. Tech wants to maximize total tuition

while limiting the total attrition to 600 first-year students.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

[Page 68]

Janet Lopez is establishing an investment portfolio that will include stock and bond funds. She has \$720,000 to invest, and she does not want the portfolio to include more than 65% stocks. The average annual return for the stock fund she plans to invest in is 18%, whereas the average annual return for the bond fund is 6%.

51.

She further estimates that the most she could lose in the next year in the stock fund is 22%, whereas the most she could lose in the bond fund is 5%. To reduce her risk, she wants to limit her potential maximum losses to \$100,000.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

Professor Smith teaches two sections of business statistics, which combined will result in 120 final exams to be graded.

Professor Smith has two graduate assistants. Brad and Sarah, who will grade the final exams. There is

a 3-day period between the time the exam is administered and when final grades must be posted. During this period Brad has 12 hours available and Sarah has 10 hours available to grade the exams. It takes Brad an average of 7.2 minutes to grade an exam, and it takes Sarah 12 minutes to grade an exam; however, Brad's exams will have errors that will require Professor Smith to ultimately regrade 10% of the exams, while only 6% of Sarah's exams will require regrading. Professor Smith wants to know how many exams to assign to each graduate assistant to grade in order to minimize the number of exams to regrade.

52.

- 1.** Formulate a linear programming model for this problem.

2. Solve this model by using graphical analysis.

53. In [Problem 52](#), if Professor Smith could hire Brad or Sarah to work 1 additional hour, which should she choose? What would be the effect of hiring the selected graduate assistant for 1 additional hour?

Starbright Coffee Shop at the Galleria Mall serves two coffee blends it brews on a daily basis, Pomona and Coastal. Each is a blend of three high-quality coffees from Colombia, Kenya, and Indonesia. The coffee shop has 10 pounds of each of these coffees available each day. Each pound of coffee will produce sixteen 16-

ounce cups of coffee. The shop has enough brewing capacity to brew 30 gallons of these two coffee blends each day. Pomona is a blend of 20% Colombian, 35% Kenyan, and 45% Indonesian, while Coastal is a blend of 60% Colombian, 10% Kenyan, and 30% Indonesian. The shop sells 1.5 times more Pomona than Coastal each day. Pomona sells for \$2.05 per cup, and Coastal sells for \$1.85 per cup. The manager wants to know how many cups of each blend to sell each day in order to maximize sales.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using graphical analysis.

In [Problem 54](#):

55.

- 1.** If Starbright Coffee Shop could get 1 more pound of coffee, which one should it be? What would be the effect on sales of getting 1 more pound of this coffee? Would it benefit the shop to increase its brewing capacity from 30 gallons to 40 gallons?
- 2.** If the shop spent \$20 per day on advertising that would increase the relative demand for Pomona to twice that of Coastal, should it be done?

Solve the following linear

programming model graphically
and explain the solution result:

- 56.** minimize $Z = \$3,000x_1 + 1,000x_2$
subject to
- $$60x_1 + 20x_2 \geq 1,200$$
- $$10x_1 + 10x_2 \geq 400$$
- $$40x_1 + 160x_2 \geq 2,400$$
- $$x_1, x_2 \geq 0$$

[Page 69]

Solve the following linear
programming model graphically
and explain the solution result:

57.

maximize $Z = 60x_1 + 90x_2$

subject to

$$60x_1 + 30x_2 \leq 1,500$$

$$100x_1 + 100x_2 \geq 6,000$$

$$x_2 \geq 30$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming model graphically and explain the solution result:

maximize $Z = 110x_1 + 75x_2$

58. subject to

$$2x_1 + x_2 \geq 40$$

$$-6x_1 + 8x_2 \leq 120$$

$$70x_1 + 105x_2 \geq 2,100$$

$$x_1, x_2 \geq 0$$

Case Problem

METROPOLITAN POLICE PATROL

The Metropolitan Police Department had recently been criticized in the local media for not responding to police calls in the downtown area rapidly enough. In several recent cases, alarms had sounded for break-ins, but by the time the police car arrived, the perpetrators had left, and in one instance a store owner had

been shot. Sergeant Joe Davis had been assigned by the chief as head of a task force to find a way to determine the optimal patrol area (dimensions) for their cars that would minimize the average time it took to respond to a call in the downtown area.

Sergeant Davis solicited help from Angela Maris, an analyst in the operations area for the police department. Together they began to work through the problem.

Joe noted to Angela that

normal patrol sectors are laid out in rectangles, with each rectangle including a number of city blocks. For illustrative purposes he defined the dimensions of the sector as x in the horizontal direction and as y in the vertical direction. He explained to Angela that cars traveled in straight lines either horizontally or vertically and turned at right angles. Travel in a horizontal direction must be accompanied by travel in a vertical direction, and the total distance traveled is the sum of the horizontal and vertical

segments. He further noted that past research on police patrolling in urban areas had shown that the average distance traveled by a patrol car responding to a call in either direction was one-third of the dimensions of the sector, or $x/3$ and $y/3$. He also explained that the travel time it took to respond to a call (assuming that a car left immediately upon receiving the call) is simply the average distance traveled divided by the average travel speed.

Angela told Joe that now that

she understood how average travel time to a call was determined, she could see that it was closely related to the size of the patrol area. She asked Joe if there were any restrictions on the size of the area sectors that cars patrolled. He responded that for their city, the department believed that the perimeter of a patrol sector should not be less than 5 miles or exceed 12 miles. He noted several policy issues and staffing constraints that required these specifications. Angela wanted to know if any

additional restrictions existed, and Joe indicated that the distance in the vertical direction must be at least 50% more than the horizontal distance for the sector. He explained that laying out sectors in that manner meant that the patrol areas would have a greater tendency to overlap different residential, income, and retail areas than if they ran the other way. He said that these areas were layered from north to south in the city. So if a sector area were laid out east to west, all of it would

tend to be in one demographic layer.

Angela indicated that she had almost enough information to develop a model, except that she also needed to know the average travel speed the patrol cars could travel. Joe told her that cars moving vertically traveled an average of 15 miles per hour, whereas cars traveled horizontally an average of 20 miles per hour. He said that the difference was due to different traffic flows.

Develop a linear programming

model for this problem and solve it by using the graphical method.

Case Problem

"THE POSSIBILITY" RESTAURANT

Angela Fox and Zooey Caulfield were food and nutrition majors at State University, as well as close friends and roommates. Upon graduation Angela and Zooey decided to open a French restaurant in Draperton, the small town where the university was located. There were no other French restaurants in Draperton, and

the possibility of doing something new and somewhat risky intrigued the two friends. They purchased an old Victorian home just off Main Street for their new restaurant, which they named "The Possibility."

Angela and Zooey knew in advance that at least initially they could not offer a full, varied menu of dishes. They had no idea what their local customers' tastes in French cuisine would be, so they decided to serve only two full-course meals each night, one

with beef and the other with fish. Their chef, Pierre, was confident he could make each dish so exciting and unique that two meals would be sufficient, at least until they could assess which menu items were most popular. Pierre indicated that with each meal he could experiment with different appetizers, soups, salads, vegetable dishes, and desserts until they were able to identify a full selection of menu items.

The next problem for Angela and Zooey was to determine

how many meals to prepare for each night so they could shop for ingredients and set up the work schedule. They could not afford too much waste. They estimated that they would sell a maximum of 60 meals each night. Each fish dinner, including all accompaniments, requires 15 minutes to prepare, and each beef dinner takes twice as long. There is a total of 20 hours of kitchen staff labor available each day. Angela and Zooey believe that because of the health consciousness of their potential

clientele, they will sell at least three fish dinners for every two beef dinners. However, they also believe that at least 10% of their customers will order beef dinners. The profit from each fish dinner will be approximately \$12, and the profit from a beef dinner will be about \$16.

Formulate a linear programming model for Angela and Zooey that will help them estimate the number of meals they should prepare each night and solve this model graphically.

If Angela and Zooey increased the menu price on the fish dinner so that the profit for both dinners was the same, what effect would that have on their solution? Suppose Angela and Zooey reconsidered the demand for beef dinners and decided that at least 20% of their customers would purchase beef dinners. What effect would this have on their meal preparation plan?

Case Problem

ANNABELLE INVESTS IN THE MARKET

Annabelle Sizemore has cashed in some treasury bonds and a life insurance policy that her parents had accumulated over the years for her. She has also saved some money in certificates of deposit and savings bonds during the 10 years since she graduated from college. As a result, she has \$120,000 available to invest.

Given the recent rise in the stock market, she feels that she should invest all of this amount there. She has researched the market and has decided that she wants to invest in an index fund tied to S&P stocks and in an Internet stock fund. However, she is very concerned about the volatility of Internet stocks. Therefore, she wants to balance her risk to some degree.

She has decided to select an index fund from Shield Securities and an Internet

stock fund from Madison Funds, Inc. She has also decided that the proportion of the dollar amount she invests in the index fund relative to the Internet fund should be at least one-third but that she should not invest more than twice the amount in the Internet fund that she invests in the index fund. The price per share of the index fund is \$175, whereas the price per share of the Internet fund is \$208. The average annual return during the last 3 years for the index fund has been 17%, and for

the Internet stock fund it has been 28%. She anticipates that both mutual funds will realize the same average returns for the coming year that they have in the recent past; however, at the end of the year she is likely to reevaluate her investment strategy anyway. Thus, she wants to develop an investment strategy that will maximize her return for the coming year.

Formulate a linear programming model for Annabelle that will indicate how much money she should invest in each fund and solve this

model by using the graphical method.

Suppose Annabelle decides to change her risk-balancing formula by eliminating the restriction that the proportion of the amount she invests in the index fund to the amount that she invests in the Internet fund must be at least one-third. What will the effect be on her solution? Suppose instead that she eliminates the restriction that the proportion of money she invests in the Internet fund relative to the stock fund not exceed a ratio of 2 to 1. How

will this affect her solution?

If Annabelle can get \$1 more to invest, how will that affect her solution? \$2 more? \$3 more? What can you say about her return on her investment strategy, given these successive changes?

Chapter 3. Linear Programming: Computer Solution and Sensitivity Analysis

[Page 72]

[Chapter 2](#) demonstrated how a linear programming model is formulated and how a solution can be derived from a graph of the model. Graphing can provide valuable insight into

linear programming and linear programming solutions in general. However, the fact that this solution method is limited to problems with only two decision variables restricts its usefulness as a *general* solution technique.

In this chapter we will show how linear programming problems can be solved using several personal computer software packages. We will also describe how to use a computer solution result to experiment with a linear programming model to see what effect

parameter changes have on the optimal solution, referred to as *sensitivity analysis*.

Computer Solution

When linear programming was first developed in the 1940s, virtually the only way to solve a problem was by using a lengthy manual mathematical solution procedure called the **simplex method**. However, during the next six decades, as computer technology evolved, the computer was used more and more to solve linear programming models. The mathematical steps of the

simplex method were simply programmed in prewritten software packages designed for the solution of linear programming problems. The ability to solve linear programming problems quickly and cheaply on the computer, regardless of the size of the problem, popularized linear programming and expanded its use by businesses. There are currently dozens of software packages with linear programming capabilities. Many of these are general-purpose management science or

quantitative methods packages with linear programming modules, among many other modules for other techniques. There are also numerous software packages that are devoted exclusively to linear programming and its derivatives. These packages are generally cheap, efficient, and easy to use.

*The **simplex method** is a procedure involving a set of mathematical steps to solve linear*

*programming
problems.*

As a result of the easy and low-cost availability of personal computers and linear programming software, the simplex method has become less of a focus in the teaching of linear programming. Thus, at this point in our presentation of linear programming, we focus exclusively on computer solution. However, knowledge

of the simplex method is useful in gaining an overall, in-depth understanding of linear programming for those who are interested in this degree of understanding. As noted, computer solution itself is based on the simplex method. Thus, while we present linear programming in the text in a manner that does not require use of the simplex method, we also provide in-depth coverage of this topic on the CD that accompanies this text.

In the next few sections we demonstrate how to solve

linear programming problems by using Excel spreadsheets and QM for Windows, a typical general-purpose quantitative methods software package.

Excel Spreadsheets

Excel can be used to solve linear programming problems, although the data input requirements can be more time-consuming and tedious than with a software package like QM for Windows that is specifically designed for the

purpose. A spreadsheet requires that column and row headings for the specific model be set up and that constraint and objective function formulas be input in their entirety, as opposed to just the model parameters, as with QM for Windows. However, this is also an advantage of spreadsheets, in that it enables the problem to be set up in an attractive format for reporting and presentation purposes. In addition, once a spreadsheet is set up for one problem, it can often be used as a template for

others. [Exhibit 3.1](#) shows an Excel spreadsheet set up for our Beaver Creek Pottery Company example introduced in [Chapter 2](#). [Appendix B](#) at the end of this text is the tutorial "Setting Up and Editing a Spreadsheet," using [Exhibit 3.5](#) as an example.

[Page 73]

Exhibit 3.1.

[\[View full size image\]](#)

Objective function

Click on "Tools"
to invoke "Solver."

The screenshot shows an Excel spreadsheet for 'The Beaver Creek Pottery Company'. The spreadsheet is divided into two main sections: 'Products' and 'Resources'. The 'Products' section (rows 3-4) lists 'Bowls' and 'Mugs' with their respective profit per unit (40 and 50). The 'Resources' section (rows 5-7) lists 'Labor (hours)' and 'Clay (pounds)' with their respective usage per unit of product. The 'Production' section (rows 8-10) lists 'Bowls' and 'Mugs' with their respective production quantities (B10 and B11). The 'Profit' cell (B12) is currently empty. The spreadsheet includes several callouts: 'Objective function' points to the formula bar showing '=C6*B10+D6*B11'; 'Click on "Tools" to invoke "Solver."' points to the 'Tools' menu; 'Decision variables—bowls (x_1)=B10; mugs (x_2)=B11' points to the 'Production' section; '=G6-E6' points to the formula bar showing '=G6-E6'; '=G7-E7' points to the formula bar showing '=G7-E7'; and '=C7*B10+D7*B11' points to the formula bar showing '=C7*B10+D7*B11'.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	The Beaver Creek Pottery Company														
2															
3	Products		Bowl	Mug											
4	Profit per unit		40	50											
5	Resources				Usage	Constraint	Available	Left over							
6	Labor (hours)		1	2	0	4x	40	40							
7	Clay (pounds)		4	3	0	0	120	120							
8															
9	Production														
10	Bowls =														
11	Mugs =														
12	Profit =														
13															

The values for bowls and mugs and for profit are contained in cells B10, B11, and B12, respectively. These cells are currently empty because the model has not yet been solved.

The objective function for profit, **=C4*B10+D4*B11**, is embedded in cell B12 and shown on the formula bar at the top of the screen. This formula is essentially the same as $Z = 40x_1 + 50x_2$, where B10 and B11 represent x_1 and x_2 , and B12 equals Z . The objective function coefficients, 40 and 50, are in cells C4 and D4. Similar formulas for the constraints for labor and clay are embedded in cells E6 and E7. For example, in cell E6 we input the formula **=C6*B10+D6*B11**. The $\leq$

signs in cells F6 and F7 are for cosmetic purposes only; they have no real effect.

To solve this problem, first bring down the "Tools" menu from the toolbar at the top of the screen and then select "Solver" from the list of menu items. (If "Solver" is not shown on the "Tools" menu, then it can be activated by clicking on "Add-ins" on the "Tools" menu and then "Solver." If Solver is not available from the "Add-ins" menu, it must be installed on the "Add-ins" menu directly from your Office or Excel

software. For example, in Office you would access "Office Setup," then the "Office Applications" menu, then "Excel," then "Add-ins," and then activate "Solver.") The window Solver Parameters will appear, as shown in [Exhibit 3.2](#). Initially all the windows on this screen are blank, and we must input the objective function cell, the cells representing the decision variables, and the cells that make up the model constraints.

Exhibit 3.2.

(This item is displayed on page 74 in the print version)

[\[View full size image\]](#)

The image shows the "Solver Parameters" dialog box in Microsoft Excel. The dialog box has a title bar with a close button (X). The main area contains the following fields and controls:

- Set Target Cell:** A text box containing "\$B\$12" and a small icon.
- Equal To:** Three radio buttons labeled "Max", "Min", and "Value of:", followed by a text box containing "0".
- By Changing Cells:** A text box containing "\$B\$10:\$B\$11" and a small icon.
- Subject to the Constraints:** A list box containing two constraints: "\$E\$6 <= \$G\$6" and "\$E\$7 <= \$G\$7".
- Buttons:** "Solve", "Close", "Options", "Reset All", "Help", "Add", "Change", and "Delete".

Callouts (blue rounded rectangles) point to specific parts of the dialog box:

- Objective function:** Points to the "Set Target Cell" field.
- Decision variables:** Points to the "By Changing Cells" field.
- Click on "Add" to add model constraints:** Points to the "Add" button.
- Constraint 1:** A callout points to the first constraint "\$E\$6 <= \$G\$6", with a corresponding formula box below it: $=C7*B10+D7*B11 \leq 120$.
- Constraint 2:** A callout points to the second constraint "\$E\$7 <= \$G\$7", with a corresponding formula box below it: $=C6*B10+D6*B11 \leq 40$.

When inputting the Solver parameters as shown in [Exhibit 3.2](#), we first input the "target cell" that contains our objective function, which is B12 for our example. (Excel automatically inserts the \$ sign next to cell addresses; you should not type it in.) Next we indicate that we want to maximize the target cell by clicking on "Max." We achieve our objective by changing cells B10 and B11, which represent our model decision variables. The

designation "B10:B11" means all the cells between B10 and B11, inclusive. We next input our model constraints by clicking on "Add," which will access the screen shown in [Exhibit 3.3](#).

Exhibit 3.3.

(This item is displayed on page 74
in the print version)

[\[View full size image\]](#)

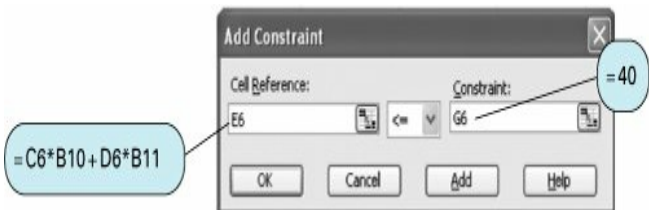


Exhibit 3.3 shows our labor constraint. Cell E6 contains the constraint formula for labor ($=\mathbf{C6*B10+D6*B11}$), whereas cell G6 contains the labor hours available (i.e., 40). We continue to add constraints until the model is complete. Note that we could have input our constraints by adding a

single constraint formula,
E6:E7 <= **G6:G7**, which means that the constraints in cells E6 and E7 are less than or equal to the values in cells G6 and G7, respectively. It is also not necessary to input the nonnegativity constraints for our decision variables, **B10:B11** >= **0**. This can be achieved in the Options screen, as explained next.

[Page 74]

Click on "OK" on the Add Constraint window after all

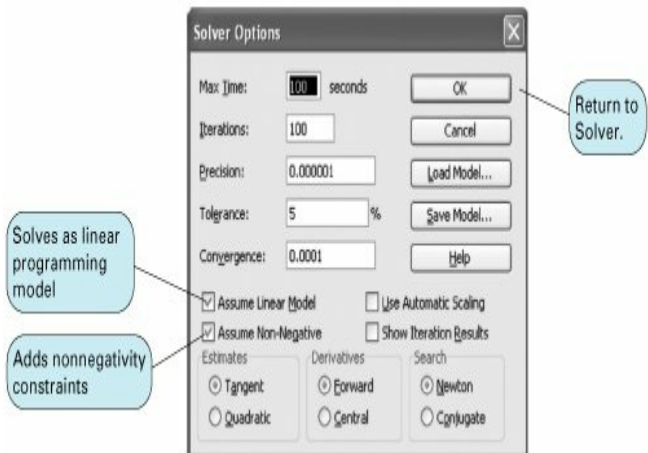
constraints have been added. This will return us to the Solver Parameters screen. There is one more necessary step before proceeding to solve the problem. Select "Options" from the Solver Parameters screen and then when the Options screen appears, as shown in [Exhibit 3.4](#), click on "Assume Linear Models." This will ensure that Solver uses the simplex procedure to solve the model and not some other numeric method (which Excel has available). This is not too important for now, but later it

will ensure that we get the right reports for sensitivity analysis, a topic we will take up next. Notice that the Options screen also enables you to establish the nonnegativity conditions by clicking on "Assume Non-Negative." Click on "OK" to return to Solver.

Exhibit 3.4.

(This item is displayed on page 75
in the print version)

[\[View full size image\]](#)



Once the complete model is input, click on "Solve" in the upper-right corner of the Solver Parameters screen ([Exhibit 3.2](#)). First, a screen

will appear, titled Solver Results, which will provide you with the opportunity to select the reports you want and then when you click on "OK," the solution screen shown in [Exhibit 3.5](#) will appear.

Exhibit 3.5.

(This item is displayed on page 75
in the print version)

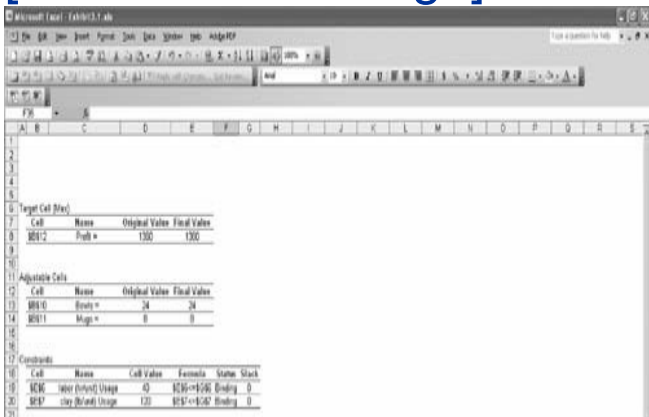
[\[View full size image\]](#)

We can also generate several reports that summarize the model results. When you click on "OK" from the Solver screen, an intermediate screen will appear before the original spreadsheet, with the solution results. This screen is titled Solver Results, and it provides an opportunity for you to select several reports, including the answer report, shown in [Exhibit 3.6](#). This report provides a summary of the solution results.

Exhibit 3.6.

(This item is displayed on page 76
in the print version)

[\[View full size image\]](#)



Microsoft Excel - Exhibit3.6.xls

File Edit Insert Format Tools Data Window Help

Formulas > Solver Parameters

Target Cell (Max)

Cell	Name	Original Value	Final Value
\$B\$12	Profit =	1300	1300

Adjustable Cells

Cell	Name	Original Value	Final Value
\$B\$10	Units =	24	24
\$B\$11	Mugs =	0	0

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
\$B\$6	water (left) Usage	40	\$B\$6<=\$B\$6	Binding	0
\$B\$7	clay (left) Usage	120	\$B\$7<=\$B\$7	Binding	0

QM for Windows

Before demonstrating how to use QM for Windows, we must first make a few comments about the proper form that constraints must be in before a linear programming model can be solved with QM for Windows. The constraints formulated in the linear programming models presented in [Chapter 2](#) and in this chapter have followed a consistent form. All the variables in the constraint have appeared to the left of the inequality, and all numerical

values have been on the right-hand side of the inequality. For example, in the pottery company model, the constraint for labor is

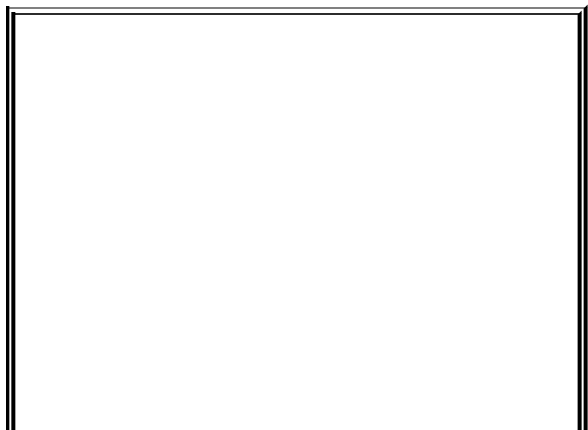
$$x_1 + 2x_2 \leq 40$$

The value, 40, is referred to as the constraint quantity, or *right-hand-side*, value.

[Page 76]

The standard form for a linear programming problem requires constraints to be in this form,

with variables on the left side and numeric values to the right of the inequality or equality sign. This is a necessary condition to input problems into some computer programs, and specifically QM for Windows, for linear programming solution.



Management Science

Application: Optimizing

Production Quantities at GE Plastics

GE Plastics is a \$5 billion global business that supplies plastics and raw materials from plants around the world to companies such as automotive, appliance, computer, and medical equipment companies.

Among its seven major divisions, the High Performance Polymers (HPP) division is the fastest growing. HPP is a very heat-tolerant polymer that is used in the manufacture of microwave cookware, fire helmets, utensils, and aircraft. HPP has a supply chain that consists of two levels of manufacturing plants and

distribution channels that is similar to all GE Plastics divisions. First-level plants convert feedstocks into resins and then ship them to finishing plants, where they are combined with additives to produce different grades of end products. The products are then shipped to GE Polymerland (a commercial front for GE plastics), which distributes them to customers. Each physical plant has multiple plant lines that operate as independent manufacturing facilities. HPP has 8 resin plant lines that feed 21 finishing plant lines, which produce 24 grades of HPP products. HPP uses a linear programming model to optimize production along this supply chain of manufacturing plant lines and distribution channels. The linear programming model objective function maximizes the total contribution margin, which consists

of revenues minus the sum of manufacturing, additive, and distribution costs, subject to demand, manufacturing capacity, and flow constraints. The decision variables are the amount of each resin produced at each resin plant line that will be used at each finishing plant line and the amount of each product at each finishing plant line. The company uses the model to develop a four-year planning horizon by solving four single-year models; each single-year LP model has 3,100 variables and 1,100 constraints. The model is solved within an Excel framework, using LINGO, a commercial optimization solver.



Source: R. Tyagi, P. Kalish, K. Akbay,
and G. Munshaw, "GE Plastics
Optimizes the Two-Echelon Global
Fulfillment Network at Its High
Performance Polymers Division,"
Interfaces 34, no. 5
(September/October 2004): 359366.

Consider a model requirement which states that the production of product 3 (x_3) must be as much as or more than the production of products 1 (x_1) and 2 (x_2). The model constraint for this requirement is formulated as

$$x_3 \geq x_1 + x_2$$

This constraint is not in proper form and could not be input

into QM for Windows as it is. It must first be converted to

$$x_3 - x_1 - x_2 \geq 0$$

This constraint can now be input for computer solution.

Next consider a problem requirement that the ratio of the production of product 1 (x_1) to the production of products 2 (x_2) and 3 (x_3) must be at least 2 to 1. The model constraint for this requirement is formulated as

$$\frac{x_1}{x_2 + x_3} \geq 2$$

Although this constraint does meet the condition that variables be on the left side of the inequality and numeric values on the right, it is not in proper form. The fractional relationship of the variables, $x_1/(x_2 + x_3)$, cannot be input into the most commonly used linear programming computer programs in that form. It must be converted to

$$x_1 \geq 2(x_2 + x_3)$$

and

$$x_1 - 2x_2 - 2x_3 \geq 0$$

Fractional relationships between variables in constraints must be eliminated.

We will demonstrate how to use QM for Windows by solving our Beaver Creek Pottery Company example. The linear

programming module in QM for Windows is accessed by clicking on "Module" at the top of the initial window. This will bring down a menu with all the program modules available in QM for Windows, as shown in [Exhibit 3.7](#). By clicking on "Linear Programming," a window for this program will come up on the screen, and by clicking on "File" and then "New," a screen for inputting problem dimensions will appear. [Exhibit 3.8](#) shows the data entry screen with the model type and the number of

constraints and decision variables for our Beaver Creek Pottery Company example.

Exhibit 3.7.

[\[View full size image\]](#)

Click on "File" and then "New" to enter a new problem.

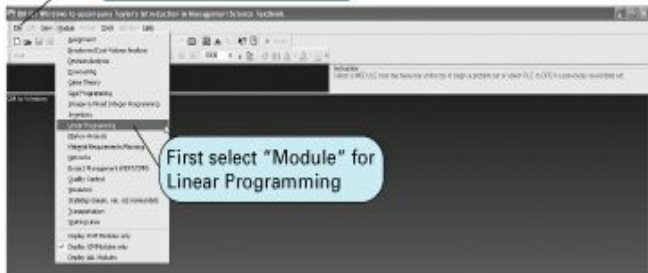


Exhibit 3.8.

(This item is displayed on page 78
in the print version)

[\[View full size image\]](#)

Set number of constraints and decision variables.

Create data set for Linear Programming

Title: Beaver Creek Pottery Company Modify default title

Number of Constraints:

Number of Variables:

Objective

☒ Maximize

☐ Minimize

Row names

☒ Constraint 1, Constraint 2, Constraint 3, ...

☐ a, b, c, d, e, ...

☐ A, B, C, D, E, ...

☐ 1, 2, 3, 4, 5, ...

☐ January, February, March, April, ...

☐ Other

Click here when finished.

Exhibit 3.9 shows the data table for our example, with the model parameters, including the objective function and

constraint coefficients, and the constraint quantity values. Notice that we also customized the row headings for the constraints by typing in "Labor (hrs)" and "Clay (lbs)." Once the model parameters have been input, click on "Solve" to get the model solution, as shown in [Exhibit 3.10](#). It is not necessary to put the model into standard form by adding slack variables.

full size image]

Exhibit 3.10.

[\[View full size image\]](#)

Beaver Creek Pottery Company Solution

	X1	X2		RHS	Dual	
Maximize	40	50			Max $40X_1 +$	
Labor (hrs)	1	2	$\leq$	40	18	
Clay (lbs)	4	3	$\leq$	120	8	
Solution $\rightarrow$	24	8	Optimal Z $\rightarrow$	1360	$4X_1 + 3X_2 \leq$	

The marginal value is the dollar amount one would be willing to pay for one additional resource unit.

Notice the values 16 and 6 under the column labeled "Dual" for the "Labor" and "Clay" rows. These dual values are the **marginal values** of labor and clay in our problem. This is useful information that is provided in addition to the normal model solution values when you solve a linear programming model. We talk about dual values in more detail later in this chapter, but for now it is sufficient to say that the marginal value is the

dollar amount the company would be willing to pay for one additional unit of a resource. For example, the dual value of 16 for the labor constraint means that if 1 additional hour of labor could be obtained by the company, it would increase profit by \$16. Likewise, if 1 additional pound of clay could be obtained, it would increase profit by \$6. Thus, the company would be willing to pay up to \$16 for 1 more hour of labor and \$6 for 1 more pound of clay. The dual value is not the purchase price of one of

these resources; it is the maximum amount the company would pay to get more of the resource. These dual values are helpful to the company in making decisions about acquiring additional resources.

[Page 79]

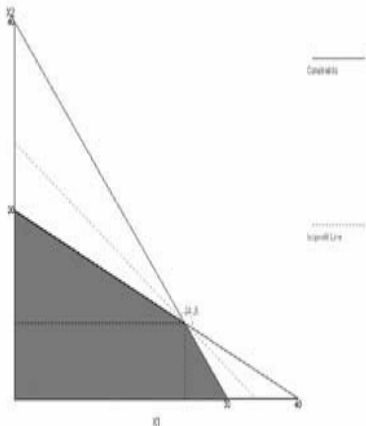
QM for Windows provides solution results in several different formats, including a graphical solution by clicking on "Window" and then selecting "Graph." [Exhibit 3.11](#) shows the graphical solution for our

Beaver Creek Pottery Company
example.

Exhibit 3.11.

[\[View full size image\]](#)

Beaver Creek Pottery Company



Constraint Display

☐ Max $40X_1 + 30X_2$

☐ $10X_1 + 20X_2 \leq 800$

☐ $40X_1 + 30X_2 \leq 120$

☐ none

Corner Points

X_1	X_2	Z
0	0	0
0	20	1,000
20	0	1,200
14	1	1,360

Sensitivity Analysis

When linear programming models were formulated in [Chapter 2](#), it was implicitly assumed that the *parameters* of the model were known with certainty. These parameters include the objective function coefficients, such as profit per bowl; model constraint quantity values, such as available hours of labor; and constraint coefficients, such as pounds of clay per bowl. In the examples

presented so far, the models have been formulated as if these parameters are known exactly or with certainty. However, rarely does a manager know all these parameters exactly. In reality, the model parameters are simply estimates (or "best guesses") that are subject to change. For this reason, it is of interest to the manager to see what effect a change in a parameter will have on the solution to the model. Changes may be either reactions to anticipated uncertainties in the

parameters or reactions to information. The analysis of parameter changes and their effects on the model solution is known as **sensitivity analysis**.

Sensitivity analysis is the analysis of the effect of parameter changes on the optimal solution.

The most obvious way to ascertain the effect of a change

in the parameter of a model is to make the change in the original model, *resolve* the model, and compare the solution results with the original. However, as we will demonstrate in this chapter, in some cases the effect of changes on the model can be determined without solving the problem again.

[Page 80]

Changes in Objective Function Coefficients

The first model parameter change we will analyze is a change in an objective function coefficient. We will use our now-familiar Beaver Creek Pottery Company example to illustrate this change:

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

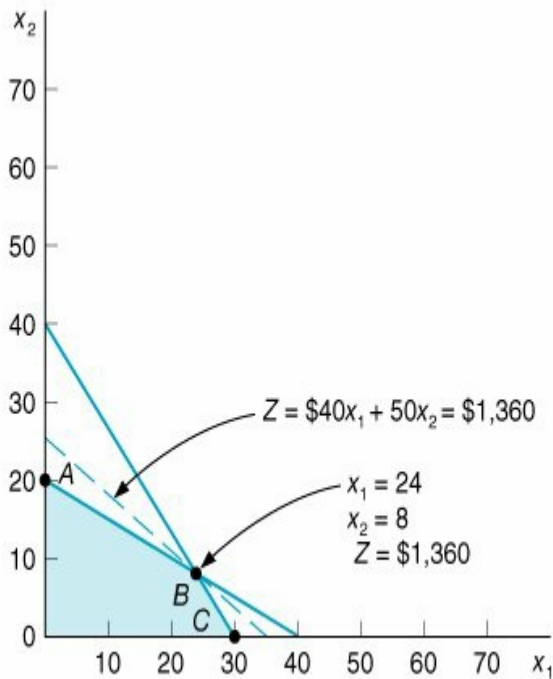
$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$

$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$

$$x_1, x_2 \geq 0$$

The graphical solution for this problem is shown in [Figure 3.1](#).

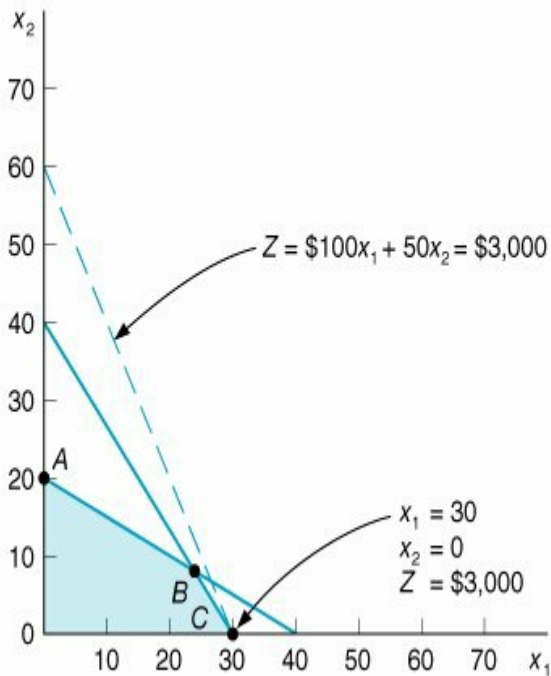
Figure 3.1. Optimal solution point



In [Figure 3.1](#) the optimal solution point is shown to be at point B ($x_1 = 24$ and $x_2 = 8$), which is the last point the objective function, denoted by the dashed line, touches as it leaves the feasible solution area. However, what if we changed the profit of a bowl, x_1 , from \$40 to \$100? How would that affect the solution identified in [Figure 3.1](#)? This change is shown in [Figure 3.2](#).

Figure 3.2. Changing the objective function x_1 coefficient

(This item is displayed on page 81
in the print version)



Increasing profit for a bowl (i.e., the x_1 coefficient) from \$40 to \$100 makes the objective function line steeper, so much so that the optimal solution point changes from point B to point C .

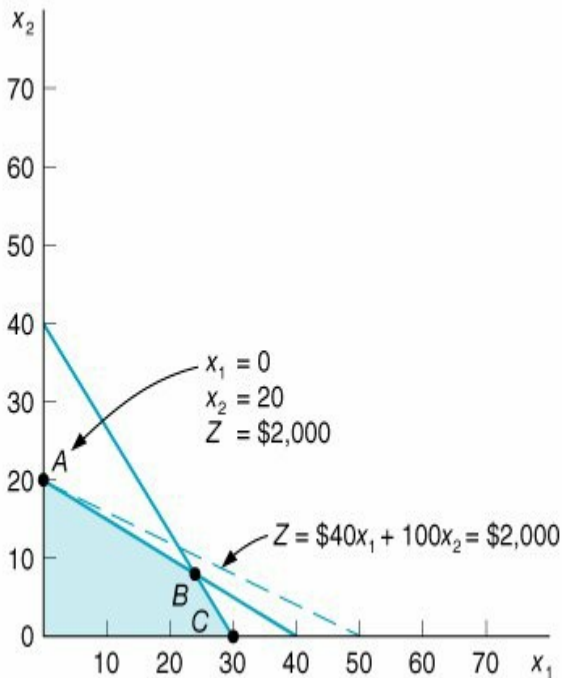
Alternatively, if we increased the profit for a mug, the x_2 coefficient, from \$50 to \$100, the objective function line would become flatter, to the extent that point A would become optimal, with

$x_1 = 0$, $x_2 = 20$, and $Z =$
\$2,000

This is shown in [Figure 3.3](#).

Figure 3.3. Changing the objective function x_2 coefficient

(This item is displayed on page 81
in the print version)



The sensitivity range for an objective coefficient is the range of values over which the current optimal solution point will remain optimal.

The objective of sensitivity analysis in this case is to determine the range of values for a specific objective function

coefficient over which the optimal solution point, x_1 and x_2 , will remain optimal. For example, the coefficient of x_1 in the objective function is originally \$40, but at some value greater than \$40, point C will become optimal, and at some value less than \$40, point A will become optimal. The focus of sensitivity analysis is to determine those two values, referred to as the sensitivity range for the x_1 coefficient, which we will designate as c_1 .

For our simple example, we can look at the graph in [Figure 3.1](#) and determine the sensitivity range for the x_1 coefficient. The slope of the objective function is currently $4/5$, determined as follows:

$$Z = 40x_1 + 50x_2$$

or

$$50x_2 = Z - 40x_1$$

and

$$x_2 = \frac{Z}{50} - \frac{4x_1}{5}$$

[Page 82]

Management Science

Application: Grape Juice

Management at Welch's

With annual sales over \$550 million, Welch's is one of the world's largest grape-processing companies. Founded in 1869 by Dr. Thomas B. Welch, it processes raw grapes (nearly 300,000 tons per year) into juice, jellies, and frozen concentrates. Welch's is owned by the National Grape Cooperative Association (NGCA), which has a membership of 1,400 growers. Welch's is NGCA's production, distribution, and marketing organization. Welch's operates its grape-processing plants near its growers. Because of the dynamic nature of product demand and customer service, Welch's holds finished goods inventory as a buffer and maintains a large raw

materials inventory, stored as grape juice in refrigerated tank farms. Packaging operations at each plant draw juice from the tank farms during the year as needed. The value of the stored grape juice often exceeds \$50 million. Harvest yields and grape quality vary between localities. In order to maintain national quality and consistency in its products, Welch's transfers juices between plants and adjusts product recipes. To do this Welch's uses a spreadsheet-based linear programming model. The juice logistics model (JLM) encompasses 324 decision variables and 361 constraints to minimize the combined costs of transporting grape juice between plants and the product recipes at each plant and the carrying cost of storing grape juice. The model decision variables include the grape juice shipped to customers for different product groups, the grape juice transferred between plants, and inventory levels at each plant.

Constraints are for recipe requirements, inventories, and grape juice usage and transfers. During the first year the linear programming model was used, it saved Welch's between \$130,000 and \$170,000 in carrying costs alone by showing Welch's it did not need to purchase extra grapes that were then available. The model has enabled Welch's management to make quick decisions regarding inventories, purchasing grapes, and adjusting product recipes when grape harvests are higher or lower than expected and when demand changes, resulting in significant cost savings.



Source: E. W. Schuster and S. J. Allen,
"Raw Material Management at Welch's,
Inc.," *Interfaces* 28, no. 5
(SeptemberOctober 1998): 1324.

The objective function is now in the form of the equation of a straight line, $y = a + bx$, where the intercept, a , equals $Z/50$ and the slope, b , is $4/5$.

If the slope of the objective function increases to $4/3$, the objective function line becomes exactly parallel to the constraint line,

$$4x_1 + 3x_2 = 120$$

and point C becomes optimal (along with B). The slope of this constraint line is $4/3$, so we ask ourselves what

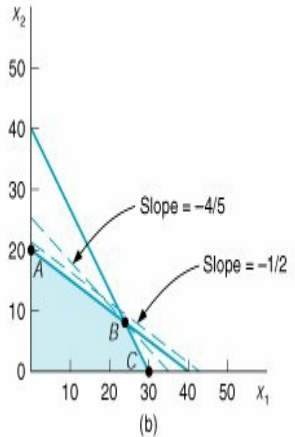
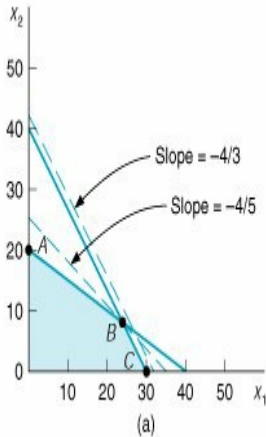
objective function coefficient for x_1 will make the objective function slope equal $4/3$? The answer is determined as follows, where c_1 is the objective function coefficient for x_1 :

$$\begin{aligned}\frac{-c_1}{50} &= \frac{-4}{3} \\ -3c_1 &= -200 \\ c_1 &= \$66.67\end{aligned}$$

If the coefficient of x_1 is 66.67, then the objective function will have a slope of $66.67/50$ or $4/3$. This is illustrated in [Figure 3.4\(a\)](#).

Figure 3.4.
Determining the
sensitivity range for
 C_1

[\[View full size image\]](#)



We have determined that the upper limit of the sensitivity range for c_1 , the x_1 coefficient,

is 66.67. If profit for a bowl increases to exactly \$66.67, the solution points will be both *B* and *C*. If the profit for a bowl is more than \$66.67, point *C* will be the optimal solution point.

The lower limit for the sensitivity range can be determined by observing [Figure 3.4\(b\)](#). In this case, if the objective function line slope decreases (becomes flatter) from 4/5 to the same slope as the constraint line,

$$x_1 + 2x_2 = 40$$

then point A becomes optimal (along with B). The slope of this constraint line is $1/2$, that is, $x_2 = 20 (1/2)x_1$. In order to have an objective function slope of $1/2$, the profit for a bowl would have to decrease to \$25, as follows:

$$\begin{aligned}\frac{-c_1}{50} &= \frac{-1}{2} \\ -2c_1 &= -50 \\ c_1 &= \$25\end{aligned}$$

This is the lower limit of the sensitivity range for the x_1 coefficient.

The complete sensitivity range for the x_1 coefficient can be expressed as

$$25 \leq c_1 \leq 66.67$$

This means that the profit for a bowl can vary anywhere between \$25.00 and \$66.67, and the optimal solution point, $x_1 = 24$ and $x_2 = 8$, will not change. Of course, the total profit, or Z value, will change, depending on whatever value c_1 actually is.

For the manager this is useful

information. Changing the production schedule in terms of how many items are produced can have a number of ramifications in an operation. Packaging, logistical, and marketing requirements for the product might need to be altered. However, with the preceding sensitivity range, the manager knows how much the profit, and hence the selling price and costs, can be altered without resulting in a change in production.

Performing the same type of graphical analysis will provide

the sensitivity range for the x_2 objective function coefficient, c_2 . This range is $30 \leq c_2 \leq 80$. This means that the profit for a mug can vary between \$30 and \$80, and the optimal solution point, B , will not change. However, for this case and the range for c_1 , the sensitivity range generally applies only if one coefficient is varied and the other held constant. Thus, when we say that profit for a mug can vary between \$30 and \$80, this is true only if c_1 remains constant.

Simultaneous changes can be made in the objective function coefficients as long as the changes taken together do not change the optimal solution point. However, determining the effect of these simultaneous changes is overly complex and time-consuming using graphical analysis. In fact, using graphical analysis is a tedious way to perform sensitivity analysis in general, and it is impossible when the linear programming model contains three or more

variables, thus requiring a three-dimensional graph. However, Excel and QM for Windows provide sensitivity analysis for linear programming problems as part of their standard solution output. Determining the effect of simultaneous changes in model parameters and performing sensitivity analysis in general are much easier and more practical using the computer. Later in this chapter we will show the sensitivity analysis output for Excel and QM for Windows.

However, before moving on to computer-generated sensitivity analysis, we want to look at one more aspect of the sensitivity ranges for objective function coefficients. Recall that the model for our fertilizer minimization model from [Chapter 2](#) is

$$\text{minimize } Z = \$6x_1 + 3x_2$$

subject to

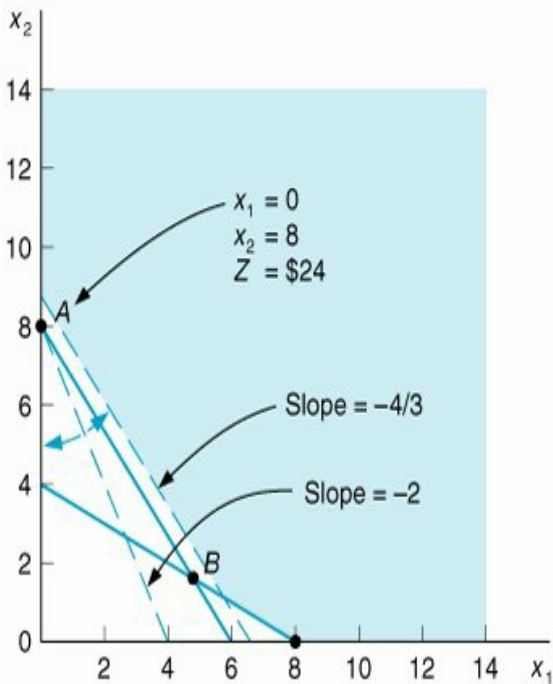
$$2x_1 + 4x_2 \geq 16$$

$$4x_1 + 3x_2 \geq 24$$

$$x_1, x_2 \geq 0$$

and the solution shown graphically in [Figure 3.5](#) is $x_1 = 0$, $x_2 = 8$, and $Z = 24$.

Figure 3.5. Fertilizer example: sensitivity range for c_1



The sensitivity ranges for the objective function coefficients are

$$4 \leq c_1 \leq \infty$$

$$0 \leq c_2 \leq 4.5$$

Notice that the upper bound for the x_1 coefficient range is infinity. The reason for this upper limit can be seen in the graphical solution of the problem shown in [Figure 3.5](#).

As the objective function coefficient for x_1 decreases from \$6, the objective function slope of 2 decreases, and the objective function line gets flatter. When the coefficient, c_1 , equals \$4, then the slope of the objective function is $-4/3$, which is the same as the constraint line, $4x_1 + 3x_2 = 24$. This makes point B optimal (as well as A). Thus, the lower limit of the sensitivity range for c_1 is \$4. However, notice that as c_1 increases from \$6, the objective function simply becomes steeper and steeper

as it rotates toward the x_2 axis of the graph. The objective function will not come into contact with another feasible solution point. Thus, no matter how much we increase cost for Super-gro fertilizer (x_1), point A will always be optimal.

[Page 85]

Objective Function Coefficient Ranges with the Computer

When we provided the Excel spreadsheet solution for the Beaver Creek Pottery Company example earlier in this chapter, we did not include sensitivity analysis. However, Excel will also generate a sensitivity report that provides the sensitivity ranges for the objective function coefficients. When you click on "Solve" from the Solver Parameters window, you will momentarily go to a Solver Results screen, shown in [Exhibit 3.12](#), which provides you with an opportunity to select different reports before

proceeding to the solution. This is how we selected our answer report earlier. The sensitivity report for our Beaver Creek Pottery Company example is shown in [Exhibit 3.13](#).

However, note that if you have not already clicked on "Assume Linear Models" from the Options screen, as we earlier warned, Excel will not generate this version of the sensitivity report.

Exhibit 3.12.

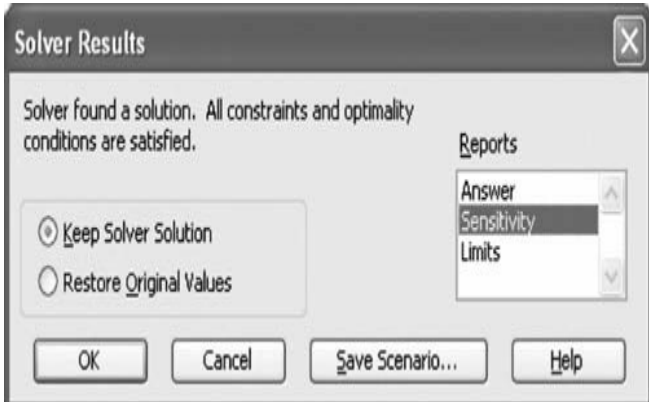
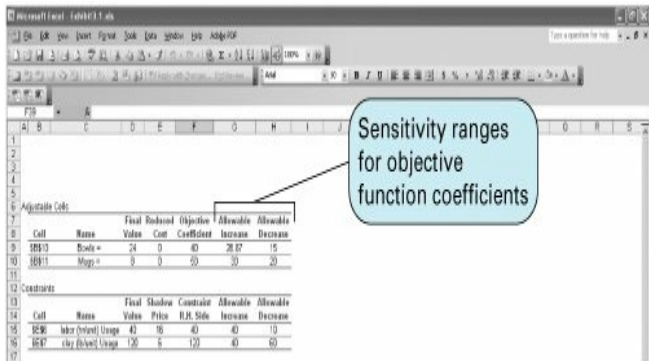


Exhibit 3.13.

[\[View full size image\]](#)



Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$10	Bowls =	24	0	40	28.87	15
\$B\$11	Mugs =	8	0	50	30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$6	labor (inland) Usage	40	16	40	40	10
\$E\$7	clay (inland) Usage	120	5	120	40	60

Sensitivity ranges for objective function coefficients

Notice that the sensitivity ranges for the objective function coefficients (40 and 50) are not provided as an upper and lower limit but instead show an allowable increase and an allowable

decrease. For example, for the coefficient \$40 for bowls (B10), the allowable increase of 26.667 results in an upper limit of 66.667, whereas the allowable decrease of 15 results in a lower limit of 25.

[Page 86]

The sensitivity ranges for the objective function coefficients for our example are presented in QM for Windows in [Exhibit 3.14](#). Notice that this output provides the upper and lower limits of the sensitivity ranges

for both variables, x_1 and x_2 .

Exhibit 3.14.

[\[View full size image\]](#)

Sensitivity ranges
for objective
function coefficients

Beaver Creek Pottery Company Solution					
Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	24	0	40	25	66.6667
X2	8	0	50	30	80
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Labor (hrs)	16	0	40	30	80
Clay (lbs)	6	0	120	60	160

Changes in Constraint Quantity Values

The second type of sensitivity analysis we will discuss is the sensitivity ranges for the constraint quantity values that is, the values to the right of the inequality signs in the constraints. For our Beaver Creek Pottery Company model,

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

$$x_1 + 2x_2 + s_1 = 40 \text{ (labor, hr.)}$$

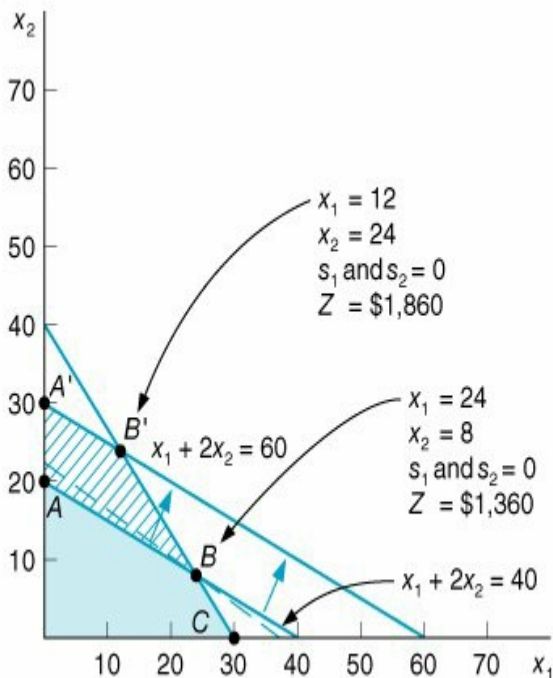
$$4x_1 + 3x_2 + s_2 = 120 \text{ (clay, lb.)}$$

$$x_1, x_2 \geq 0$$

the constraint quantity values are 40 and 120.

Consider a change in which the manager of the pottery company can increase the labor hours from 40 to 60. The effect of this change in the model is graphically displayed in [Figure 3.6](#).

Figure 3.6. Increasing the labor constraint quantity



Increasing the available labor hours from 40 to 60 causes the feasible solution space to change. It was originally $OABC$, and now it is $OA'B'C$. B' is the new optimal solution, instead of B . However, the important consideration in this type of sensitivity analysis is that the solution *mix* (or variables that do not have zero values), including slack variables, did not change, even though the

values of x_1 and x_2 did change (from $x_1 = 24$, $x_2 = 8$ to $x_1 = 12$, $x_2 = 24$). The focus of sensitivity analysis for constraint quantity values is to determine the range over which the constraint quantity values can change without changing the solution variable mix, specifically including the slack variables.

The sensitivity range for a right-hand-side value is the range of values over which the quantity values

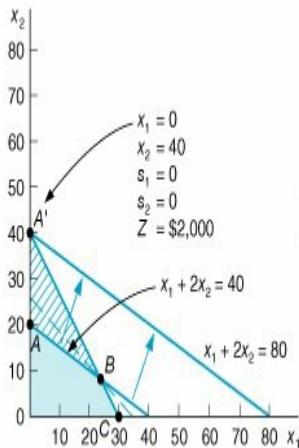
can change without changing the solution variable mix, including slack variables.

If the quantity value for the labor constraint is increased from 40 to 80 hours, the new solution space is $OA'C$, and a new solution variable mix occurs at A' , as shown in [Figure 3.7\(a\)](#). Whereas at the original optimal point, B , both x_1 and x_2

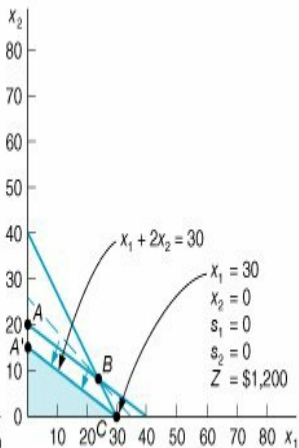
are in the solution, at the new optimal point, A' , only x_2 is produced (i.e., $x_1 = 0$, $x_2 = 40$, $s_1 = 0$, $s_2 = 0$).

Figure 3.7.
Determining the
sensitivity range for
labor quantity

[\[View full size image\]](#)



(a)



(b)

Thus, the upper limit of the sensitivity range for the quantity value for the first

constraint, which we will refer to as q_1 , is 80 hours. At this value the solution mix changes such that bowls are no longer produced. Furthermore, as q_1 increases past 80 hours, s_1 increases (i.e., slack hours are created). Similarly, if the value for q_1 is decreased to 30 hours, the new feasible solution space is $OA'C$, as shown in [Figure 3.7\(b\)](#). The new optimal point is at C , where no mugs (x_2) are produced. The new solution is $x_1 = 30$, $x_2 = 0$, $s_1 = 0$, $s_2 = 0$, and $Z = \$1,200$. Again, the

variable mix is changed.
Summarizing, the sensitivity range for the constraint quantity value for labor hours is

$$30 \leq q_1 \leq 80 \text{ hr.}$$

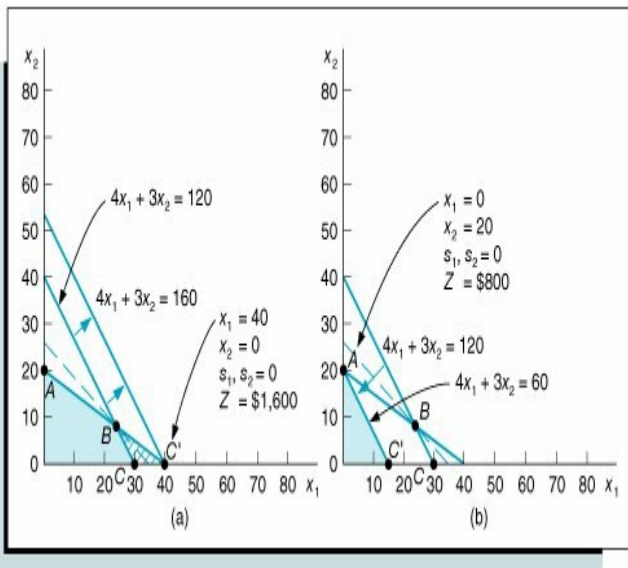
The sensitivity range for clay can be determined graphically in a similar manner. If the quantity value for the clay constraint, $4x_1 + 3x_2 \leq 120$, is increased from 120 to 160, shown in [Figure 3.8\(a\)](#), then a new solution space, OAC' , results, with a new optimal

point, C' . Alternatively, if the quantity value is decreased from 120 to 60, as shown in [Figure 3.8](#)(b), the new solution space is OAC' , and the new optimal point is A ($x_1 = 0$, $x_2 = 20$, $s_1 = 0$, $s_2 = 0$, $Z = \$800$).

Figure 3.8.
Determining the
sensitivity range for
clay quantity

(This item is displayed on page 88
in the print version)

[\[View full size image\]](#)



Summarizing, the sensitivity

ranges for q_1 and q_2 are

$$30 \leq q_1 \leq 80 \text{ hr.}$$

$$60 \leq q_2 \leq 160 \text{ lb.}$$

[Page 88]

As was the case with the sensitivity ranges for the objective function coefficient, these sensitivity ranges are valid for only one q_i value and assumes all other q_i values are held constant. However, simultaneous changes can occur, as long as they do not

change the variable mix.

These ranges for constraint quantity values provide useful information for the manager, especially regarding production scheduling and planning. If resources are reduced at the pottery company, then at some point one of the products will no longer be produced, and the support facilities and apparatus for that production will not be needed, or extra hours of resources will be created that are not needed. A similar result, albeit a better one, will occur if resources are increased

because profit will be more than with a reduction in resources.

Constraint Quantity Value Ranges with Excel and QM for Windows

Previously we showed how to generate the sensitivity report for Excel, resulting in [Exhibit 3.13](#). This report is shown again in [Exhibit 3.15](#), with the sensitivity ranges for the constraint quantity values highlighted. As mentioned

previously, the ranges are expressed in terms of an allowable increase and decrease instead of upper and lower limits.

Exhibit 3.15.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 3.1.xls

File Edit Format Tools Data Window Help

Type a question for help

100%

File Edit Format Tools Data Window Help

Ad

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

A B C D E F G H I J K L M N O P Q R S

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$10	Bears =	24	0	40	26.67	15
\$B\$11	Mugs =	8	0	50	30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$C\$6	labor (hr) and Usage	40	15	40	40	10
\$C\$7	clay (lb) and Usage	120	6	120	40	80

Sensitivity ranges for constraint quantity values

The sensitivity ranges for the constraint quantity values with QM for Windows can be observed in [Exhibit 3.16](#).

Exhibit 3.16.

[\[View full size image\]](#)

Ranging

Beaver Creek Pottery Company Solution

Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	24	0	40	25	66.6667
X2	8	0	50	30	80

Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Labor (hrs)	16	0	40	30	80
Clay (lbs)	6	0	120	60	160

Sensitivity ranges for constraint quantity values.

Other Forms of Sensitivity Analysis

Excel and QM for Windows provide sensitivity analysis ranges for objective function coefficients and constraint quantity values as part of the standard solution output.

However, there are other forms of sensitivity analysis, including changing individual constraint parameters, adding new constraints, and adding new variables.

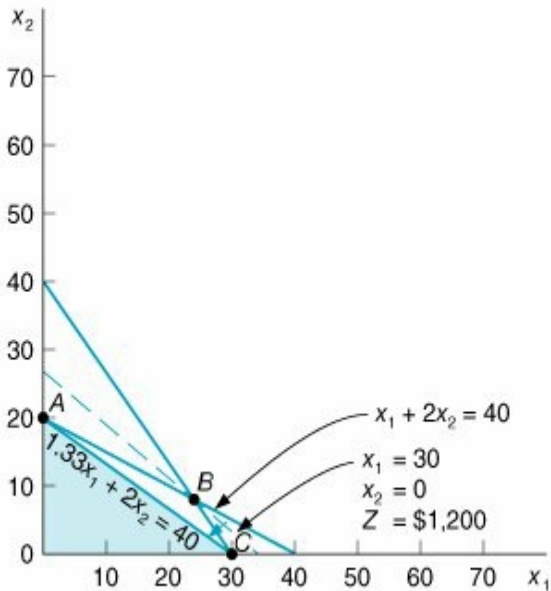
Other forms of sensitivity analysis include changing constraint parameter

values, adding new constraints, and adding new variables.

For instance, in our Beaver Creek Pottery example, if a new, less-experienced artisan were hired to make pottery, it might take this individual 1.33 hours to produce a bowl instead of 1 hour. Thus, the labor constraint would change from $x_1 + 2x_2 \leq 40$ to $1.33x_1 + 2x_2 \leq 40$. This change is illustrated

in [Figure 3.9](#).

Figure 3.9. Changing the x_1 coefficient in the labor constraint



Note that a change in the coefficient for x_1 in the labor constraint rotates the constraint line, changing the solution space from $OABC$ to OAC . It also results in a new optimal solution point, C , where $x_1 = 30$, $x_2 = 0$, and $Z = \$1,200$. Then 1.33 hours would be the logical upper limit for this constraint coefficient. However, as we pointed out, this type of sensitivity analysis for constraint variable coefficients is not typically provided in the standard linear programming computer output.

As a result, the most logical way to ascertain the effect of this type of change is simply to run the computer program with different values.

[Page 90]

Other types of sensitivity analysis are to add a new constraint to the model or to add a new variable. For example, suppose the Beaver Creek Pottery Company added a third constraint for packaging its pottery, as follows:

$$0.20x_1 + 0.10x_2 \leq 5 \text{ hr.}$$

This would require the model to be solved again with the new constraint. This new constraint does, in fact, change the optimal solution, as shown in the Excel spreadsheet in [Exhibit 3.17](#). This spreadsheet requires a new row (8) added to the original spreadsheet (first shown in [Exhibit 3.1](#)) with our new constraint parameter values, and a new constraint, $E8 \leq G8$, added to Solver. In the original model, the solution (shown in [Exhibit 3.5](#)) was 24

bowls and 8 mugs, with a profit of \$1,360. With the new constraint for packaging added, the solution is now 20 bowls and 10 mugs, with a profit of \$1,300.

Exhibit 3.17.

[\[View full size image\]](#)

Microsoft Excel - Exhibit12.17.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

Formulas: =A4*B11+D4*B12

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	The Beaver Creek Pottery Company														
2															
3	Products:		Bowl	Mug											
4	Profit per unit		40	50											
5	Resources:				Usage	Constraints	Available	Left over							
6	labor (hr/unit)		1	2	40	<=	40	0							
7	clay (lb/unit)		4	3	110	<=	110	0							
8	packaging (hr/unit)		0.2	0.1	5	<=	5	0							
9															
10	Production														
11	Bowls =	20													
12	Mugs =	10													
13	Profit =	1300													
14															

Added constraint for packaging

Constraint E8 ≤ G8 added to Solver

If a new variable is added to the model, this would also require that the model be solved again to determine the effect of the change. For example, suppose the pottery

company was contemplating producing a third product, cups. It can secure no additional resources, and the profit for a cup is estimated to be \$30. This change is reflected in the following model reformulation:

$$\text{maximize } Z = \$40x_1 + 50x_2 + 30x_3$$

subject to

$$x_1 + 2x_2 + 1.2x_3 \leq 40 \text{ (labor, hr.)}$$

$$4x_1 + 3x_2 + 2x_3 \leq 120 \text{ (clay, hr.)}$$

lb.)

$$x_1, x_2, x_3 \geq 0$$

Solving this new formulation with the computer will show that this prospective change will have no effect on the original solution that is, the model is not sensitive to this change. The estimated profit from cups was not enough to offset the production of bowls and mugs, and the solution remained the same.

Shadow Prices

We briefly discussed dual values (also called *shadow prices*) earlier in this chapter, in our discussion of QM for Windows. You will recall that a dual value was defined as the marginal value of one additional unit of resource. We need to mention shadow prices again at this point in our discussion of sensitivity analysis because decisions are often made regarding resources by considering the marginal value of resources in conjunction with their sensitivity ranges.

Consider again the Excel sensitivity report for the Beaver Creek Pottery Company example shown in [Exhibit 3.18](#).

Exhibit 3.18.

[\[View full size image\]](#)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$10	Bowls =	24	0	40	26.67	15
\$B\$11	Mugs =	8	0	50	30	20

Shadow prices
(dual values)

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$E\$6	labor (hr/unit) Usage	40	16	40	40	10
\$E\$7	clay (lb/unit) Usage	120	6	120	40	60

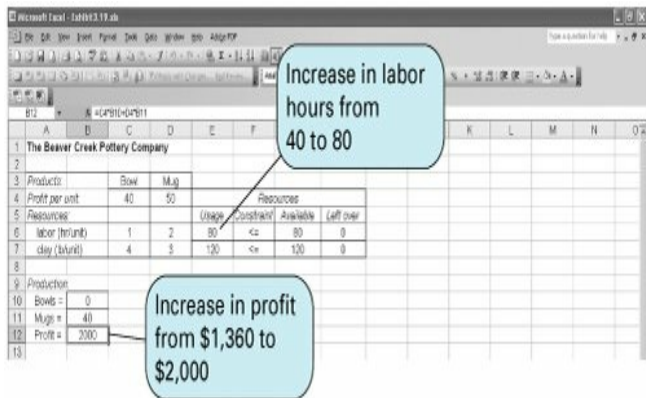
The shadow price (or marginal value) for labor is \$16 per hour, and the shadow price for clay is \$6 per pound. This means that for every additional hour of labor that can be obtained, profit will increase by \$16. If

the manager of the pottery company can secure more labor at \$16 per hour, how much more can be obtained before the optimal solution mix will change and the current shadow price is no longer valid? The answer is at the upper limit of the sensitivity range for the labor constraint value. A maximum of 80 hours of labor can be used before the optimal solution mix changes. Thus, the manager can secure 40 more hours, the allowable increase shown in the Excel sensitivity output for the labor constraint.

If 40 extra hours of labor can be obtained, what is its total value? The answer is (\$16/hr.) (40 hr.) = \$640. In other words, profit is increased by \$640 if 40 extra hours of labor can be obtained. This is shown in the Excel output in [Exhibit 3.19](#), where increasing the labor constraint from 40 to 80 hours has increased profit from \$1,360 to \$2,000, or \$640.

Exhibit 3.19.

[\[View full size image\]](#)



Looking back to [Figure 3.8\(a\)](#), this is the solution point at C'. Increasing the labor hours to more than 80 hours (the upper limit of the sensitivity range)

will not result in an additional increase in profit or a new solution; it will only result in slack hours of labor.

[Page 92]

Alternatively, what would be the effect on profit if one of the Native American artisans were sick one day during the week, and the available labor hours decreased from 40 to 32? Profit in this case would decrease by \$16 per hour, or a total amount of \$128. Thus, total profit would fall from \$1,360 to

\$1,232.

Similarly, if the pottery company could obtain only 100 pounds of clay instead of its normal weekly allotment of 120 pounds, what would be the effect on profit? Profit would decrease by \$6 per pound for 20 pounds, or a total of \$120. This would result in an overall reduction in profit from \$1,360 to \$1,240.

Thus, another piece of information that is provided by the sensitivity ranges for the constraint quantity values is

the range over which the shadow price remains valid. When q_i increases past the upper limit of the sensitivity range or decreases below the lower limit, the shadow price will change, specifically because slack (or surplus) will be created. Therefore, the sensitivity range for the constraint quantity value is the range over which the shadow price is valid.

*The sensitivity range
for a constraint
quantity value is also*

*the range over which
the shadow price is
valid.*

The shadow price of \$16 for 1 hour of labor is not necessarily what the manager would *pay* for an hour of labor. This depends on how the objective function is defined. In the Beaver Creek Pottery Company example, we are assuming that all the resources available, 40 hours of labor and 120 pounds

of clay, are already paid for. Even if the company does not use all the resources, it still must pay for them. These are *sunk* costs. Therefore, the individual profit values in the objective function for each product are not affected by how much of a resource is actually used; the total profit is independent of the resources used. In this case, the shadow prices are the maximum amounts the manager would pay for additional units of resource. The manager would pay up to \$16 for 1 extra hour

of labor and up to \$6 for an extra pound of clay.

Alternatively, if each hour of labor and each pound of clay were purchased separately, and thus were not sunk costs, profit would be a function of the cost of the resources. In this case, the shadow price would be the additional amount, over and above the original cost of the resource, that would be paid for one more unit of the resource.

Summary

This chapter has focused primarily on the computer solution of linear programming problems. This required us first to show how a linear programming model is put into standard form, with the addition of slack variables or the subtraction of surplus variables. Computer solution also enabled us to consider the topic of sensitivity analysis, the analysis of the effect of model

parameter changes on the solution of a linear programming model. In the next chapter we will provide some examples of more complex linear programming model formulations than the simple ones we have described so far.

References

See the references at the end of [Chapter 2](#).

Example Problem Solution

This example demonstrates the transformation of a linear programming model into standard form, sensitivity analysis, computer solution, and shadow prices.

Problem Statement

The Xecker Tool Company is considering bidding on a job for

two airplane wing parts. Each wing part must be processed through three manufacturing stages: stamping, drilling, and finishing for which the company has limited available hours. The linear programming model to determine how many of part 1 (x_1) and part 2 (x_2) the company should produce in order to maximize its profit is as follows:

$$\text{maximize } Z = \$650x_1 + 910x_2$$

subject to

$$4x_1 + 7.5x_2 \leq 105 \text{ (stamping, hr.)}$$

$$6.2x_1 + 4.9x_2 \leq 90 \text{ (drilling, hr.)}$$

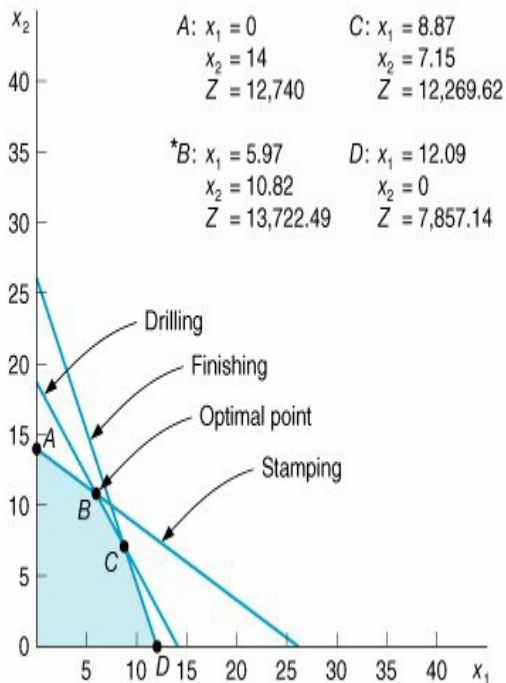
$$9.1x_1 + 4.1x_2 \leq 110 \text{ (finishing, hr.)}$$

$$x_1, x_2 \geq 0$$

- 1.** Solve the model graphically.
- 2.** Indicate how much slack resource is available at the optimal solution point.

- 3.** Determine the sensitivity ranges for the profit for wing part 1 and the stamping hours available.
- 4.** Solve this model by using Excel.

Solution



1.

2. The slack at point B , where $x_1 = 5.97$ and $x_2 = 10.82$, is computed as follows:

$$4(5.97) + 7.5(10.82) + s_1 = 105 \text{ (stamping, hr.)}$$

$$s_1 = 0 \text{ hr.}$$

$$6.2(5.97) + 4.9(10.82) + s_2 = 90 \text{ (drilling, hr.)}$$

$$s_2 = 0 \text{ hr.}$$

$$9.1(5.97) + 4.1(10.82) + s_3 = 110 \text{ (finishing, hr.)}$$

$$s_3 = 11.35 \text{ hr.}$$

[Page 94]

- 3.** The sensitivity range for the profit for part 1 is determined by observing the graph of the model and computing how much the slope of the objective function must increase to make the optimal point move from B to C . This is the upper limit of the range and is determined by computing the value of c_1 that will make the slope of

the objective function equal with the slope of the constraint line for drilling, $6.2x_1 + 4.9x_2 = 90$:

$$\frac{-c_1}{910} = \frac{-6.2}{4.9}$$
$$c_1 = 1,151.43$$

The lower limit is determined by computing the value of c_1 that will equate the slope of the objective function with the slope of the constraint line for stamping, $4x_1 + 7.5x_2 = 105$:

$$\frac{-c_1}{910} = \frac{-4}{7.5}$$

$$c_1 = 485.33$$

Summarizing,

$$485.33 \leq c_1 \leq 1,151.43$$

The upper limit of the range for stamping hours is determined by first computing the value for q_1 that would move the solution point from B to where the drilling constraint intersects with the x_2 axis, where $x_1 = 0$ and $x_2 = 18.37$:

$$4(0) + 7.5(18.37) = q_1$$
$$q_1 = 137.76$$

The lower limit of the sensitivity range occurs where the optimal point B moves to C , where $x_1 = 8.87$ and $x_2 = 7.15$:

$$4(8.87) + 7.5(7.15) = q_1$$
$$q_1 = 89.10$$

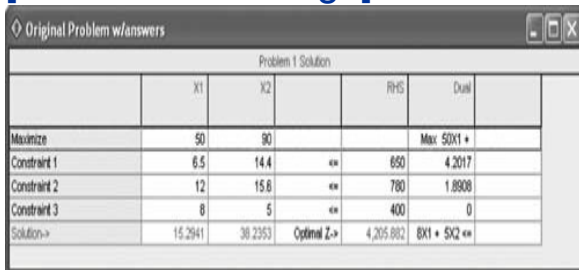
Summarizing, $89.10 \leq q_1 \leq 137.76$.

- 4.** The Excel spreadsheet solution to this example problem is as follows:

Problems

Given the following QM for Windows computer solution of a linear programming model, graph the problem and identify the solution point, including variable values and slack, from the computer output:

[\[View full size image\]](#)



The screenshot shows a window titled "Original Problem w/answers" with standard Windows window controls. Inside, a table titled "Problem 1 Solution" displays the results of a linear programming model. The table has columns for the objective function, decision variables (X1, X2), RHS, Dual values, and slack/surplus. The objective is to maximize 50X1 + 8X2. The optimal solution is found at X1 = 15.2941 and X2 = 38.2353, with an optimal Z value of 4,205.682. Constraint 1 is binding with a slack of 0, while Constraints 2 and 3 have positive slacks of 4,201.7 and 0 respectively.

	X1	X2		RHS	Dual	
Maximize	50	80			Max 50X1 +	
Constraint 1	6.5	14.4	≤	650	4.2017	
Constraint 2	12	15.6	≤	780	1.8908	
Constraint 3	8	5	≤	400	0	
Solution ->	15.2941	38.2353	Optimal Z ->	4,205.682	8X1 + 5X2 ≤	

Linear Programming Results		
Problem 1 Solution		
Variable	Status	Value
X1	Basic	15.2941
X2	Basic	38.2353
slack 1	NONBasic	0
slack 2	NONBasic	0
slack 3	Basic	86.4706
Optimal Value (Z)		4,205.882

- Explain the primary differences between a software package such as
- 2.** QM for Windows and Excel spreadsheets for solving linear programming problems.

Given the following Excel spreadsheet for a linear programming model and Solver window, indicate the formula for cell B13 and fill in the Solver window with the appropriate information to solve the problem:

[\[View full size image\]](#)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Problem #3														
2															
3	Products		1	2	3										
4	Profit per unit		115	90	130										
5	Resources					Available	Usage	Left over							
6	A		0.3	4.7	5.9	345	0.00	345.00							
7	B		10.1	11.0	14.0	710	0	710							
8															
9	Production														
10	Product 1 =														
11	Product 2 =														
12	Product 3 =														
13	Profit =	0.00													
14															

Solver Parameters [X]

Set Target Cell: [] [icon]

Equal To: ☒ Max ☐ Min ☐ Value of: [0]

By Changing Cells: [] [icon]

[Guess]

Subject to the Constraints: []

[Add]

[Change]

[Delete]

[Options]

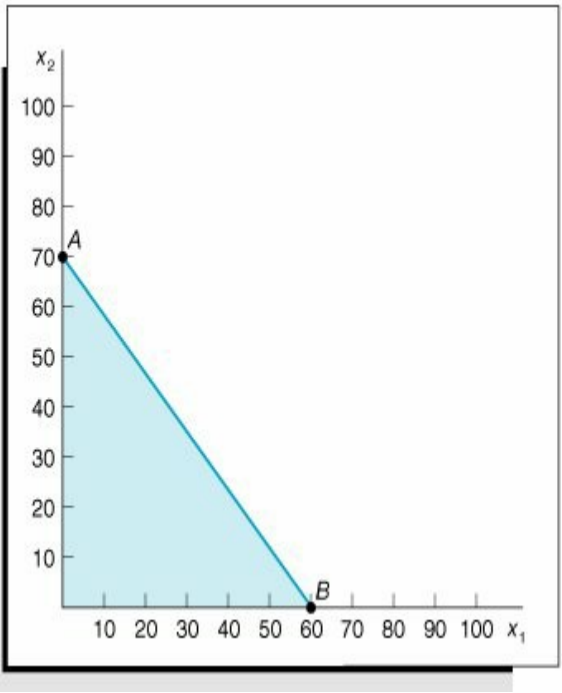
[Reset All]

[Help]

Given the following graph of a linear programming model with a single constraint and the objective function maximize $Z = 30x_1 + 50x_2$,

determine the optimal solution point:

4.



Determine the values by which c_1 and

c_2 must decrease or increase in order to change the current solution point to the other extreme point.

Southern Sporting Goods Company makes basketballs and footballs. Each product is produced from two resources rubber and leather. The resource requirements for each product and the total resources available are as follows:

Resource Requirements per Unit		
Product	Rubber (lb.)	Leather (ft. ²)

Basketball	3	4
------------	---	---

Football	2	5
----------	---	---

5.

Total resources available	500 lb.	800 ft. ²
---------------------------------	---------	----------------------

[Page 97]

Each basketball produced results in a profit of \$12, and each football earns \$16 in profit.

- 1.** Formulate a linear programming model to determine the number of basketballs and footballs to

produce in order to maximize profit.

- 2.** Transform this model into standard form.

Solve the model formulated in [Problem 5](#) for Southern Sporting Goods Company graphically.

- 1.** Identify the amount of unused resources (i.e., slack) at each of the graphical extreme points.
 - 2.** What would be the effect on the optimal solution if the profit for a basketball changed from \$12 to \$13? What would be the effect if the profit for a football changed from \$16 to \$15?
- 6.**
- 3.** What would be the effect on the optimal solution if 500 additional

pounds of rubber could be obtained? What would be the effect if 500 additional square feet of leather could be obtained?

For the linear programming model for Southern Sporting Goods Company, formulated in [Problem 5](#) and solved graphically in [Problem 6](#):

- 1.** Determine the sensitivity ranges for the objective function coefficients and constraint quantity values, using graphical analysis.
- 2.** Verify the sensitivity ranges determined in (a) by using the computer.
- 3.** Using the computer, determine the shadow prices for the resources and explain their

meaning.

A company produces two products, A and B, which have profits of \$9 and \$7, respectively. Each unit of product must be processed on two assembly lines, where the required production times are as follows.

Product	Hours/Unit	
	Line 1	Line 2
A	12	4
8. B	4	8

Total hours

60

40

- 1.** Formulate a linear programming model to determine the optimal product mix that will maximize profit.
- 2.** Transform this model into standard form.

Solve [Problem 8](#) graphically.

- 1.** Identify the amount of unused resources (i.e., slack) at each of the graphical extreme points.
- 2.** What would be the effect on the optimal solution if the production time on line 1 were reduced to

9.

40 hours?

- 3.** What would be the effect on the optimal solution if the profit for product B were increased from \$7 to \$15? to \$20?

For the linear programming model formulated in [Problem 8](#) and solved graphically in [Problem 9](#):

- 1.** Determine the sensitivity ranges for the objective function coefficients, using graphical analysis.
 - 2.** Verify the sensitivity ranges determined in (a) by using the computer.
 - 3.** Using the computer, determine the shadow prices for additional hours of production time on line
- 10.**

1 and line 2 and indicate whether the company would prefer additional line 1 or line 2 hours.

[Page 98]

Irwin Textile Mills produces two types of cotton cloth: denim and corduroy. Corduroy is a heavier grade of cotton cloth and, as such, requires 7.5 pounds of raw cotton per yard, whereas denim requires 5 pounds of raw cotton per yard. A yard of corduroy requires 3.2 hours of processing time; a yard of denim requires 3.0 hours. Although the demand for denim is practically unlimited, the maximum demand for corduroy is 510 yards per month. The manufacturer has 6,500 pounds of cotton and 3,000 hours of processing time available each month. The

11.

manufacturer makes a profit of \$2.25 per yard of denim and \$3.10 per yard of corduroy. The manufacturer wants to know how many yards of each type of cloth to produce to maximize profit.

- 1.** Formulate a linear programming model for this problem.
- 2.** Transform this model into standard form.

Solve the model formulated in [Problem 11](#) for Irwin Textile Mills graphically.

- 1.** How much extra cotton and processing time are left over at the optimal solution? Is the demand for corduroy met?
- 2.** What is the effect on the optimal

- 12.** solution if the profit per yard of denim is increased from \$2.25 to \$3.00? What is the effect if the profit per yard of corduroy is increased from \$3.10 to \$4.00?
- 3.** What would be the effect on the optimal solution if Irwin Mills could obtain only 6,000 pounds of cotton per month?

Solve the linear programming model formulated in [Problem 11](#) for Irwin Mills by using the computer.

- 1.** If Irwin Mills can obtain additional cotton or processing time, but not both, which should it select? How much? Explain your answer.

13.

- 2.** Identify the sensitivity ranges for the objective function coefficients and for the constraint quantity

values. Then explain the sensitivity range for the demand for corduroy.

United Aluminum Company of Cincinnati produces three grades (high, medium, and low) of aluminum at two mills. Each mill has a different production capacity (in tons per day) for each grade, as follows:

Mill		
Aluminum Grade	1	2
High	6	2

14.	Medium	2	2
	Low	4	10

The company has contracted with a manufacturing firm to supply at least 12 tons of high-grade aluminum, 8 tons of medium-grade aluminum, and 5 tons of low-grade aluminum. It costs United \$6,000 per day to operate mill 1 and \$7,000 per day to operate mill 2. The company wants to know the number of days to operate each mill in order to meet the contract at the minimum cost.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 14](#) for United Aluminum Company graphically.

1. How much extra (i.e., surplus) high-, medium-, and low-grade aluminum does the company produce at the optimal solution?
2. What would be the effect on the optimal solution if the cost of operating mill 1 increased from \$6,000 to \$7,500 per day?

15.

[Page 99]

3. What would be the effect on the optimal solution if the company could supply only 10 tons of high-grade aluminum?

Solve the linear programming model

formulated in [Problem 14](#) for United Aluminum Company by using the computer.

1. Identify and explain the shadow prices for each of the aluminum grade contract requirements.
- 16.**
2. Identify the sensitivity ranges for the objective function coefficients and the constraint quantity values.
 3. Would the solution values change if the contract requirements for high-grade aluminum were increased from 12 tons to 20 tons? If yes, what would the new solution values be?

The Bradley family owns 410 acres of farmland in North Carolina on which they grow corn and tobacco. Each

acre of corn costs \$105 to plant, cultivate, and harvest; each acre of tobacco costs \$210. The Bradleys have a budget of \$52,500 for next year. The government limits the

17. number of acres of tobacco that can be planted to 100. The profit from each acre of corn is \$300; the profit from each acre of tobacco is \$520. The Bradleys want to know how many acres of each crop to plant in order to maximize their profit.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 17](#) for the Bradley family farm graphically.

1. How many acres of farmland will not be cultivated at the optimal

solution? Do the Bradleys use the entire 100-acre tobacco allotment?

18.

- 2.** What would the profit for corn have to be for the Bradleys to plant only corn?
- 3.** If the Bradleys can obtain an additional 100 acres of land, will the number of acres of corn and tobacco they plan to grow change?
- 4.** If the Bradleys decide not to cultivate a 50-acre section as part of a crop recovery program, how will it affect their crop plans?

Solve the linear programming model formulated in [Problem 17](#) for the Bradley farm by using the computer.

19.

- 1.** The Bradleys have an opportunity to lease some extra land from a neighbor. The neighbor is offering the land to them for \$110 per acre. Should the Bradleys lease the land at that price? What is the maximum price the Bradleys should pay their neighbor for the land, and how much land should they lease at that price?
- 2.** The Bradleys are considering taking out a loan to increase their budget. For each dollar they borrow, how much additional profit would they make? If they borrowed an additional \$1,000, would the number of acres of corn and tobacco they plant change?

The manager of a Burger Doodle

franchise wants to determine how many sausage biscuits and ham biscuits to prepare each morning for breakfast customers. The two types of biscuits require the following resources:

Biscuit	Labor (hr.)	Sausage (lb.)	Ham (lb.)	Flour (lb.)
Sausage	0.010	0.10		0.04
Ham	0.024		0.15	0.04

20.

The franchise has 6 hours of labor available each morning. The manager has a contract with a local grocer for

30 pounds of sausage and 30 pounds of ham each morning. The manager also purchases 16 pounds of flour. The profit for a sausage biscuit is \$0.60; the profit for a ham biscuit is \$0.50. The manager wants to know the number of each type of biscuit to prepare each morning in order to maximize profit.

[Page 100]

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 20](#) for the Burger Doodle restaurant graphically.

- 1.** How much extra sausage and ham is left over at the optimal solution point? Is there any idle

- 21.** labor time?
- 2.** What would the solution be if the profit for a ham biscuit were increased from \$0.50 to \$0.60?
 - 3.** What would be the effect on the optimal solution if the manager could obtain 2 more pounds of flour?

Solve the linear programming model developed in [Problem 20](#) for the Burger Doodle restaurant by using the computer.

- 1.** Identify and explain the shadow prices for each of the resource constraints.
- 22.**
- 2.** Which of the resources constraints profit the most?

- 3.** Identify the sensitivity ranges for the profit of a sausage biscuit and the amount of sausage available. Explain these sensitivity ranges.

Rucklehouse Public Relations has been contracted to do a survey following an election primary in New Hampshire. The firm must assign interviewers to conduct the survey by telephone or in person. One person can conduct 80 telephone interviews or 40 personal interviews in a single day. The following criteria have been established by the firm to ensure a representative survey:

- At least 3,000 interviews must be conducted.

23.

- At least 1,000 interviews must

be by telephone.

- At least 800 interviews must be personal.

An interviewer conducts only one type of interview each day. The cost is \$50 per day for a telephone interviewer and \$70 per day for a personal interviewer. The firm wants to know the minimum number of interviewers to hire in order to minimize the total cost of the survey.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 23](#) for Rucklehouse Public Relations graphically.

- 1.** Determine the sensitivity ranges for the daily cost of a telephone interviewer and the number of personal interviews required.
- 24.**
- 2.** Does the firm conduct any more telephone and personal interviews than are required, and if so, how many more?
 - 3.** What would be the effect on the optimal solution if the firm were required by the client to increase the number of personal interviews conducted from 800 to a total of 1,200?

Solve the linear programming model formulated in [Problem 23](#) for Rucklehouse Public Relations by using the computer.

- 1.** If the firm could reduce the

minimum interview requirement for either telephone or personal interviews, which should the firm select? How much would a reduction of one interview in the requirement you selected reduce total cost? Solve the model again, using the computer, with the reduction of this one interview in the constraint requirement to verify your answer.

25.

- 2.** Identify the sensitivity ranges for the cost of a personal interview and the number of total interviews required.

The Bluegrass Distillery produces custom-blended whiskey. A particular blend consists of rye and bourbon whiskey. The company has received

an order for a minimum of 400 gallons of the custom blend. The customer specified that the order must contain at least 40% rye and not more than 250 gallons of bourbon. The customer also specified that the blend should be mixed in the ratio of two parts rye to one part bourbon. The distillery can produce 500 gallons per week, regardless of the blend. The

- 26.** production manager wants to complete the order in 1 week. The blend is sold for \$5 per gallon.

[Page 101]

The distillery company's cost per gallon is \$2 for rye and \$1 for bourbon. The company wants to determine the blend mix that will meet customer requirements and maximize profits.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 26](#) for the Bluegrass Distillery graphically.

27.

- 1.** Indicate the slack and surplus available at the optimal solution point and explain their meanings.
- 2.** What increase in the objective function coefficients in this model would change the optimal solution point? Explain your answer.

Solve the linear programming model formulated in [Problem 26](#) for the Bluegrass Distillery by using the computer.

28.

- 1.** Identify the sensitivity ranges for the objective function coefficients and explain what the upper and lower limits are.
- 2.** How much would it be worth to the distillery to obtain additional production capacity?
- 3.** If the customer decided to change the blend requirement for its custom-made whiskey to a mix of three parts rye to one part bourbon, how would this change the optimal solution?

Alexis Harrington received an inheritance of \$95,000, and she is considering two speculative investments: the purchase of land and the purchase of cattle. Each investment would be for 1 year. Under the present (normal) economic

conditions, each dollar invested in land will return the principal plus 20% of the principal; each dollar invested in cattle will return the principal plus 30%.

29. However, both investments are relatively risky. If economic conditions were to deteriorate, there is an 18% chance she would lose everything she invested in land and a 30% chance she would lose everything she invested in cattle. Alexis does not want to lose more than \$20,000 (on average). She wants to know how much to invest in each alternative to maximize the cash value of the investments at the end of 1 year.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 29](#) for Alexis

Harrington graphically.

30.

- 1.** How much would the return for cattle have to increase in order for Alexis to invest only in cattle?
- 2.** Should all of Alexis's inheritance be invested according to the optimal solution?
- 3.** How much "profit" would the optimal solution earn Alexis over and above her investment?

Solve the linear programming model formulated in [Problem 29](#) for Alexis Harrington by using the computer.

31.

- 1.** If Alexis decided to invest some of her own savings along with the money from her inheritance, what return would she realize for each dollar of her own money

that she invested? How much of her own savings could she invest before this return would change?

2. If the risk of losing the investment in land increased to 30%, how would this change the optimal investment mix?

[Page 102]

Transform the following linear programming model into standard form and solve by using the computer:

32.

maximize $Z = 140x_1 + 205x_2 + 190x_3$

subject to

$$10x_1 + 15x_2 + 8x_3 \leq 610$$

$$\frac{x_1}{x_2} \leq 3$$

$$x_1 \geq 0.4 (x_1 + x_2 + x_3)$$

$$x_2 \geq x_3$$

$$x_1, x_2, x_3 \geq 0$$

Chemco Corporation produces a chemical mixture for a specific customer in 1,000-pound batches. The mixture contains three ingredients: zinc, mercury, and potassium. The mixture must conform to formula specifications that are supplied by the customer. The company wants to know the amount of each ingredient it needs to put in the mixture that will meet all the

requirements of the mix and minimize total cost.

The customer has supplied the following formula specifications for each batch of mixture:

- The mixture must contain at least 200 pounds of mercury.
- The mixture must contain at least 300 pounds of zinc.
- The mixture must contain at least 100 pounds of potassium.
- The ratio of potassium to the other two ingredients cannot exceed 1 to 4.

The cost per pound of mercury is \$400; the cost per pound of zinc,

\$180; and the cost per pound of potassium, \$90.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve the model formulated in (a) by using the computer.

The following linear programming model formulation is used for the production of four different products, with two different manufacturing processes and two different material requirements:

maximize $Z = \$50x_1 + 58x_2 + 46x_3 + 62x_4$

subject to

34. $4x_1 + 3.5x_2 + 4.6x_3 + 3.9x_4 \leq 600$ hr. (process 1)

$2.1x_1 + 2.6x_2 + 3.5x_3 + 1.9x_4 \leq 500$ hr. (process 2)

$15x_1 + 23x_2 + 18x_3 + 25x_4 \leq 3,600$ lb. (material A)

$8x_1 + 12.6x_2 + 9.7x_3 + 10.5x_4 \leq 1,700$ lb. (material B)

$$\frac{x_1 + x_2}{x_1 + x_2 + x_3 + x_4} \geq .60$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- 1.** Solve this problem by using the computer.
- 2.** Identify the sensitivity ranges for the objective function coefficients and the constraint quantity values.
- 3.** Which is the most valuable resource to the firm?

- 4.** One of the four products is not produced in the optimal solution. How much would the profit for this product have to be for it to be produced?

Island Publishing Company publishes two types of magazines on a monthly basis: a restaurant and entertainment guide and a real estate guide. The company distributes the magazines free to businesses, hotels, and stores on Hilton Head Island in South Carolina. The company's profits come exclusively from the paid advertising in the magazines. Each of the restaurant and entertainment guides distributed generates \$0.50 per magazine in advertising revenue, whereas the real estate guide generates \$0.75 per magazine. The real estate magazine is a more sophisticated publication that

includes color photos, and accordingly it costs \$0.25 per magazine to print, compared with only \$0.17 for the restaurant and entertainment guide. The publishing company has a printing budget of \$4,000 per month. There is enough rack space to distribute at most 18,000 magazines each month. In order to entice businesses to place advertisements, Island Publishing promises to distribute at least 8,000 copies of each magazine. The company wants to determine the number of copies of each magazine it should print each month in order to maximize advertising revenue.

[Page 103]

Formulate a linear programming model for this problem.

Solve the linear programming model formulation in [Problem 35](#) for Island Publishing Company graphically.

1. Determine the sensitivity range for the advertising revenue generated by the real estate guide.

- 36.**
2. Does the company spend all of its printing budget? If not, how much slack is left over?
 3. What would be the effect on the optimal solution if the local real estate agents insisted that 12,000 copies of the real estate guide be distributed instead of the current 8,000 copies, or they would withdraw their advertising?

Solve the linear programming model formulated in [Problem 35](#) for Island

Publishing Company by using the computer.

37.

- 1.** How much would it be worth to Island Publishing Company to obtain enough additional rack space to distribute 18,500 copies instead of the current 18,000 copies? 20,000 copies?
- 2.** How much would it be worth to Island Publishing to reduce the requirement to distribute the entertainment guide from 8,000 to 7,000 copies?

Mega-Mart, a discount store chain, is to build a new store in Rock Springs. The parcel of land the company has purchased is large enough to accommodate a store with 140,000 square feet of floor space. Based on marketing and demographic surveys

of the area and historical data from its other stores, Mega-Mart estimates its annual profit per square foot for each of the store's departments to be as shown in the following table:

Department	Profit per ft. ²
Men's clothing	\$4.25
Women's clothing	5.10
Children's clothing	4.50
Toys	5.20

38. Housewares	4.10
Electronics	4.90
Auto supplies	3.80

Each department must have at least 15,000 ft.² of floor space, and no department can have more than 20% of the total retail floor space. Men's, women's, and children's clothing plus housewares keep all their stock on the retail floor; however, toys, electronics, and auto supplies keep some items (such as bicycles, televisions, and tires) in inventory. Thus, 10% of the total retail floor space devoted to these three departments must be set

aside outside the retail area for stocking inventory. Mega-Mart wants to know the floor space that should be devoted to each department in order to maximize profit.

1. Formulate a linear programming model for this problem.
2. Solve this model by using the computer.

[Page 104]

1. In [Problem 38](#) Mega-Mart is considering purchasing a parcel of land adjacent to the current site on which it plans to build its store. The cost of the parcel is \$190,000, and it would enable Mega-Mart to increase the size of its store to 160,000 ft.²

39.

Discuss whether Mega-Mart should purchase the land and increase the planned size of the store.

2. Suppose that the profit per ft.² will decline in all departments by 20% if the store size increases to 160,000 ft.² (If the stock does not turn over as fast, increasing inventory costs will reduce profit.) How might this affect Mega-Mart's decision in (a)?

The Food Max grocery store sells three brands of milk in half-gallon cartons: its own brand, a local dairy brand, and a national brand. The profit from its own brand is \$0.97 per carton, the profit from the local dairy brand is \$0.83 per carton, and the

profit from the national brand is \$0.69 per carton. The total refrigerated shelf space allotted to half-gallon cartons of milk is 36 square feet per week. A half-gallon carton takes up 16 square inches of shelf space. The store manager knows that each week Food Max always sells more of the national brand than of the local dairy brand and its own brand combined and at least three times as much of the national brand as its own brand. In addition, the local dairy can supply only 10 dozen cartons per week. The store manager wants to know how many half-gallon cartons of each brand to stock each week in order to maximize profit.

1. Formulate a linear programming model for this problem.
2. Solve this model by using the computer.

1. If Food Max in [Problem 40](#) could increase its shelf space for half-gallon cartons of milk, how much would profit increase per carton?
2. If Food Max could get the local dairy to increase the amount of milk it could supply each week, would it increase profit?

- 41.**
3. Food Max is considering discounting its own brand in order to increase sales. If it were to do so, it would decrease the profit margin for its own brand to \$0.86 per carton, but it would cut the demand for the national brand relative to its own brand in half. Discuss whether the store should implement the price discount.

John Hoke owns Hoke's Spokes, a bicycle shop. Most of John's bicycle sales are customer orders; however, he also stocks bicycles for walk-in customers. He stocks three types of bicycles: road-racing, cross-country, and mountain. A road-racing bike costs \$1,200, a cross-country bike costs \$1,700, and a mountain bike costs \$900. He sells road-racing bikes for \$1,800, cross-country bikes for \$2,100, and mountain bikes for \$1,200. He has \$12,000 available this month to purchase bikes. Each bike

- 42.** must be assembled; a road-racing bike requires 8 hours to assemble, a cross-country bike requires 12 hours, and a mountain bike requires 16 hours. He estimates that he and his employees have 120 hours available to assemble bikes. He has enough space in his store to order 20 bikes

this month. Based on past sales, John wants to stock at least twice as many mountain bikes as the other two combined because mountain bikes sell better.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 42](#) for Hoke's Spokes by using the computer.

1. Should John Hoke try to increase his budget for purchasing bikes, increase space to stock bikes, or increase labor hours to assemble bikes? Why?

- 43.** 2. If John were to hire an additional worker for 30 hours at \$10 per hour, how much additional profit would he make, if any?

- 3.** If John were to purchase a cheaper cross-country bike for \$1,200 and sell it for \$1,900, would this affect the original solution?

Metro Food Services Company delivers fresh sandwiches each morning to vending machines throughout the city. The company makes three kinds of sandwiches: ham and cheese, bologna, and chicken salad. A ham and cheese sandwich requires a worker 0.45 minutes to assemble, a bologna sandwich requires 0.41 minutes, and a chicken salad sandwich requires 0.50 minutes to make. The company has 960 available minutes each night for sandwich assembly. Vending machine capacity is available for 2,000

sandwiches each day. The profit for a ham and cheese sandwich is \$0.35, the profit for a bologna sandwich is \$0.42, and the profit for a chicken salad sandwich is \$0.37. The company knows from past sales records that its customers buy as many or more of the ham and cheese sandwiches than the other two sandwiches combined, but customers need a variety of sandwiches available, so Metro stocks at least 200 of each. Metro management wants to know how many of each sandwich it should stock to maximize profit.

[Page 105]

Formulate a linear programming model for this problem.

Solve the linear programming problem

formulated in [Problem 44](#) for Metro Food Services Company by using the computer.

- 1.** If Metro Food Services could hire another worker and increase its available assembly time by 480 minutes or increase its vending machine capacity by 100 sandwiches, which should it do? Why? How much additional profit would your decision result in?

45.

- 2.** What would the effect be on the optimal solution if the requirement that at least 200 sandwiches of each kind be stocked were eliminated? Compare the profit between the optimal solution and this solution. Which solution would you recommend?
- 3.** What would the effect be on the

optimal solution if the profit for a ham and cheese sandwich were increased to \$0.40? to \$0.45?

Mountain Laurel Vineyards produces three kinds of wine: Mountain Blanc, Mountain Red, and Mountain Blush. The company has 17 tons of grapes available to produce wine this season. A cask of Blanc requires 0.21 tons of grapes, a cask of Red requires 0.24 tons, and a cask of Blush requires 0.18 tons. The vineyard has enough storage space in its aging room to store 80 casks of wine.

- 46.** The vineyard has 2,500 hours of production capacity, and it requires 12 hours to produce a cask of Blanc, 14.5 hours to produce a cask of Red, and 16 hours to produce a cask of Blush. From past sales the vineyard knows that demand for the Blush will

be no more than half of the sales of the other two wines combined. The profit for a cask of Blanc is \$7,500, the profit for a cask of Red is \$8,200, and the profit for a cask of Blush is \$10,500.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 46](#) for Mountain Laurel Vineyards by using the computer.

- 1.** If the vineyard were to determine that the profit from Red was \$7,600 instead of \$8,200, how would that affect the optimal solution?
- 2.** If the vineyard could secure one

47.

additional unit of any of the resources used in the production of wine, which one should it select?

- 3.** If the vineyard could obtain 0.5 more tons of grapes, 500 more hours of production capacity, or enough storage capacity to store 4 more casks of wine, which should it choose?
- 4.** All three wines are produced in the optimal solution. How little would the profit for Blanc have to be for it to no longer be produced?

Exeter Mines produces iron ore at four different mines; however, the ores extracted at each mine are different in their iron content. Mine 1 produces magnetite ore, which has a 70% iron

content; mine 2 produces limonite ore, which has a 60% iron content; mine 3 produces pyrite ore, which has a 50% iron content; and mine 4 produces taconite ore, which has only a 30% iron content. Exeter has three customers that produce steel: Armco, Best, and Corcom. Armco needs 400 tons of pure (100%) iron, Best requires 250 tons of pure iron, and Corcom requires 290 tons. It costs \$37 to extract and process 1 ton of

- 48.** magnetite ore at mine 1, \$46 to produce 1 ton of limonite ore at mine 2, \$50 per ton of pyrite ore at mine 3, and \$42 per ton of taconite ore at mine 4. Exeter can extract 350 tons of ore at mine 1; 530 tons at mine 2; 610 tons at mine 3; and 490 tons at mine 4. The company wants to know how much ore to produce at each mine in order to minimize cost and meet its customers' demand for pure

(100%) iron.

[Page 106]

Formulate a linear programming model for this problem

Solve the linear programming problem formulated in [Problem 48](#) for Exeter Mines by using the computer.

- 1.** Do any of the mines have slack capacity? If yes, which one(s)?
- 2.** If Exeter Mines could increase production capacity at any one of its mines, which should it be? Why?

- 49.** **3.** If Exeter were to decide to increase capacity at the mine identified in (b), how much could

it increase capacity before the optimal solution point (i.e., the optimal set of variables) would change?

4. If Exeter were to determine that it could increase production capacity at mine 1 from 350 tons to 500 tons, at an increase in production costs to \$43 per ton, should it do so?

Given the following linear programming model:

50.

minimize $Z = 8.2x_1 + 7.0x_2 + 6.5x_3 + 9.0x_4$
subject to

$$6x_1 + 2x_2 + 5x_3 + 7x_4 \geq 820$$

$$\frac{x_1}{x_1 + x_2 + x_3 + x_4} \geq .3$$

$$\frac{x_2 + x_3}{x_1 + x_4} \leq .2$$

$$x_3 \geq x_1 + x_4$$

$$x_1, x_2, x_3, x_4 \geq 0$$

transform the model into standard form and solve by using the computer.

Marie McCoy has committed to the local PTA to make some items for a bake sale on Saturday. She has decided to make some combination of chocolate cakes, loaves of white bread, custard pies, and sugar cookies. Thursday evening she went to the store and purchased 20 pounds

of flour, 10 pounds of sugar, and 3 dozen eggs, which are the three main ingredients in all the baked goods she is thinking about making. The following table shows how much of each of the main ingredients is required for each baked good:

Ingredient				
	Flour (cups)	Sugar (cups)	Eggs	Baking Time (min.)
Cake	2.5	2	2	45
Bread	9	0.25	0	35

Pie	1.3	1	5	50
Cookies	2.5	1	2	16

There are 18.5 cups in a 5 pound bag of flour and 12 cups in a 5 pound bag of sugar. Marie plans to get up and start baking on Friday morning after her kids leave for school and finish before they return after soccer practice (8 hours). She knows that the PTA will sell a chocolate cake for \$12, a loaf of bread for \$8, a custard pie for \$10, and a batch of cookies for \$6. Marie wants to decide how many of each type of baked good she should make in order for the PTA to make the most money possible.

Formulate a linear programming model for this problem.

Solve the linear programming model formulated in [Problem 51](#) for Marie McCoy.

- 1.** Are any of the ingredients left over?
- 2.** If Marie could get more of any ingredient, which should it be? Why?
- 52. 3.** If Marie could get 6 more eggs, 20 more cups of flour, or 30 more minutes of oven time, which should she choose? Why?
- 4.** The solution values for this problem should logically be

integers. If the solution values are not integers, discuss how Marie should decide how many of each item to bake. How do total sales for this integer solution compare with those in the original, non-integer solution?

Case Problem

MOSSAIC TILES, LTD.

Gilbert Moss and Angela Pasaic spent several summers during their college years working at archaeological sites in the Southwest. While at those digs, they learned how to make ceramic tiles from local artisans. After college they made use of their college experiences to start a tile manufacturing firm called

Mossaic Tiles, Ltd. They opened their plant in New Mexico, where they would have convenient access to a special clay they intend to use to make a clay derivative for their tiles. Their manufacturing operation consists of a few relatively simple but precarious steps, including molding the tiles, baking, and glazing.

Gilbert and Angela plan to produce two basic types of tile for use in home bathrooms, kitchens, sunrooms, and laundry rooms. The two types of tile are a larger, single-

colored tile and a smaller, patterned tile. In the manufacturing process, the color or pattern is added before a tile is glazed. Either a single color is sprayed over the top of a baked set of tiles or a stenciled pattern is sprayed on the top of a baked set of tiles.

The tiles are produced in batches of 100. The first step is to pour the clay derivative into specially constructed molds. It takes 18 minutes to mold a batch of 100 larger tiles and 15 minutes to prepare a mold for a batch of 100 smaller tiles. The

company has 60 hours available each week for molding. After the tiles are molded, they are baked in a kiln: 0.27 hour for a batch of 100 larger tiles and 0.58 hour for a batch of 100 smaller tiles. The company has 105 hours available each week for baking. After baking, the tiles are either colored or patterned and glazed. This process takes 0.16 hour for a batch of 100 larger tiles and 0.20 hour for a batch of 100 smaller tiles. Forty hours are available each week for the glazing process. Each

batch of 100 large tiles requires 32.8 pounds of the clay derivative to produce, whereas each batch of smaller tiles requires 20 pounds. The company has 6,000 pounds of the clay derivative available each week.

Mosaic Tiles earns a profit of \$190 for each batch of 100 of the larger tiles and \$240 for each batch of 100 smaller patterned tiles. Angela and Gilbert want to know how many batches of each type of tile to produce each week to maximize profit. In addition,

they have some questions about resource usage they would like answered.

- 1.** Formulate a linear programming model for Mossaic Tiles, Ltd., and determine the mix of tiles it should manufacture each week.
- 2.** Transform the model into standard form.
- 3.** Solve the linear programming model graphically.

- 4.** Determine the resources left over and not used at the optimal solution point.
- 5.** Determine the sensitivity ranges for the objective function coefficients and constraint quantity values by using the graphical solution of the model.
- 6.** For artistic reasons, Gilbert and Angela like to produce the smaller, patterned tiles better. They also believe that in the long run, the smaller tiles will be a more successful product. What

must the profit be for the smaller tiles in order for the company to produce only the smaller tiles?

- 7.** Solve the linear programming model by using the computer and verify the sensitivity ranges computed in (E).
- 8.** Mossaic believes it may be able to reduce the time required for molding to 16 minutes for a batch of larger tiles and 12 minutes for a batch of the smaller tiles. How will this affect

the solution?

- 9.** The company that provides Mossaic with clay has indicated that it can deliver an additional 100 pounds each week. Should Mossaic agree to this offer?
- 10.** Mossaic is considering adding capacity to one of its kilns to provide 20 additional glazing hours per week, at a cost of \$90,000. Should it make the investment?
- 11.** The kiln for glazing had to

be shut down for 3 hours, reducing the available kiln hours from 40 to 37. What effect will this have on the solution?

Case Problem

"THE POSSIBILITY"

RESTAURANT CONTINUED

In "The Possibility" Restaurant case problem in [Chapter 2](#), Angela Fox and Zooey Caulfield opened a French restaurant called "The Possibility." Initially, Angela and Zooey could not offer a full, varied menu, so their chef, Pierre, prepared two full-course dinners with beef and fish each evening. In the

case problem, Angela and Zooey wanted to develop a linear programming model to help determine the number of beef and fish meals they should prepare each night. Solve Zooey and Angela's linear programming model by using the computer.

- 1.** Angela and Zooey are considering investing in some advertising to increase the maximum number of meals they serve. They estimate that if they spend \$30 per day on a newspaper ad, it will

increase the maximum number of meals they serve per day from 60 to 70. Should they make the investment?

- 2.** Zooey and Angela are concerned about the reliability of some of their kitchen staff. They estimate that on some evenings they could have a staff reduction of as much as 5 hours. How would this affect their profit level?
- 3.** The final question they would like to explore is

raising the price of the fish dinner. Angela believes the price for a fish dinner is a little low and that it could be closer to the price of a beef dinner without affecting customer demand. However, Zooey has noted that Pierre has already made plans based on the number of dinners recommended by the linear programming solution. Angela has suggested a price increase that will increase profit for the fish dinner to \$14. Would this

be acceptable to Pierre, and how much additional profit would be realized?

Case Problem

JULIA'S FOOD BOOTH

Julia Robertson is a senior at Tech, and she's investigating different ways to finance her final year at school. She is considering leasing a food booth outside the Tech stadium at home football games. Tech sells out every home game, and Julia knows, from attending the games herself, that everyone eats a lot of

food. She has to pay \$1,000 per game for a booth, and the booths are not very large.

Vendors can sell either food or drinks on Tech property, but not both. Only the Tech athletic department concession stands can sell both inside the stadium. She thinks slices of cheese pizza, hot dogs, and barbecue sandwiches are the most popular food items among fans and so these are the items she would sell.

Most food items are sold during the hour before the game starts and during half time;

thus it will not be possible for Julia to prepare the food while she is selling it. She must prepare the food ahead of time and then store it in a warming oven. For \$600 she can lease a warming oven for the six-game home season. The oven has 16 shelves, and each shelf is 3 feet by 4 feet. She plans to fill the oven with the three food items before the game and then again before half time.

Julia has negotiated with a local pizza delivery company to deliver 14-inch cheese pizzas twice each game 2 hours before

the game and right after the opening kickoff. Each pizza will cost her \$6 and will include 8 slices. She estimates it will cost her \$0.45 for each hot dog and \$0.90 for each barbecue sandwich if she makes the barbecue herself the night before. She measured a hot dog and found it takes up about 16 in.² of space, whereas a barbecue sandwich takes up about 25 in.² She plans to sell a slice of pizza and a hot dog for \$1.50 apiece and a barbecue sandwich for \$2.25. She has \$1,500 in cash

available to purchase and prepare the food items for the first home game; for the remaining five games she will purchase her ingredients with money she has made from the previous game.

Julia has talked to some students and vendors who have sold food at previous football games at Tech as well as at other universities. From this she has discovered that she can expect to sell at least as many slices of pizza as hot dogs and barbecue sandwiches combined. She also anticipates that she

will probably sell at least twice as many hot dogs as barbecue sandwiches. She believes that she will sell everything she can stock and develop a customer base for the season if she follows these general guidelines for demand.

If Julia clears at least \$1,000 in profit for each game after paying all her expenses, she believes it will be worth leasing the booth.

- 1.** Formulate and solve a linear programming model for Julia that will help you

advise her if she should lease the booth.

- 2.** If Julia were to borrow some more money from a friend before the first game to purchase more ingredients, could she increase her profit? If so, how much should she borrow and how much additional profit would she make? What factor constrains her from borrowing even more money than this amount (indicated in your answer to the previous question)?

- 3.** When Julia looked at the solution in (A), she realized that it would be physically difficult for her to prepare all the hot dogs and barbecue sandwiches indicated in this solution. She believes she can hire a friend of hers to help her for \$100 per game. Based on the results in (A) and (B), is this something you think she could reasonably do and should do?

4. Julia seems to be basing her analysis on the assumption that everything will go as she plans. What are some of the uncertain factors in the model that could go wrong and adversely affect Julia's analysis? Given these uncertainties and the results in (A), (B), and (C), what do you recommend that Julia do?

Chapter 4. Linear Programming: Modeling Examples

[Page 111]

In [Chapters 2](#) and [3](#), two basic linear programming models, one for a maximization problem and one for a minimization problem, were used to demonstrate model formulation, graphical solution, computer solution, and sensitivity analysis. Most of

these models were very straightforward, consisting of only two decision variables and two constraints. They were necessarily simple models so that the linear programming topics being introduced could be easily understood.

In this chapter, more complex examples of model formulation are presented. These examples have been selected to illustrate some of the more popular application areas of linear programming. They also provide guidelines for model formulation for a variety of

problems and computer solutions with Excel and QM for Windows.

You will notice as you go through each example that the model formulation is presented in a systematic format. First, decision variables are identified, then the objective function is formulated, and finally the model constraints are developed. Model formulation can be difficult and complicated, and it is usually beneficial to follow this set of steps in which you identify a specific model component at

each step instead of trying to "see" the whole formulation after the first reading.

A Product Mix Example

Quick-Screen is a clothing manufacturing company that specializes in producing commemorative shirts immediately following major sporting events such as the World Series, Super Bowl, and Final Four. The company has been contracted to produce a standard set of shirts for the winning team, either State University or Tech, following a college football bowl game on

New Year's Day. The items produced include two sweatshirts, one with silk-screen printing on the front and one with print on both sides, and two T-shirts of the same configuration. The company has to complete all production within 72 hours after the game, at which time a trailer truck will pick up the shirts. The company will work around the clock. The truck has enough capacity to accommodate 1,200 standard-size boxes. A standard-size box holds 12 T-shirts, and a box of

12 sweatshirts is three times the size of a standard box. The company has budgeted \$25,000 for the production run. It has 500 dozen blank sweatshirts and T-shirts each in stock, ready for production. This scenario is illustrated in [Figure 4.1](#).

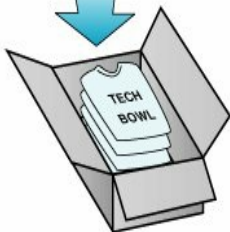
Figure 4.1. Quick-Screen Shirts

(This item is displayed on page 112 in the print version)

T-shirts (500)

Front (\$25/dz.)

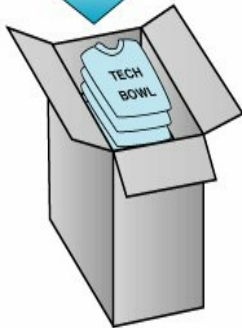
Back/front (\$35/dz.)



Sweatshirts (500)

Front (\$36/dz.)

Back/front (\$48/dz.)



The resource requirements, unit costs, and profit per dozen for each type of shirt are shown in the following table:

	Processing Time (hr.) per Dozen	Cost per Dozen	Profit per Dozen
SweatshirtF	0.10	\$36	\$ 90
SweatshirtB/F	0.25	48	125
T-shirtF	0.08	25	45

T-shirtB/F

0.21

35

65

The company wants to know how many dozen (boxes) of each type of shirt to produce in order to maximize profit.

[Page 112]

Following is a review of the model formulation steps for this problem:

Summary of Linear Programming Model Formulation Steps

Define the decision variables

Step 1. How many (dozens of) T-shirts and sweatshirts of each type to produce

Define the objective function

Step 2. Maximize profit

Define the constraints

Step 3. The resources available, including processing time, blank shirts, budget, and

shipping capacity

Decision Variables

This problem contains four decision variables, representing the number of dozens (boxes) of each type of shirt to produce:

x_1 = sweatshirts, front printing

x_2 = sweatshirts, back and front printing

x_3 = T-shirts, front printing

x_4 = T-shirts, back and front printing

The Objective Function

The company's objective is to maximize profit. The total profit is the sum of the individual profits gained from each type of shirt. The objective function is expressed as

$$\text{maximize } Z = \$90x_1 + 125x_2 + 45x_3 + 65x_4$$

Model Constraints

The first constraint is for processing time. The total available processing time is the 72-hour period between the end of the game and the truck pickup:

$$0.10x_1 + 0.25x_2 + 0.08x_3 + 0.21x_4 \leq 72 \text{ hr}$$

The second constraint is for the available shipping capacity, which is 1,200 standard-size boxes. A box of sweatshirts is three times the size of a

standard-size box. Thus, each box of sweatshirts is equivalent in size to three boxes of T-shirts. This relative size differential is expressed in the following constraint:

$$3x_1 + 3x_2 + x_3 + x_4 \leq 1,200$$

boxes

The third constraint is for the cost budget. The total budget available for production is \$25,000:

$$\$36x_1 + 48x_2 + 25x_3 + 35x_4 \leq \$25,000$$

The last two constraints reflect the available blank sweatshirts and T-shirts the company has in storage:

$$x_1 + x_2 \leq 500 \text{ dozen sweatshirts}$$

$$x_3 + x_4 \leq 500 \text{ dozen T-shirts}$$

Model Summary

The linear programming model for Quick-Screen is summarized as follows:

maximize $Z = 90x_1 + 125x_2 + 45x_3 + 65x_4$
subject to

$$0.10x_1 + 0.25x_2 + 0.08x_3 + 0.21x_4 \leq 72$$

$$3x_1 + 3x_2 + x_3 + x_4 \leq 1,200$$

$$36x_1 + 48x_2 + 25x_3 + 35x_4 \leq 25,000$$

$$x_1 + x_2 \leq 500$$

$$x_3 + x_4 \leq 500$$

$$x_1, x_2, x_3, x_4 \geq 0$$

*This model can be
input as shown for
computer solution.*

Computer Solution with

Excel

The Excel spreadsheet solution for this product mix example is shown in [Exhibit 4.1](#). The decision variables are located in cells **B14:B17**. The profit is computed in cell B18, and the formula for profit, **=B14*D5+B15*E5+B16*F5+** is shown on the formula bar at the top of the spreadsheet. The constraint formulas are embedded in cells H7 through H11, under the column titled "Usage." For example, the constraint formula for

processing time in cell H7 is
=D7*B14+E7*B15+F7*B16+
Cells H8 through H11 have
similar formulas.

[Page 114]

Exhibit 4.1.

[\[View full size image\]](#)

Click on "Tools" to access Solver

Objective function

These cells have no effect added for "cosmetic" purposes

=J7-H7

=D7*B14+E7*B15+F7*B16+G7*B17

=F11*B16+G11*B17

Microsoft Excel - 1440161.xls

File Edit View Insert Format Tools Solver Help

Formulas: =D7*B14+E7*B15+F7*B16+G7*B17

A Product Mix Example

Products	X1	X2	X3	X4	Resources			
Profit per box	90	125	45	65				
Resources					Usage	Constraints	Available	Left over
processing time (hr/box)	0.1	0.35	0.08	0.21	72.0	<=	72	0.0
shipping capacity (boxes)	3	3	1	1	1200.0	<=	1200	0.0
budget (\$/box)	30	40	25	35	21600.0	<=	25000	3400.0
blank sweatshirts (boxes)	1	1			285.0	<=	500	214.7
blank T-shirts (boxes)			1	1	600.0	<=	600	0.0
Production								
X1 = 175.58	sweatshirts, front printing							
X2 = 27.75	sweatshirts, back and front printing							
X3 = 600.00	T-shirts, front printing							
X4 = 0.00	T-shirts, back and front printing							
Profit = 45622.22								

Cells K7 through K11 contain the formulas for the leftover resources, or slack. For

example, cell K7 contains the formula **=J7H7**. These formulas for leftover resources enable us to demonstrate a spreadsheet operation that can save you time in developing the spreadsheet model. First, enter the formula for leftover resources, **=J7H7**, in cell K7, as we have already shown. Next, using the right mouse button, click on "Copy." Then cover cells K8:K11 with the cursor (by holding the left mouse button down). Click the right mouse button again and then click on "Paste." This will

automatically insert the correct formulas for leftover resources in cells K8 through K11 so that you do not have to type them all in individually. This copying operation can be used when the variables in the formula are all in the same row or column. The copying operation simply increases the row number for each cell that the formulas are copied into (i.e., J8 and H8, J9 and H9, J10 and H10, and J11 and H11).

The Solver window for this model is shown in [Exhibit 4.2](#). Notice that we were able to

insert all five constraint formulas with one line in the "Subject to the Constraints:" window. We used the constraint **H7:H11 <= J7:J11**, which means that all the constraint usage values computed in cells H7 through H11 are less than the corresponding available resource values computed in cells J7 through J11.

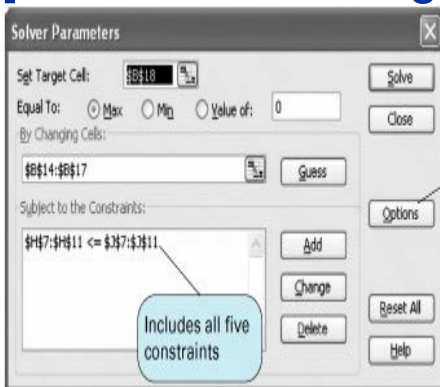
[Page 115]

Exhibit 4.2.

(This item is displayed on page

114 in the print version)

[\[View full size image\]](#)



Computer Solution With

QM for Windows

The QM for Windows solution for this problem is shown in [Exhibits 4.3](#) and [4.4](#).

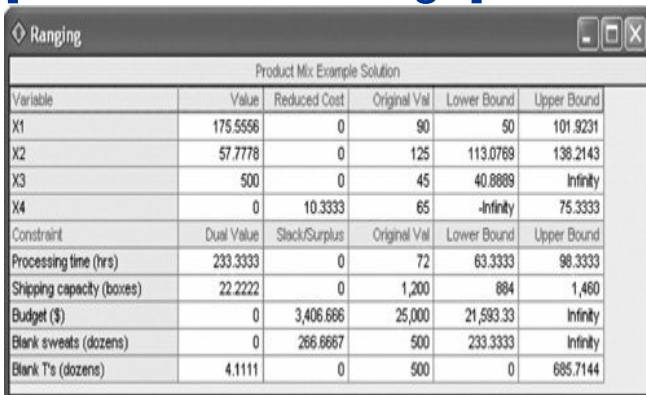
Exhibit 4.3.

[\[View full size image\]](#)

Original Problem answers							
Product Mix Example Solution							
	X1	X2	X3	X4		RHS	Dual
Maximize	90	125	45	65			Max: $90X1 + 125X2 + 45X3 + 65X4$
Processing time (hrs)	.1	.25	.08	.21	≤	72	233.3333
Shipping capacity (boxes)	3	3	1	1	≤	1,200	22.2222
Budget (\$)	36	48	25	35	≤	25,000	0
Blank sweets (dozens)	1	1	0	0	≤	500	0
Blank T's (dozens)	0	0	1	1	≤	500	4.1111
Solution >	175.5556	57.7778	500	0	Optimal Z >	45,522.22	$X3 + X4 \leq 500$

Exhibit 4.4.

[\[View full size image\]](#)



The screenshot shows a software window titled "Ranging" with standard window controls (minimize, maximize, close) in the top right corner. The window contains a table titled "Product Mix Example Solution". The table is divided into two sections. The first section lists variables (X1 through X4) with columns for Value, Reduced Cost, Original Val, Lower Bound, and Upper Bound. The second section lists constraints (Processing time, Shipping capacity, Budget, Blank sweats, Blank T's) with columns for Dual Value, Slack/Surplus, Original Val, Lower Bound, and Upper Bound.

Product Mix Example Solution					
Variable	Value	Reduced Cost	Original Val	Lower Bound	Upper Bound
X1	175.5556	0	90	50	101.9231
X2	57.7778	0	125	113.0769	138.2143
X3	500	0	45	40.8889	Infinity
X4	0	10.3333	65	-Infinity	75.3333
Constraint	Dual Value	Slack/Surplus	Original Val	Lower Bound	Upper Bound
Processing time (hrs)	233.3333	0	72	63.3333	98.3333
Shipping capacity (boxes)	22.2222	0	1,200	884	1,460
Budget (\$)	0	3,406.666	25,000	21,593.33	Infinity
Blank sweats (dozens)	0	266.6667	500	233.3333	Infinity
Blank T's (dozens)	4.1111	0	500	0	685.7144

Solution Analysis

The model solution is

$x_1 = 175.56$ boxes of front-only sweatshirts

$x_2 = 57.78$ boxes of front and back sweatshirts

$x_3 = 500$ boxes of front-only T-shirts

$Z = \$45,522.22$ profit

The manager of Quick-Screen might have to round off the solution to send whole boxes for example, 175 boxes of front-only sweatshirts, 57 of front and back sweatshirts, and 500 of front-only T-shirts. This would result in a profit of \$45,375.00, which is only \$147.22 less than the optimal profit value of \$45,522.22.

After formulating and solving this model, Quick-Screen might decide that it needs to produce and ship at least some of each type of shirt. Management could evaluate this possibility

by adding four constraints that establish minimum levels of production for each type of shirt, including front and back T-shirts, x_4 , none of which are produced in the current solution. The manager might also like to experiment with the constraints to see the effect on the solution of adding resources. For example, the dual value for processing time shows profit would increase by \$233.33 per hour (up to 98.33 hours, the upper limit of the sensitivity range for this constraint quality value).

Although the 72-hour limit seems pretty strict, it might be possible to reduce individual processing times and achieve the same result.

A Diet Example

Breathtakers, a health and fitness center, operates a morning fitness program for senior citizens. The program includes aerobic exercise, either swimming or step exercise, followed by a healthy breakfast in the dining room. Breathtakers' dietitian wants to develop a breakfast that will be high in calories, calcium, protein, and fiber, which are especially important to senior

citizens, but low in fat and cholesterol. She also wants to minimize cost. She has selected the following possible food items, whose individual nutrient contributions and cost from which to develop a standard breakfast menu are shown in the following table:

Breakfast	Fat	Cholesterol	Iron	Cost
-----------	-----	-------------	------	------

Food	Calories (g)	(mg)	(mg)	
------	--------------	------	------	--

1. Bran cereal (cup)	90	0	0	6
----------------------	----	---	---	---

2. Dry cereal (cup)	110	2	0	4
3. Oatmeal (cup)	100	2	0	2
4. Oat bran (cup)	90	2	0	3
5. Egg	75	5	270	1
6. Bacon (slice)	35	3	8	0
7. Orange	65	0	0	1

8. Milk2% (cup)	100	4	12	0
9. Orange juice (cup)	120	0	0	0
10. Wheat toast (slice)	65	1	0	1

The dietitian wants the breakfast to include at least 420 calories, 5 milligrams of iron, 400 milligrams of calcium, 20 grams of protein, and 12

grams of fiber. Furthermore, she wants to limit fat to no more than 20 grams and cholesterol to 30 milligrams.

Decision Variables

This problem includes 10 decision variables, representing the number of standard units of each food item that can be included in each breakfast:

x_1 = cups of bran cereal

x_2 = cups of dry cereal

x_3 = cups of oatmeal

x_4 = cups of oat bran

x_5 = eggs

x_6 = slices of bacon

x_7 = oranges

x_8 = cups of milk

x_9 = cups of orange juice

x_{10} = slices of wheat toast

The Objective Function

The dietitian's objective is to minimize the cost of a breakfast. The total cost is the sum of the individual costs of each food item:

$$\begin{aligned}\text{minimize } Z = & \$0.18x_1 + 0.22x_2 + 0.10x_3 + 0.12x_4 + 0.10x_5 + 0.09x_6 + 0.40x_7 \\ & + 0.16x_8 + 0.50x_9 + 0.07x_{10}\end{aligned}$$

[Page 117]

Model Constraints

The constraints are the

requirements for the nutrition items:

[\[View full size image\]](#)

$$90x_1 + 110x_2 + 100x_3 + 90x_4 + 75x_5 + 35x_6 + 65x_7 + 100x_8 + 120x_9 + 65x_{10} \geq 420 \text{ calories}$$

$$2x_2 + 2x_3 + 2x_4 + 5x_5 + 3x_6 + 4x_8 + x_{10} \leq 20 \text{ g of fat}$$

$$270x_5 + 8x_6 + 12x_8 \leq 30 \text{ mg of cholesterol}$$

$$6x_1 + 4x_2 + 2x_3 + 3x_4 + x_5 + x_7 + x_{10} \geq 5 \text{ mg of iron}$$

$$20x_1 + 48x_2 + 12x_3 + 8x_4 + 30x_5 + 52x_7 + 250x_8 + 3x_9 + 26x_{10} \geq 400 \text{ mg of calcium}$$

$$3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 + 2x_6 + x_7 + 9x_8 + x_9 + 3x_{10} \geq 20 \text{ g of protein}$$

$$5x_1 + 2x_2 + 3x_3 + 4x_4 + x_7 + 3x_{10} \geq 12 \text{ g of fiber}$$

Model Summary

The linear programming model for this problem can be summarized as follows:

minimize $Z = 0.18x_1 + 0.22x_2 + 0.10x_3 + 0.12x_4 + 0.10x_5 + 0.09x_6 + 0.40x_7$
 $+ 0.16x_8 + 0.50x_9 + 0.07x_{10}$

subject to

$$90x_1 + 110x_2 + 100x_3 + 90x_4 + 75x_5 + 35x_6 + 65x_7 + 100x_8 + 120x_9 + 65x_{10} \geq 420$$

$$2x_2 + 2x_3 + 2x_4 + 5x_5 + 3x_6 + 4x_8 + x_{10} \leq 20$$

$$270x_5 + 8x_6 + 12x_8 \leq 30$$

$$6x_1 + 4x_2 + 2x_3 + 3x_4 + x_5 + x_7 + x_{10} \geq 5$$

$$20x_1 + 48x_2 + 12x_3 + 8x_4 + 30x_5 + 52x_7 + 250x_8 + 3x_9 + 26x_{10} \geq 400$$

$$3x_1 + 4x_2 + 5x_3 + 6x_4 + 7x_5 + 2x_6 + x_7 + 9x_8 + x_9 + 3x_{10} \geq 20$$

$$5x_1 + 2x_2 + 3x_3 + 4x_4 + x_7 + 3x_{10} \geq 12$$

$$x_i \geq 0$$

Computer Solution with Excel

The solution to our diet example using an Excel spreadsheet is shown in [Exhibit](#)

4.5. The decision variables (i.e., "servings" of each menu item) for our problem are contained in cells **C5:C14** inclusive, and the constraint formulas are in cells F15 through L15. Cells F17 through L17 contain the constraint (right-hand-side) values. For example, cell F15 contains the constraint formula for calories,

=SUMPRODUCT(C5:C14,F5:F:

Then when the Solver Parameters screen is accessed, the constraint **F15>=F17** will be added in. This constraint formula in cell F15 could have

been constructed by multiplying each of the values in column C by each of the corresponding column F cell values. The equivalent constraint formula in cell F15 to our SUMPRODUCT formula is **=C5*F5+C6*F6+C7*F7+C8*F**

Exhibit 4.5.

(This item is displayed on page 118 in the print version)

[\[View full size image\]](#)

function value is also computed by using the Excel SUMPRODUCT command rather than by individually multiplying each cell value in column C by each cell value in column E and then summing them. This would be a tedious formula to type into cell C19.

Alternatively, the SUMPRODUCT() formula multiplies all the values in cells C5 through C14 by all the corresponding values in cells E5 through E14 and then sums them.

The Solver dialog box for this

problem is shown in [Exhibit 4.6](#). Notice that the constraint box on this screen is not large enough to show all the constraints in the problem.

Exhibit 4.6.

(This item is displayed on page 118 in the print version)

[\[View full size image\]](#)

Decision variables;
"servings" in column C

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Variable Cells:

Subject to the Constraints:

-
-
-
-
-
-

Buttons: Solve, Close, Options, Reset All, Help

For nonnegativity constraints click on "Options" and then "Assume Non-Negative."

Constraint for "calories" in column F;
 $\text{=SUMPRODUCT}(C5:C14, F5:F14) \leq 420$

Solution Analysis

The solution is

$x_3 = 1.025$ cups of oatmeal

$x_8 = 1.241$ cups of milk

$x_{10} = 2.975$ slices of wheat
toast

$Z = \$0.509$ cost per meal

[Page 119]

The result of this simplified
version of a real menu planning
model (see the application box
"The Evolution of the Diet

Problem") is interesting in that it suggests a very practical breakfast menu. This would be a healthy breakfast for anyone.

This model includes a daily minimum requirement of only 420 calories. The recommended daily calorie requirement for an adult is approximately 2,000 calories. Thus, the breakfast requirement is only about 21% of normal daily adult needs. In this model the dietitian must have felt a low-calorie breakfast was needed because the senior citizens had high-calorie lunches and dinners. An

alternative approach to a healthy diet is a high-calorie breakfast followed by low-calorie, light meals and snacks the rest of the day. However, in this model, as the calorie requirements are increased above 420, the model simply increases the cups of oatmeal. For example, a 700-calorie requirement results in about 5 or 6 cups of oatmeal not a very appetizing breakfast. This difficulty can be alleviated by establishing upper limits on the servings for each food item and solving the model again with a

higher calorie requirement.

An Investment Example

Kathleen Allen, an individual investor, has \$70,000 to divide among several investments. The alternative investments are municipal bonds with an 8.5% annual return, certificates of deposit with a 5% return, treasury bills with a 6.5% return, and a growth stock fund with a 13% annual return. The investments are all evaluated after 1 year. However, each investment

alternative has a different perceived risk to the investor; thus, it is advisable to diversify. Kathleen wants to know how much to invest in each alternative in order to maximize the return.

The following guidelines have been established for diversifying the investments and lessening the risk perceived by the investor:

- 1.** No more than 20% of the total investment should be in municipal bonds.

- 2.** The amount invested in certificates of deposit should not exceed the amount invested in the other three alternatives.
- 3.** At least 30% of the investment should be in treasury bills and certificates of deposit.
- 4.** To be safe, more should be invested in CDs and treasury bills than in municipal bonds and the growth stock fund, by a ratio of at least 1.2 to 1.

Kathleen wants to invest the entire \$70,000.

Decision Variables

Four decision variables represent the monetary amount invested in each investment alternative:

x_1 = amount (\$) invested in municipal bonds

x_2 = amount (\$) invested in certificates of deposit

x_3 = amount (\$) invested in treasury bills

x_4 = amount (\$) invested in growth stock fund

The Objective Function

The objective of the investor is to maximize the total return from the investment in the four alternatives. The total return is the sum of the individual returns from each alternative. Thus, the objective function is expressed as

$$\text{maximize } Z = \$0.085x_1 + 0.05x_2 + 0.065x_3 + 0.130x_4$$

where

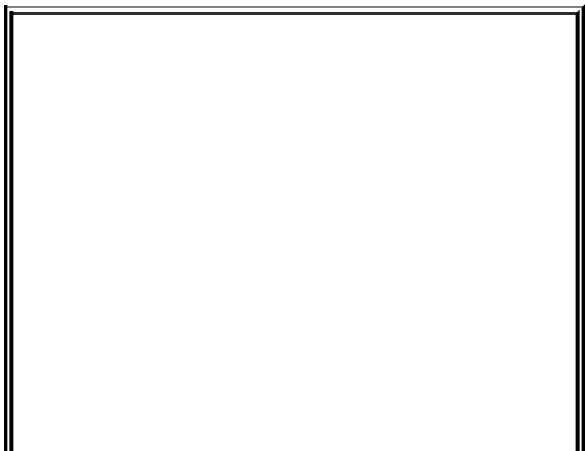
Z = total return from all investments

$0.085x_1$ = return from the investment in municipal bonds

$0.05x_2$ = return from the investment in certificates of deposit

$0.065x_3$ = return from the investment in
treasury bills

$0.130x_4$ = return from the investment in
growth stock fund



Management Science

Application: The Evolution of the Diet Problem

When George Dantzig developed the simplex method for solving linear programming models, he illustrated the new technique by using the "diet problem" developed in 1945 by George Stigler. The problem was to determine a menu of foods that would satisfy nutritional requirements at a minimum cost. The diet problem has evolved over the years and continues to be a popular application of linear programming as institutional feeding programs at hospitals, military installations, nursing homes, schools, prisons, shelters, and restaurants have proliferated. Menu planning represents a formidable

problem, with more than 30 nutritional constraints and thousands of variables, as well as consumer food preferences. Recommended daily allowances include minimum levels of 29 nutrients, while various health groups recommend maximum daily levels of fat, cholesterol, and sodium. Today, software packages are available for menu planning, including computer-assisted menu planning (CAMP), part of the IBM contributed programs library.

CAMP has had numerous reported applications around the world, including the development of a 90-day planned menu cycle at a state hospital and planned menus for schools that lower fat content while meeting nutritional requirements at a lower cost. In general, mathematical programmingbased meal planning models have been formed to reduce

food costs by approximately 10% while meeting government-established nutritional standards. Other traditional menu planning methods have generally not achieved this level of success.



Source: L. M. Lancaster, "The Evolution of the Diet Model in Managing Food Systems," *Interfaces* 22, no. 5 (September/October

Model Constraints

In this problem the constraints are the guidelines established for diversifying the total investment. Each guideline is transformed into a mathematical constraint separately.

The first guideline states that

no more than 20% of the total investment should be in municipal bonds. The total investment is \$70,000; 20% of \$70,000 is \$14,000. Thus, this constraint is

$$x_1 \leq \$14,000$$

The second guideline indicates that the amount invested in certificates of deposit should not exceed the amount invested in the other three alternatives. Because the investment in certificates of deposit is x_2 and the amount

invested in the other alternatives is $x_1 + x_3 + x_4$, the constraint is

[Page 121]

$$x_2 \leq x_1 + x_3 + x_4$$

This constraint is not in what we referred to in [Chapter 3](#) as *standard form* for a computer solution. In standard form, all the variables would be on the left-hand side of the inequality ($\leq$), and all the numeric values would be on the right side. This type of constraint can be used

in Excel just as it is shown here; however, for solution with QM for Windows, all constraints must be in standard form. We will go ahead and convert this constraint and others in this model to standard form, but when we solve this model with Excel, we will explain how the model constraints could be used in their original (nonstandard) form. To convert this constraint to standard form, $x_1 + x_3 + x_4$ must be subtracted from both sides of the $\leq$ sign to put this constraint in proper form:

$$x_2 \ x_1 \ x_3 \ x_4 \leq 0$$

*Standard form
requires all variables
to be to the left of
the inequality and
numeric values to the
right.*

The third guideline specifies that at least 30% of the investment should be in treasury bills and certificates of deposit. Because 30% of

\$70,000 is \$21,000 and the amount invested in certificates of deposit and treasury bills is represented by $x_2 + x_3$, the constraint is

$$x_2 + x_3 \geq \$21,000$$

The fourth guideline states that the ratio of the amount invested in certificates of deposit and treasury bills to the amount invested in municipal bonds and the growth stock fund should be at least 1.2 to 1:

$$[(x_2 + x_3)/(x_1 + x_4)] \geq 1.2$$

This constraint is not in standard linear programming form because of the fractional relationship of the decision variables, $(x_2 + x_3)/(x_1 + x_4)$. It is converted as follows:

$$\begin{aligned} x_2 + x_3 &\geq 1.2 (x_1 + x_4) \\ -1.2x_1 + x_2 + x_3 - 1.2x_4 &\geq 0 \end{aligned}$$

Standard form requires that fractional relationships between variables be eliminated.

Finally, the investor wants to invest the entire \$70,000 in the four alternatives. Thus, the sum of all the investments in the four alternatives must *equal* \$70,000:

$$x_1 + x_2 + x_3 + x_4 = \$70,000$$

Model Summary

The complete linear programming model for this problem can be summarized as

maximize $Z = \$0.085x_1 + 0.05x_2 + 0.065x_3 + 0.130x_4$
subject to

$$x_1 \leq 14,000$$

$$x_2 - x_1 - x_3 - x_4 \leq 0$$

$$x_2 + x_3 \geq 21,000$$

$$-1.2x_1 + x_2 + x_3 - 1.2x_4 \geq 0$$

$$x_1 + x_2 + x_3 + x_4 = 70,000$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Computer Solution with Excel

The Excel spreadsheet solution for the investment example is shown in [Exhibit 4.7](#), and its Solver window is shown in [Exhibit 4.8](#). The spreadsheet is

set up very similarly to the spreadsheet for our product mix example in [Exhibit 4.1](#). The decision variables are located in cells **B13:B16**. The total return (Z) is computed in cell B17, and the formula for Z is shown on the formula bar at the top of the spreadsheet. The constraint formulas for the investment guidelines are embedded in cells H6 through H10. For example, the first guideline formula, in cell H6, is **=D6*B13**, and the second guideline formula, in cell H7, is **=D7*B13+E7*B14+F7*B15+**

(Note that it would probably have been easier just to type the guideline formulas directly into cells H6 through H10 rather than create the array of constraint coefficients in **D6:G10**; however, for demonstration purposes, we wanted to show all the parameter values.)

[Page 122]

Exhibit 4.7.

[\[View full size image\]](#)

Formula for Z,
total return

Microsoft Excel - Exhibit 4.8.xls

Formulas: =D6*B13+E10*B14+F10*B15+G10*B16

An Investment Example					Guidelines			
	X1	X2	X3	X4	Objective	Constraints	Requirements	Black/White
1. municipal bonds	1				0.00	≤	14000.00	14000.00
2. CDs	-1	1	-1	-1	-2000.00	≤	0.00	2000.00
3. treasury bills and CDs			1	1	30000.00	≥	21000.00	17000.00
4. ratio	-1.2	1	1	-1.2	0.00	≤	0.00	0.00
5. total invested	1	1	1	1	70000.00	=	70000.00	0.00
6. Returns:								
10 =	B				invested in municipal bonds			
14 =	B				invested in CDs			
15 =	30000.00				invested in treasury bills			
16 =	70000.00				invested in stock fund			
17 =	2 =	1000.00			total return			

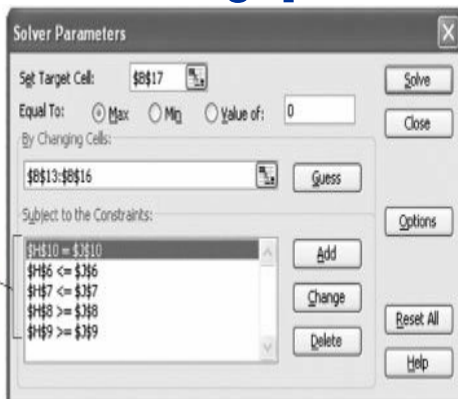
First guideline,
=D6*B13

Total investment requirement,
=D10*B13+E10*B14+F10*B15+G10*B16

Exhibit 4.8.

[\[View full size image\]](#)

Guideline constraints



The image shows the 'Solver Parameters' dialog box in Microsoft Excel. The 'Set Target Cell' is '\$B\$17'. The 'Equal To' section has three radio buttons: 'Max' (selected), 'Min', and 'Value of:'. The 'Value of' field is set to '0'. The 'By Changing Cells' field is '\$B\$13:\$B\$16'. The 'Subject to the Constraints' list contains five constraints: '\$H\$10 = \$J\$10', '\$H\$6 <= \$J\$6', '\$H\$7 <= \$J\$7', '\$H\$8 >= \$J\$8', and '\$H\$9 >= \$J\$9'. A callout box labeled 'Guideline constraints' points to this list. On the right side of the dialog, there are buttons for 'Solve', 'Close', 'Options', 'Add', 'Change', 'Delete', 'Reset All', and 'Help'.

Solver Parameters

Set Target Cell:

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-
-
-

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

As mentioned earlier, it is not necessary to convert the original model constraints into standard form to solve this model using Excel. For

example, the constraint for certificates of deposit, $x_2 \leq x_1 + x_3 + x_4$, could be entered in the spreadsheet in row 7 in [Exhibit 4.7](#) as follows. The value, 1, the coefficient for x_2 , could be entered in cell E7, **=E7*B14** could be entered in cell J7, and the remainder of the constraint to the right side of the inequality could be entered as **=B13+B15+B16** in cell H7. In the Solver window this constraint is entered as it originally was, **H7<=J7**. The fourth guideline constraint could be entered in its original

form similarly.

[Page 123]

Solution Analysis

The solution is

$x_3 = \$38,181.82$ invested in
treasury bonds

$x_4 = \$31,818.18$ invested in a
growth stock fund

$Z = \$6,818.18$

The sensitivity report for our Excel spreadsheet solution to this problem is shown in [Exhibit 4.9](#).

Exhibit 4.9.

[\[View full size image\]](#)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$13	X1 =	0	-0.045	0.005	0.045	1E+30
\$B\$14	X2 =	0	-0.015	0.050	0.015	1E+30
\$B\$15	X3 =	38181.82	0	0.065	0.065	0.015
\$B\$16	X4 =	31818.18	0	0.13	45354.805	0.045

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$H\$6	1. municipal bonds Achievement	0.00	0.000	14000	1E+30	14000
\$H\$7	2. CDs Achievement	-70000.00	0.000	0	1E+30	70000
\$H\$9	3. treasury bills and CDs Achievement	38181.82	0.000	21000	17181.82	1E+30
\$H\$9	4. ratio Achievement	0.00	-0.030	0	70000	37800
\$H\$10	total invested Achievement	70000.00	0.095	70000	1E+30	31500

Shadow price for the amount available to invest

Notice that the dual (shadow price) value for constraint 5 (i.e., the sum of the investments must equal \$70,000) is 0.095. This indicates that for each

additional \$1 Kathleen Allen invests (above \$70,000), according to the existing investment guidelines she has established, she could expect a return of 9.5%. The sensitivity ranges show that there is no upper bound on the amount she could invest and still receive this return.

An interesting variation of the problem is to not specify that the entire amount available (in this case, \$70,000) must be invested. This changes the constraints for the first and third guidelines and the

constraint that requires that the entire \$70,000 be invested.

Recall that the first guideline is "no more than 20% of the total investment should be in municipal bonds." The total investment is no longer exactly \$70,000, but the sum of all four investments, $x_1 + x_2 + x_3 + x_4$. The constraint showing that the amount invested in municipal bonds, x_1 , as a percentage (or ratio) of this total cannot exceed 20% is written as

$$\frac{x_1}{x_1 + x_2 + x_3 + x_4} \leq 0.20$$

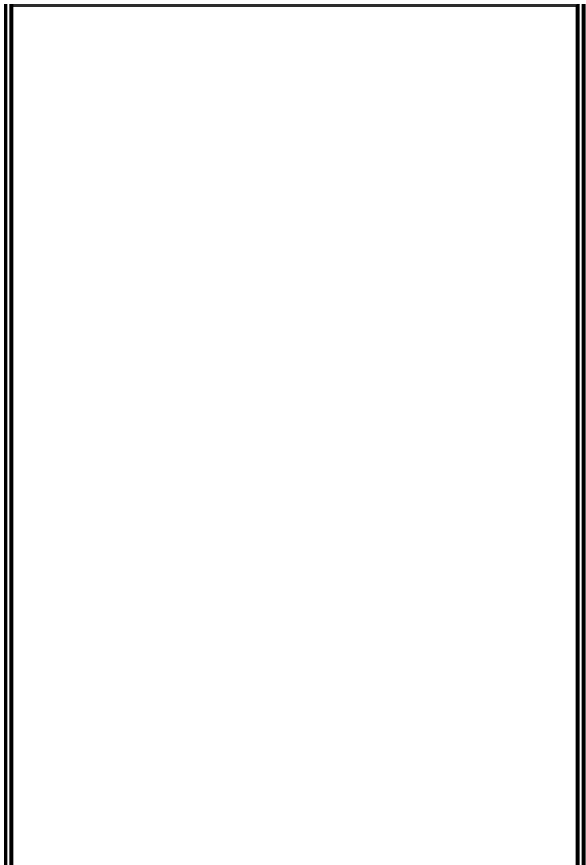
Rewriting this constraint in a form more consistent with a linear programming solution (to eliminate the fractional relationship between variables) results in

$$x_1 \leq 0.2(x_1 + x_2 + x_3 + x_4)$$

[Page 124]

and

$$0.8x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 \leq 0$$



Management Science

Application: A Linear Programming Model for Optimal Portfolio Selection at Prudential Securities, Inc

In the secondary mortgage market, government agencies purchase mortgage loans from the original mortgage lenders and pool them to create mortgage-backed securities (MBSs). These securities are traded in capital markets along with other fixed-income securities, such as treasury and corporate bonds. The total size of this market for MBSs is well over \$1 trillion. There are a number of types of MBSs. The market for MBSs is maintained by a network of dealers, including

Prudential Securities, Inc. This firm and others like it underwrite and issue new market-based securities. This market is somewhat more complex than standard bond investments, such as treasury or corporate securities, because the principal on mortgages is returned gradually over the life of the security rather than in a lump sum at the end. In addition, cash flows fluctuate because of the homeowner's right to prepay mortgages. In order to deal with these complexities, Prudential Securities has developed and implemented a number of management science models to reduce investment risk and properly value securities for its investors. One such model employs linear programming to design an optimal securities portfolio to meet various investors' criteria under different

interest rate environments. The linear programming model determines the amount to invest in different MBSs to meet a client's objectives for portfolio performance. Constraints might include maximum and minimum percentages (of the total portfolio investment) that could be invested in any one or more securities, the duration of the securities, the amount to be invested, and the amount of the different securities available. This linear programming model and other management science models are used hundreds of times each day at Prudential Securities by traders, salespeople, and clients. The models allow the firm to participate successfully in the mortgage market. Prudential's secondary market trading in MBSs typically exceeds \$5 billion per week.



Source: Y. Ben-Dow, L. Hayre, and V. Pica, "Mortgage Valuation Models at Prudential Securities," *Interfaces* 22, no. 1 (JanuaryFebruary 1992): 5571.

The constraint for the third guideline, which stipulates that

at least 30% of the total investment ($x_1 + x_2 + x_3 + x_4$) should be in treasury bills and CDs ($x_2 + x_3$), is formulated similarly as

$$\frac{x_2 + x_3}{x_1 + x_2 + x_3 + x_4} \geq 0.30$$

and

$$-0.3x_1 + 0.7x_2 + 0.7x_3 - 0.3x_4 \geq 0$$

Because the entire \$70,000 does not have to be invested, the last constraint becomes

$$x_1 + x_2 + x_3 + x_4 \leq 70,000$$

The complete linear programming model is summarized as

[Page 125]

maximize $Z = \$0.085x_1 + 0.05x_2 + 0.065x_3 + 0.130x_4$
subject to

$$\begin{aligned}0.8x_1 - 0.2x_2 - 0.2x_3 - 0.2x_4 &\leq 0 \\x_2 - x_1 - x_3 - x_4 &\leq 0 \\-0.3x_1 + 0.7x_2 + 0.7x_3 - 0.3x_4 &\geq 0 \\-1.2x_1 + x_2 + x_3 - 1.2x_4 &\geq 0 \\x_1 + x_2 + x_3 + x_4 &\leq 70,000 \\x_1, x_2, x_3, x_4 &\geq 0\end{aligned}$$

The solution to this altered model is exactly the same as

the solution to our original model, wherein the entire \$70,000 must be invested. This is logical because only positive returns are achieved from investing, and thus the investor would leave no money not invested. However, if losses could be realized from some investments, then it might be a good idea to construct the model so that the entire amount would not have to be invested.

A Marketing Example

The Biggs Department Store chain has hired an advertising firm to determine the types and amount of advertising it should invest in for its stores. The three types of advertising available are television and radio commercials and newspaper ads. The retail chain desires to know the number of each type of advertisement it should purchase in order to maximize exposure. It is

estimated that each ad or commercial will reach the following potential audience and cost the following amount:

	Exposure (people/ad or commercial)	Cost
Television commercial	20,000	\$15,000
Radio commercial	12,000	6,000
Newspaper ad	9,000	4,000

The company must consider the following resource constraints:

- 1.** The budget limit for advertising is \$100,000.
- 2.** The television station has time available for 4 commercials.
- 3.** The radio station has time available for 10 commercials.
- 4.** The newspaper has space available for 7 ads.

5. The advertising agency has time and staff available for producing no more than a total of 15 commercials and/or ads.

Decision Variables

This model consists of three decision variables that represent the number of each type of advertising produced:

x_1 = number of television commercials

x_2 = number of radio commercials

x_3 = number of newspaper ads

The Objective Function

The objective of this problem is different from the objectives in the previous examples, in which only profit was to be maximized (or cost minimized). In this problem, profit is not to be maximized; instead, audience exposure is to be maximized. Thus, this objective

function demonstrates that although a linear programming model must either maximize or minimize some objective, the objective itself can be in terms of any type of activity or valuation.

[Page 126]

For this problem the objective audience exposure is determined by summing the audience exposure gained from each type of advertising:

maximize $Z = 20,000x_1 +$

$$12,000x_2 + 9,000x_3$$

where

Z = total level of audience exposure

$20,000x_1$ = estimated number of people reached by television commercials

$12,000x_2$ = estimated number of people reached by radio commercials

$9,000x_3$ = estimated number of people reached by newspaper ads

Model Constraints

The first constraint in this model reflects the limited budget of \$100,000 allocated for advertisement:

$$\$15,000x_1 + 6,000x_2 + 4,000x_3 \leq \$100,000$$

where

$\$15,000x_1$ = amount spent for television advertising

$6,000x_2$ = amount spent for radio advertising

$4,000x_3$ = amount spent for newspaper advertising

The next three constraints represent the fact that television and radio commercials are limited to 4 and 10, respectively, and newspaper ads are limited to 7:

$x_1 \leq 4$ television commercials

$x_2 \leq 10$ radio commercials

$x_3 \leq 7$ newspaper ads

The final constraint specifies that the total number of

commercials and ads cannot exceed 15 because of the limitations of the advertising firm:

$x_1 + x_2 + x_3 \leq 15$ commercials and ads

Model Summary

The complete linear programming model for this problem is summarized as

maximize $Z = 20,000x_1 + 12,000x_2 + 9,000x_3$
subject to

$$\$15,000x_1 + 6,000x_2 + 4,000x_3 \leq \$100,000$$

$$x_1 \leq 4$$

$$x_2 \leq 10$$

$$x_3 \leq 7$$

$$x_1 + x_2 + x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Computer Solution with Excel

The solution to our marketing example using Excel is shown in [Exhibit 4.10](#). The model decision variables are contained in cells **D6:D8**. The

formula for the objective function in cell E10 is shown on the formula bar at the top of the screen. When Solver is accessed from the "Tools" menu, as shown in [Exhibit 4.11](#), it is necessary to use only one formula to enter the model constraints: **H6:H10 <= J6:J10**.

[Page 127]

Exhibit 4.10.

[\[View full size image\]](#)

Objective function

Microsoft Excel - tabl04.10.xls

File Edit Format Tools Data Window Help

Type a question for help

100%

Formulas

fx Add

ET0 = A1 =SUMPRODUCT(D6:D8,E6:E8)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	A Marketing Example																
2																	
3																	
4					Exposures	Cost											
5		Type of Advertising	Number	(people-adj)			Reserves	Actual	Cost/spot	Available							
6		Television commercials =	1,818	20,000	15,000		TV commercials	1,82	ea	4							
7		Radio commercials =	10	12,000	9,000		Radio commercials	10	ea	10							
8		Newspaper ads =	3,182	9,000	4,000		Newspaper ads	3,18	ea	7							
9							Total ads and commercials	15	ea	15							
10		Total Exposures =		35,000			Cost (\$)	106,000	ea	100,000							
11																	
12																	

$$=F6*D6+F7*D7+F8*D8$$

or

$$=SUMPRODUCT(D6:D8,F6:F8)$$

Exhibit 4.11.

[\[View full size image\]](#)

Includes all five
constraints

Solver Parameters

Set Target Cell:

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

Solution Analysis

The solution shows

$x_1 = 1.818$ television
commercials

$x_2 = 10$ radio commercials

$x_3 = 3.182$ newspaper ads

$Z = 185,000$ audience
exposure

This is a case where a non-integer solution can create difficulties. It is not realistic to round 1.818 television commercials to 2 television commercials, with 10 radio

commercials and 3 newspaper ads. Some quick arithmetic with the budget constraint shows that such a solution will exceed the \$100,000 budget limitation, although only by \$2,000. Thus, the store must either increase its advertising or plan for 1 television commercial, 10 radio commercials, and 3 newspaper ads. The audience exposure for this solution will be 167,000 people, or 18,000 fewer than the optimal number, almost a 10% decrease. There may, in fact, be a better solution than

this "rounded-down" solution.

[Page 128]

The integer linear programming technique, which restricts solutions to integer values, should be used. Although we will discuss the topic of integer programming in more detail in [Chapter 5](#), for now we can derive an integer solution by using Excel with a simple change when we input our constraints in the Solver window. We specify that our variable cells, **D6:D8**, are

integers in the Change Constraint window, as shown in [Exhibits 4.12](#) and [4.13](#). This will result in the spreadsheet solution in [Exhibit 4.14](#) when the problem is solved, which you will notice is better (i.e., more total exposures) than the rounded-down solution.

Exhibit 4.12.

[\[View full size image\]](#)

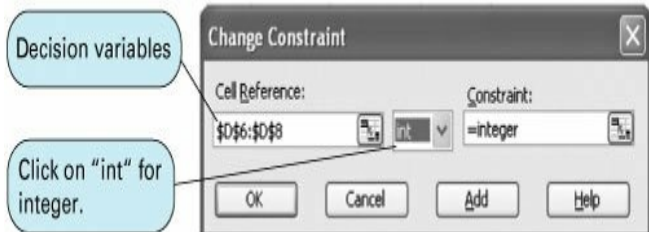


Exhibit 4.13.

[\[View full size image\]](#)

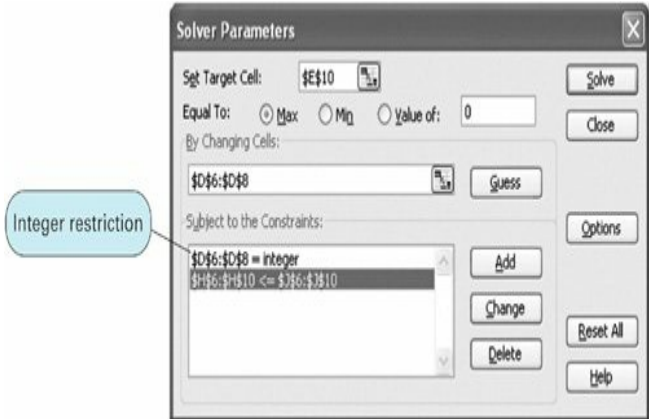


Exhibit 4.14.

[\[View full size image\]](#)

A Transportation Example

The Zephyr Television Company ships televisions from three warehouses to three retail stores on a monthly basis. Each warehouse has a fixed supply per month, and each store has a fixed demand per month. The manufacturer wants to know the number of television sets to ship from each warehouse to each store in order to minimize the total cost of transportation.

Each warehouse has the following supply of televisions available for shipment each month:

<i>Warehouse</i>	<i>Supply (sets)</i>
1. Cincinnati	300
2. Atlanta	200
3. Pittsburgh	200
	700

Each retail store has the following monthly demand for television sets:

<i>Store</i>	<i>Demand (sets)</i>
A. New York	150
B. Dallas	250
C. Detroit	200
	600

Costs of transporting television sets from the warehouses to the retail stores vary as a result of differences in modes of transportation and distances. The shipping cost per television set for each route is as follows:

	To Store		
	A	B	C
From Warehouse			
1	\$16	\$18	\$11

2	14	12	13
3	13	15	17

Decision Variables

The model for this problem consists of nine decision variables, representing the number of television sets transported from each of the three warehouses to each of the three stores:

x_{ij} = number of television sets shipped from warehouse i to store j

where $i = 1, 2, 3$, and $j = A, B, C$

The variable x_{ij} is referred to as a double-subscripted variable. The subscript, whether double or single, simply gives a "name" to the variable (i.e., distinguishes it from other decision variables). For example, the decision variable x_{3A} represents the number of television sets shipped from

warehouse 3 in Pittsburgh to store A in New York.

A double-subscripted variable is simply another form of variable name.

The Objective Function

The objective function of the television manufacturer is to minimize the total

transportation costs for all shipments. Thus, the objective function is the sum of the individual shipping costs from each warehouse to each store:

$$\begin{aligned} \text{minimize } Z = & \$16x_{1A} + 18x_{1B} \\ & + 11x_{1C} + 14x_{2A} + 12x_{2B} + \\ & 13x_{2C} + 13x_{3A} + 15x_{3B} + \\ & 17x_{3C} \end{aligned}$$

[Page 130]

Model Constraints

The constraints in this model are the number of television sets available at each warehouse and the number of sets demanded at each store. There are six constraintsone for each warehouse's supply and one for each store's demand. For example, warehouse 1 in Cincinnati is able to supply 300 television sets to any of the three retail stores. Because the number shipped to the three stores is the sum of x_{1A} , x_{1B} , and x_{1C} , the constraint for warehouse 1 is

$$X_{1A} + X_{1B} + X_{1C} \leq 300$$

This constraint is a $\leq$ inequality for two reasons. First, no more than 300 television sets can be shipped because that is the maximum available at the warehouse. Second, fewer than 300 can be shipped because 300 are not needed to meet the total demand of 600. That is, total demand is less than total supply, which equals 700. To meet the total demand at the three stores, all that can be supplied by the three warehouses is not needed.

Thus, the other two supply constraints for warehouses 2 and 3 are also inequalities:

$$x_{2A} + x_{2B} + x_{2C} \leq 200$$

$$x_{3A} + x_{3B} + x_{3C} \leq 200$$

*In a "**balanced**"
transportation
model, supply equals
demand such that all
constraints are
equalities; in an
"**unbalanced**"
transportation
model, supply does*

*not equal demand,
and one set of
constraints is $\leq$.*

The three demand constraints are developed in the same way as the supply constraints, except that the variables summed are the number of television sets supplied from each of the three warehouses. Thus, the number shipped to one store is the sum of the shipments from the three

warehouses. They are equalities because all demands can be met:

$$x_{1A} + x_{2A} + x_{3A} = 150$$

$$x_{1B} + x_{2B} + x_{3B} = 250$$

$$x_{1C} + x_{2C} + x_{3C} = 200$$

Model Summary

The complete linear programming model for this problem is summarized as follows:

$$\begin{aligned} \text{minimize } Z = & \$16x_{1A} + 18x_{1B} \\ & + 11x_{1C} + 14x_{2A} + 12x_{2B} + \\ & 13x_{2C} + 13x_{3A} + 15x_{3B} + \\ & 17x_{3C} \end{aligned}$$

subject to

$$x_{1A} + x_{1B} + x_{1C} \leq 300$$

$$x_{2A} + x_{2B} + x_{2C} \leq 200$$

$$x_{3A} + x_{3B} + x_{3C} \leq 200$$

$$x_{1A} + x_{2A} + x_{3A} = 150$$

$$x_{1B} + x_{2B} + x_{3B} = 250$$

$$x_{1C} + x_{2C} + x_{3C} = 200$$

$$x_{ij} \geq 0$$

Computer Solution with Excel

The computer solution for this model was achieved using Excel and is shown in [Exhibit 4.15](#).

Notice that the objective function contained in cell C11 is shown on the formula bar at the top of the screen. The constraints for supply are included in cells F5, F6, and F7, whereas the demand constraints are in cells C8, D8, and E8. Thus, a typical supply constraint for Cincinnati would be **C5+D5+E5≤H5**. The Solver window is shown in [Exhibit 4.16](#).

Exhibit 4.15.

(This item is displayed on page 131 in the print version)

[\[View full size image\]](#)

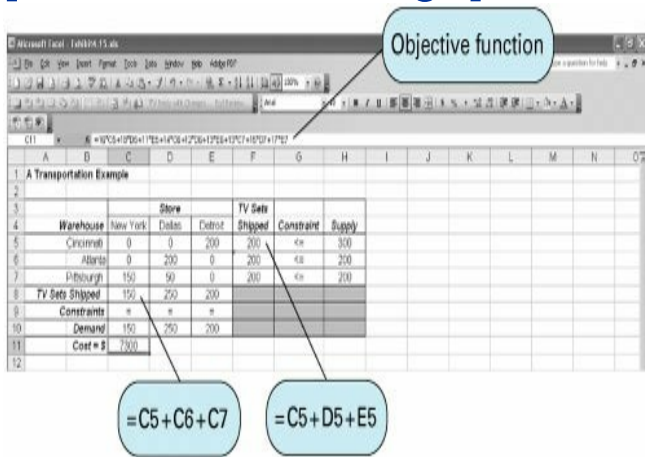


Exhibit 4.16.

(This item is displayed on page 131 in the print version)

[\[View full size image\]](#)

Decision variables

Demand constraints

Supply constraints

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

Solution Analysis

The solution is

$x_{1C} = 200$ TVs shipped from Cincinnati to Detroit

$x_{2B} = 200$ TVs shipped from Atlanta to Dallas

[Page 131]

150 TVs shipped from Pittsburgh to

x_{3A} = New York

x_{3B} = 50 TVs shipped from Pittsburgh to Dallas

Z = \$7,300 shipping cost

Note that the surplus for the first constraint, which is the supply at the Cincinnati warehouse, equals 100 TVs. This means that 100 TVs are left in the Cincinnati warehouse, which has

inventory and storage
implications for the manager.

This is an example of a type of linear programming model known as a *transportation problem*, which is the topic of [Chapter 6](#). Notice that all the solution values are integer values. This is always the case with transportation problems because of the unique characteristic that all the coefficient parameter values are ones and zeros.

A Blend Example

A petroleum company produces three grades of motor oil: super, premium, and extra from three components. The company wants to determine the optimal mix of the three components in each grade of motor oil that will maximize profit. The maximum quantities available of each component and their cost per barrel are as follows:

Component	Maximum Barrels Available/Day	Cost/Barrel
1	4,500	\$12
2	2,700	10
3	3,500	14

To ensure the appropriate blend, each grade has certain general specifications. Each grade must have a minimum amount of component 1 plus a

combination of other components, as follows:

Grade	Component Specifications	Selling Price/Barrel
Super	At least 50% of 1	\$23
	Not more than 30% of 2	
Premium	At least 40% of 1	20
	Not more than 25% of 3	

Extra	At least 60% of 1	18
	At least 10% of 2	

The company wants to produce at least 3,000 barrels of each grade of motor oil.

Decision Variables

The decision variables for this problem must specify the

quantity of each of the three components used in each grade of motor oil. This requires nine decision variables, as follows:

x_{ij} = barrels of component i used in motor oil grade j per day, where $i = 1, 2, 3$ and $j = s$ (super), p (premium), e (extra). For example, the amount of component 1 in super motor oil is x_{1s} . The total amount of each grade of motor oil will be

super:	$x_{1s} + x_{2s} + x_{3s}$
premium:	$x_{1p} + x_{2p} + x_{3p}$
extra:	$x_{1e} + x_{2e} + x_{3e}$

The Objective Function

The company's objective is to maximize profit. This requires that the cost of each barrel be subtracted from the revenue obtained from each barrel. Revenue is determined by multiplying the selling price of each grade of motor oil by the total barrels of each grade produced. Cost is achieved by multiplying the cost of each component by the total barrels of each component used:

$$\text{maximize } Z = \$23(x_{1s} + x_{2s} + x_{3s}) + 20(x_{1p} + x_{2p} + x_{3p}) + 18(x_{1e} + x_{2e} + x_{3e}) \\ - 12(x_{1s} + x_{1p} + x_{1e}) - 10(x_{2s} + x_{2p} + x_{2e}) - 14(x_{3s} + x_{3p} + x_{3e})$$

*"Profit" is maximized
in the objective
function by
subtracting cost from
revenue.*

[Page 133]

Combining terms results in the
following objective function:

$$\text{maximize } Z = 11x_{1s} + 13x_{2s} + 9x_{3s} + 8x_{1p} + 10x_{2p} + 6x_{3p} + 6x_{1e} + 8x_{2e} + 4x_{3e}$$

Model Constraints

This problem has several sets of constraints. The first set reflects the limited amount of each component available on a daily basis:

$$x_{1s} + x_{1p} + x_{1e} \leq 4,500 \text{ bbl.}$$

$$x_{2s} + x_{2p} + x_{2e} \leq 2,700 \text{ bbl.}$$

$$x_{3s} + x_{3p} + x_{3e} \leq 3,500 \text{ bbl.}$$

The next group of constraints is for the blend specifications for each grade of motor oil. The first specification is that super contain at least 50% of component 1, which is expressed as

$$\frac{x_{1s}}{x_{1s} + x_{2s} + x_{3s}} \geq 0.50$$

This constraint says that the ratio of component 1 in super to the total amount of super produced, $x_{1s} + x_{2s} + x_{3s}$, must be at least 50%. Rewriting this

constraint in a form more consistent with linear programming solution results in

Standard form requires that fractional relationships between variables be eliminated.

$$x_{1s} \geq 0.50(x_{1s} + x_{2s} + x_{3s})$$

and

$$0.50x_{1s} + 0.50x_{2s} + 0.50x_{3s} \geq 0$$

This is the general form a linear programming constraint must be in before you can enter it for computer solution. All variables are on the left-hand side of the inequality, and only numeric values are on the right-hand side.

The constraint for the other blend specification for super grade, not more than 30% of component 2, is developed in the same way:

$$\frac{x_{2s}}{x_{1s} + x_{2s} + x_{3s}} \leq 0.30$$

and

$$0.70x_{2s} - 0.30x_{1s} - 0.30x_{3s} \leq 0$$

The two blend specifications for premium motor oil are

$$60x_{1p} - 0.40x_{2p} - 0.40x_{3p} \geq 0$$

$$0.75x_{3p} - 0.25x_{1p} - 0.25x_{2p} \leq 0$$

The two blend specifications for extra motor oil are

$$0.40x_{1e} - 0.60x_{2e} - 0.60x_{3e} \geq 0$$

$$0.90x_{2e} - 0.10x_{1e} - 0.10x_{3e} \geq 0$$

The final set of constraints reflects the requirement that at least 3,000 barrels of each grade be produced:

$$x_{1s} + x_{2s} + x_{3s} \geq 3,000 \text{ bbl.}$$

$$x_{1p} + x_{2p} + x_{3p} \geq 3,000 \text{ bbl.}$$

$$x_{1e} + x_{2e} + x_{3e} \geq 3,000 \text{ bbl.}$$

[Page 134]

Model Summary

The complete linear programming model for this problem is summarized as follows:

maximize $Z = 11x_{1s} + 13x_{2s} + 9x_{3s} + 8x_{1p} + 10x_{2p} + 6x_{3p} + 6x_{1e} + 8x_{2e} + 4x_{3e}$
subject to

$$x_{1s} + x_{1p} + x_{1e} \leq 4,500$$

$$x_{2s} + x_{2p} + x_{2e} \leq 2,700$$

$$x_{3s} + x_{3p} + x_{3e} \leq 3,500$$

$$0.50x_{1s} - 0.50x_{2s} - 0.50x_{3s} \geq 0$$

$$0.70x_{2s} - 0.30x_{1s} - 0.30x_{3s} \leq 0$$

$$0.60x_{1p} - 0.40x_{2p} - 0.40x_{3p} \geq 0$$

$$0.75x_{3p} - 0.25x_{1p} - 0.25x_{2p} \leq 0$$

$$0.40x_{1e} - 0.60x_{2e} - 0.60x_{3e} \geq 0$$

$$0.90x_{2e} - 0.10x_{1e} - 0.10x_{3e} \geq 0$$

$$x_{1s} + x_{2s} + x_{3s} \geq 3,000$$

$$x_{1p} + x_{2p} + x_{3p} \geq 3,000$$

$$x_{1e} + x_{2e} + x_{3e} \geq 3,000$$

$$x_{ij} \geq 0$$

Computer Solution with Excel

The Excel spreadsheet solution for this blend example is shown in [Exhibit 4.17](#). Solver is shown in [Exhibit 4.18](#). The decision variables are located in cells **B7:B15** in [Exhibit 4.17](#). The total profit is computed in cell C16, using the formula **=SUMPRODUCT(B7:B15,C7:C** which is also shown on the formula bar at the top of the spreadsheet. The constraint formulas are embedded in cells

H6 through H17. Notice that we did not develop an array of constraint coefficients on the spreadsheet for this model; instead, we typed the constraint formulas directly into cells H6:H17, which seemed easier. For example, the constraint formula in cell H6 is **=B7+B10+B13**, and the constraint formula in cell H9 is **=.5*B7.5*B8 .5*B9**. The remaining cells in column H have similar constraint formulas.

Exhibit 4.17.

[\[View full size image\]](#)

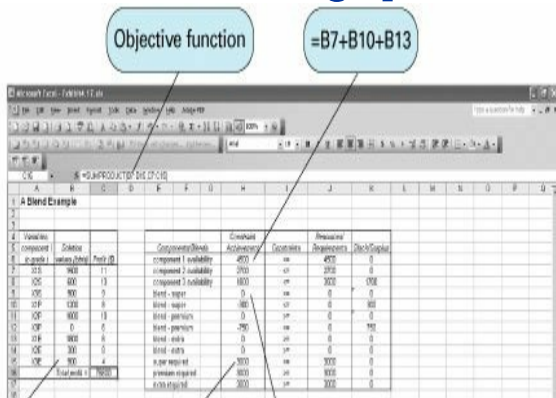


Exhibit 4.18.

Solver Parameters ✕

Set Target Cell: 

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells: 

Subject to the Constraints:

\$H\$10 <= \$J\$10

\$H\$11 >= \$J\$11

\$H\$12 <= \$J\$12

\$H\$13:\$H\$17 >= \$J\$13:\$J\$17

\$H\$6:\$H\$8 <= \$J\$6:\$J\$8

\$H\$9 = \$J\$9

^

v

Solution Analysis

The solution is

$$x_{1s} = 1,500 \text{ bbl.}$$

$$x_{2s} = 600 \text{ bbl.}$$

$$x_{3s} = 900 \text{ bbl.}$$

$$x_{1p} = 1,200 \text{ bbl.}$$

$$x_{2p} = 1,800 \text{ bbl.}$$

$$x_{1e} = 1,800 \text{ bbl.}$$

$$x_{2e} = 300 \text{ bbl.}$$

$$x_{3e} = 900 \text{ bbl.}$$

$$Z = \$76,800$$

Summarizing these results, 3,000 barrels of super grade, premium, and extra are produced. Also, 4,500 barrels of component 1, and 2,700 barrels of component 2, and 1,800 barrels of component 3 are used. (This problem also contains multiple optimal solutions.)

Exhibit 4.19 shows the sensitivity report for our Excel solution of the blend problem. Notice that the shadow price for component 1 is \$20. This indicates that component 1 is by far the most critical resource for increasing profit. For every additional barrel of component 1 that can be acquired, profit will increase by \$20. For example, if the current available amount of component 1 were increased from 4,500 barrels to 4,501 barrels, profit would increase from \$76,800 to \$76,820. (This dual price is

valid up to 6,200 barrels of component 1, the upper limit for the sensitivity range for this constraint quantity value.)

Exhibit 4.19.

(This item is displayed on page 136 in the print version)

[\[View full size image\]](#)

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
\$B\$7	X1S values (bbls)	1500	0	11	1E+30	3
\$B\$8	X2S values (bbls)	600	0	13	0	0
\$B\$9	X3S values (bbls)	900	0	9	0	0
\$B\$10	X1P values (bbls)	1200	0	8	3	1E+30
\$B\$11	X2P values (bbls)	1800	0	10	2	0
\$B\$12	X3P values (bbls)	0	0	6	0	1E+30
\$B\$13	X1E values (bbls)	1800	0	6	3	1E+30
\$B\$14	X2E values (bbls)	300	0	8	0	1E+30
\$B\$15	X3E values (bbls)	900	0	4	22.67	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$H\$13	blend - extra Achievement	0	-18	0	1.1E-05	8.5E+02
\$H\$14	blend - extra Achievement	0	0	0	6.0E+02	3.0E+02
\$H\$15	super required Achievement	3000	0	3000	2.2E-05	1.0E+30
\$H\$16	premium required Achievement	3000	-1	3000	2.7E-05	8.3E+02
\$H\$17	extra required Achievement	3000	-7	3000	1.8E-05	3.0E+03
\$H\$12	blend - premium Achievement	-750	0	0	1.0E+30	7.5E+02
\$H\$11	blend - premium Achievement	0	-18	0	1.1E-05	6.0E+02
\$H\$9	blend - super Achievement	0	-18	0	1.1E-05	8.5E+02
\$H\$10	blend - super Achievement	-300	0	0	1.0E+30	3.0E+02
\$H\$6	component 1 availability Achievement	4500	20	4500	1.7E+03	1.1E-05
\$H\$7	component 2 availability Achievement	2700	4	2700	3.0E+02	6.0E+02
\$H\$8	component 3 availability Achievement	1800	0	3500	1.0E+30	1.7E+03

The shadow price for component 1 is \$20.

The upper limit for the sensitivity range for component 1 is $4500 + 1700 = 6200$.

*Recall that the **shadow price** is the marginal economic value of one additional unit of a resource.*

In the refinery industry, different grade stocks of oil and gasoline are available based on the makeup and quality of the crude oil that is received. Thus,

as crude oil properties change, it is necessary to change blend requirements. Component availability changes as well. The general structure of this model can be used on a daily basis to plan production based on component availability and blend specification changes.

A Multiperiod Scheduling Example

PM Computer Services assembles its own brand of personal computers from component parts it purchases overseas and domestically. PM sells most of its computers locally to different departments at State University as well as to individuals and businesses in the immediate geographic region.

PM has enough regular production capacity to produce 160 computers per week. It can produce an additional 50 computers with overtime. The cost of assembling, inspecting, and packaging a computer during regular time is \$190. Overtime production of a computer costs \$260. Furthermore, it costs \$10 per computer per week to hold a computer in inventory for future delivery. PM wants to meet all customer orders, with no shortages, to provide quality

service. PM's order schedule for the next 6 weeks is as follows:

<i>Week</i>	<i>Computer Orders</i>
1	105
2	170
3	230
4	180
5	150

PM Computers wants to determine a schedule that will indicate how much regular and overtime production it will need each week to meet its orders at the minimum cost. The company wants no inventory left over at the end of the 6-week period.

Management Science

Application: Gasoline

Blending at Texaco

The petroleum industry first began using linear programming to solve gasoline blending problems in the 1950s. A single grade of gasoline can be a blend of from 3 to 10 different components. A typical refinery might have as many as 20 different components it blends into 4 or more grades of gasoline. Each grade of gasoline differs according to octane level, volatility, and area marketing requirements.

At Texaco, for example, the typical gasoline blends are Power and unleaded regular, Plus, and Power Premium. These different grades are

blended from a variety of available stocks that are intermediate refinery products, such as distilled gasoline, reformat gasoline, and catalytically cracked gasoline. Additives include, among other things, octane enhancers. As many as 15 stocks can be blended to yield up to 8 different blends. The properties or attributes of a blend are determined by a combination of properties that exist in the gasoline stocks and those of any additives. Examples of stock properties (that originally emanated from crude oil) include to some extent vapor pressure, sulfur content, aromatic content, and octane value, among other items. A linear programming model determines the volume of each blend, subject to constraints for stock availability, demand, and the property (or attribute) specifications

for each blend. A single blend may have up to 14 different characteristics. In a typical blend analysis involving 7 input stocks and 4 blends, the problem will include 40 variables and 71 constraints.



Sources: C. E. Bodington and T. E. Baker, "A History of Mathematical Programming in the Petroleum Industry," *Interfaces* 20, no. 4 (July/August 1990): 117-127; and C. W. DeWitt et al., "OMEGA: An

Improved Gasoline Blending System
for Texaco," *Interfaces* 19, no. 1
(January/February 1989): 85-101.

Decision Variables

This model consists of three different sets of decision variables for computers produced during regular time each week, overtime production each week, and extra computers carried over as inventory each week. This

results in a total of 17 decision variables. Because this problem contains a large number of decision variables, we will define them with names that will make them a little easier to keep up with:

r_j = regular production of computers per week j ($j = 1, 2, 3, 4, 5, 6$)

o_j = overtime production of computers per week j ($j = 1, 2, 3, 4, 5, 6$)

i_j = extra computers carried

over as inventory in week j ($j = 1, 2, 3, 4, 5$)

This will result in 6 decision variables for regular production (r_1, r_2 , etc.) and 6 decision variables for overtime production. There are only 5 decision variables for inventory because the problem stipulates that there is to be no inventory left over at the end of the 6-week period.

The Objective Function

The objective is to minimize total production cost, which is the sum of regular and overtime production costs and inventory costs:

$$\text{minimize } Z = \$190(r_1 + r_2 + r_3 + r_4 + r_5 + r_6) + 260(o_1 + o_2 + o_3 + o_4 + o_5 + o_6) + 10(i_1 + i_2 + i_3 + i_4 + i_5)$$

[Page 138]

Model Constraints

This model includes three sets of constraints. The first two sets reflect the available capacity for regular and overtime production, while the third set establishes the production schedule per week.

The available regular production capacity of 160 computers for each of the 6 weeks results in six constraints, as follows:

$$r_j \leq 160 \text{ computers in week } j (j = 1, 2, 3, 4, 5, 6)$$

The available overtime

production capacity of 50 computers per week also results in six constraints:

$$o_j \leq 50 \text{ computers in week } j (j = 1, 2, 3, 4, 5, 6)$$

The next set of six constraints shows the regular and overtime production and inventory necessary to meet the order requirement each week:

$$\text{week 1: } r_1 + o_1 + i_1 = 105$$

$$\text{week 2: } r_2 + o_2 + i_1 + i_2 = 170$$

$$\text{week 3: } r_3 + o_3 + i_2 + i_3 = 230$$

$$\text{week 4: } r_4 + o_4 + i_3 i_4 = 180$$

$$\text{week 5: } r_5 + o_5 + i_4 i_5 = 150$$

$$\text{week 6: } r_6 + o_6 + i_5 = 250$$

For example, 105 computers have been ordered in week 1. Regular production, r_1 , and overtime production, o_1 , will meet this order, whereas any extra production that might be needed in a future week to meet an order, i_1 , is subtracted because it does not go to meet the order. Subsequently, in week 2 the order of 110

computers is met by regular and overtime production, $r_2 + o_2$, plus inventory carried over from week 1, i_1 .

Model Summary

The complete model for this problem is summarized as follows:

$$\text{minimize } Z = \$190(r_1 + r_2 + r_3 + r_4 + r_5 + r_6) + \$260(o_1 + o_2 + o_3 + o_4 + o_5 + o_6) \\ + 10(i_1 + i_2 + i_3 + i_4 + i_5)$$

subject to

$$r_j \leq 160 \quad (j = 1, 2, 3, 4, 5, 6)$$

$$o_j \leq 50 \quad (j = 1, 2, 3, 4, 5, 6)$$

$$r_1 + o_1 - i_1 = 105$$

$$r_2 + o_2 + i_1 - i_2 = 170$$

$$r_3 + o_3 + i_2 - i_3 = 230$$

$$r_4 + o_4 + i_3 - i_4 = 180$$

$$r_5 + o_5 + i_4 - i_5 = 150$$

$$r_6 + o_6 + i_5 = 250$$

$$r_j, o_j, i_j \geq 0$$

Computer Solution with Excel

The Excel spreadsheet for this multiperiod scheduling example

is shown in [Exhibit 4.20](#), and the accompanying Solver window is shown in [Exhibit 4.21](#). The formula for the objective function is embedded in cell C13 and is shown on the formula bar at the top of the screen. The decision variables for regular production are included in cells **B6:B11**, whereas the variables for overtime production are in cells **D6:D11**. The computers available each week in column G includes the regular production plus overtime production plus the inventory

from the previous week. For example, cell G7 (computers available in week 2) has the formula =**B7+D7+I6**. Cell I7 (the inventory left over in week 2) includes the formula =**G7H7** (i.e., computers available minus computers ordered).

[Page 139]

Exhibit 4.20.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 4.21.xls

File Edit View Insert Format Tools Data Window Help

Formulas: =SUM(B6:D11)+H7*50000

1 A Multiperiod Scheduling Example

Week	Regular Production	Regular Capacity	Overtime Production	Overtime Capacity	Total Production	Computers Available	Computer Orders	Inventory
1	80	100	0	50	130	100	105	55
2	80	100	0	50	130	215	175	40
3	80	100	25	50	155	230	230	0
4	80	100	25	50	155	190	185	5
5	80	100	30	50	160	100	160	0
6	80	100	50	50	180	200	200	0
Total Cost =	270,000							

Decision variables for regular production-B6:B11

Decision variables for overtime production-D6:D11

=G7-H7

=B7+D7+I6; regular production + overtime production + inventory from previous week

Exhibit 4.21.

Solver Parameters ✕

Set Target Cell: 

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells: 

Subject to the Constraints:

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

The actual constraint for week 2 would be **G7 = H7**. It is also necessary to enter the

constraints for available regular and overtime production capacities into Solver. For example, the constraint for regular capacity would be **B6:B11 $\leq$ 160**.

Solution Analysis

The solution is

$r_1 = 160$ computers produced in regular time in week 1

$r_2 = 160$ computers produced in regular time in week 2

$r_3 = 160$ computers produced
in regular time in week 3

$r_4 = 160$ computers produced
in regular time in week 4

$r_5 = 160$ computers produced
in regular time in week 5

$r_6 = 160$ computers produced
in regular time in week 6

$o_3 = 25$ computers produced
with overtime in week 3

$o_4 = 20$ computers produced
with overtime in week 4

$o_5 = 30$ computers produced
with overtime in week 5

$o_6 = 50$ computers produced
with overtime in week 6

$i_1 = 55$ computers carried over
in inventory in week 1

$i_2 = 45$ computers carried over
in inventory in week 2

$i_5 = 40$ computers carried over
in inventory in week 5

PM Computers must operate with regular production at full capacity during the entire 6-week period. It must also resort to overtime production in weeks 3, 4, 5, and 6 in order to meet heavier order volume in weeks 3 through 6.

As each week passes, PM Computers can use this model to update its production schedule should order sizes change. For example, if PM is in week 2, and the orders for week 5 change from 150 computers to 200, the model can be altered and a new

schedule computed. The model can also be extended further out into the planning horizon to develop future production schedules.

PM may want to consider expanding its regular production capacity because of the heavy overtime requirements in weeks 3 through 6. The dual values (from the sensitivity analysis report from the computer output) for the constraints representing regular production in weeks 3 through 6 show that for each capacity increase for

regular production time that will enable the production of one more computer, cost can be reduced by between \$70 and \$80. (This total reduction would occur if processing time could be reduced without acquiring additional labor, etc. If additional resources were acquired to achieve an increase in capacity, the cost would have to be subtracted from the shadow price.)

A Data Envelopment Analysis Example

Data envelopment analysis (DEA) is a linear programming application that compares a number of service units of the same type such as banks, hospitals, restaurants, and schools based on their inputs (resources) and outputs. The model solution result indicates whether a particular unit is less productive, or inefficient, compared to other units. For

example, DEA has compared hospitals where inputs include hospital beds and staff size and outputs include patient days for different age groups.

As an example of a DEA linear programming application, consider a town with four elementary schools Alton, Beeks, Carey, and Delancey. The state has implemented a series of standards of learning (SOL) tests in reading, math, and history that all schools are required to administer to all students in the fifth grade. The average test scores are a

measurable output of the school's performance. The school board has identified three key resources, or inputs, that affect a school's SOL scores: the teacher-to-student ratio, supplementary funds per student (i.e., funding generated by the PTA and other private sources over and above the normal budget), and the average educational level of the parents (where 12 = high school graduate, 16 = college graduate, etc.). These inputs and outputs are summarized as follows:

input 1 = teacher-to-student
ratio

input 2 = supplementary
funds/student

input 3 = average educational
level of parents

output 1 = average reading
SOL score

output 2 = average math SOL
score

output 3 = average history SOL
score

The actual input and output values for each school are

School	Inputs			Outputs		
	1	2	3	1	2	3
Alton	.06	\$260	11.3	86	75	71
Beeks	.05	320	10.5	82	72	67
Carey	.08	340	12.0	81	79	80
Delancey	.06	460	13.1	81	73	69

For example, at Alton, the teacher-to-student ratio is 0.06 (or approximately 16.67 students per teacher), there is \$260 per student supplemental funds, and the average parent educational grade level is 11.3. The average scores on Alton's reading, math, and history tests are 86, 75, and 71, respectively.

Management Science

Application: Analyzing Bank Branch Efficiency by Using DEA

Data envelopment analysis (DEA) is an application of linear programming that is used to determine the less productive (i.e., inefficient) service units among a group of similar service units. DEA bases this determination on the inputs (resources) and outputs of the service units.

A major northeastern bank, with 33 branches, used DEA to identify ways to improve its productivity. Bank management did not know how to identify the more efficient branches.

The bank identified five resource inputs available to all branches: customer service (tellers), sales service, management, expenses, and office space. Five groups of branch outputs were also identified: deposits, withdrawals, and checking; bank and traveler's checks and bonds; night deposits; loans; and new accounts. The DEA model found that only 10 branches were operating efficiently, whereas 23 branches were using excess resources to provide their volume and mix of services and were, thus, inefficient. The DEA model results provided the basis for reviewing and evaluating branch operations, which revealed operating differences between inefficient branches and best-practice branches. The bank was able to improve its branch productivity and profits substantially,

reducing its total branch staff by approximately 20% within 1 year of the completion of the DEA analysis and implementing changes in branch operations that resulted in annual savings of over \$6 million.



Source: H. D. Sherman and G. Ladino, "Managing Bank Productivity

Using Data Envelopment Analysis
(DEA)," *Interfaces* 25, no. 2
(March/April 1995): 6073.

The school board wants to identify the school or schools that are less efficient in converting their inputs to outputs relative to the other elementary schools in the town. The DEA linear programming model will compare one particular school with all the others. For a complete analysis, a separate

model is necessary for each school. For this example, we will evaluate the Delancey school as compared to the other schools.

[Page 142]

Decision Variables

The formulation of a DEA linear programming model is not as readily apparent as the previous example models in this chapter. It is particularly difficult to define the decision

variables in a meaningful way.

In a DEA model, the decision variables are defined as a price per unit of each output and each input. These are not the actual prices that inputs or outputs would be valued at. In economic terms, these prices are referred to as *implicit prices*, or *opportunity costs*.

These prices are a relative valuation of inputs and outputs among the schools. They do not have much meaning for us, as will become apparent as the model is developed. For now, the decision variables will

simply be defined as

x_i = a price per unit of each output where $i = 1, 2, 3$

y_i = a price per unit of each input where $i = 1, 2, 3$

The Objective Function

The objective of our model is to determine whether Delancey is efficient. In a DEA model, when we are attempting to determine whether a unit (i.e., a school) is efficient, it simplifies things to

scale the input prices so that the total value of a unit's inputs equals 1. The efficiency of the unit will then equal the value of the unit's outputs. For this example, this means that once we scale Delancey's input prices so that they equal 1 (which we will do by formulating a constraint), the efficiency of Delancey will equal the value of its outputs. The objective, then, is to maximize the value of Delancey's outputs, which also equals efficiency:

$$\text{maximize } Z = 81x_1 + 73x_2 + 69x_3$$

Because the school's inputs will be scaled to 1, the maximum value that Delancey's outputs, as formulated in the objective function, can take on is 1. If the objective function equals 1, the school is efficient; if the objective function is less than 1, the school is inefficient.

Model Constraints

The constraint that will scale

the Delancey school's inputs to 1 is formulated as

$$.06y_1 + 460y_2 + 13.1y_3 = 1$$

This sets the value of the inputs to 1 and ultimately forces the value of the outputs to 1 or less. This also has the effect of making the values of the decision variables even less meaningful.

The values of the outputs are forced to be 1 or less by the next set of constraints. In general terms, the efficiency of a school (or any productive

unit) can be defined as

$$\text{efficiency} = \frac{\text{value of outputs}}{\text{value of inputs}}$$

It is not possible for a school or any service unit to be more than 100% efficient; thus, the efficiency of the school must be less than or equal to 1 and

$$\frac{\text{value of school's outputs}}{\text{value of school's inputs}} \leq 1$$

[Page 143]

Converting this to standard linear form,

value of school's outputs $\leq$
value of school's inputs

Substituting the model's
decision variables and
parameters for inputs and
outputs into this general
constraint form results in four
constraints, one for each
school:

$$86x_1 + 75x_2 + 71x_3 \leq .06y_1 + 260y_2 + 11.3y_3$$

$$82x_1 + 72x_2 + 67x_3 \leq .05y_1 + 320y_2 + 10.5y_3$$

$$81x_1 + 79x_2 + 80x_3 \leq .08y_1 + 340y_2 + 12.0y_3$$

$$81x_1 + 73x_2 + 69x_3 \leq .06y_1 + 460y_2 + 13.1y_3$$

Model Summary

The complete linear programming model for determining the efficiency of the Delancey school is

maximize $Z = 81x_1 + 73x_2 + 69x_3$

subject to

$$.06y_1 + 460y_2 + 13.1y_3 = 1$$

$$86x_1 + 75x_2 + 71x_3 \leq .06y_1 + 260y_2 + 11.3y_3$$

$$82x_1 + 72x_2 + 67x_3 \leq .05y_1 + 320y_2 + 10.5y_3$$

$$81x_1 + 79x_2 + 80x_3 \leq .08y_1 + 340y_2 + 12.0y_3$$

$$81x_1 + 73x_2 + 69x_3 \leq .06y_1 + 460y_2 + 13.1y_3$$

$$x_i, y_i \geq 0$$

The objective of this model is to determine whether Delancey is inefficient: If the value of the objective function equals 1, the school is efficient; if it is less than 1, it is inefficient. As mentioned previously, the values of the decision variables, x_i and y_i , have little meaning for us. They are the

implicit prices of converting an input into an output, but they have been scaled to 1 to simplify the model. The model solution selects values of x_i and y_i that will maximize the school's efficiency, the maximum of which is 1, but these values have no easily interpretable meaning beyond that.

Computer Solution with Excel

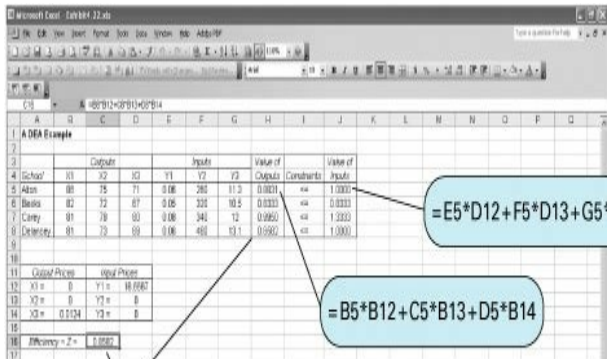
The Excel spreadsheet for this

DEA example is shown in [Exhibit 4.22](#). The Solver window is shown in [Exhibit 4.23](#). The decision variables are located in cells **B12:B14** and **D12:D14**. The objective function value, Z , which is also the indicator of whether Delancey is inefficient, is shown in cell C16. This value is computed using the formula for the value of the Delancey school's output, $=\mathbf{B8*B12+C8*B13+D8*D14}$, which is also shown in cell H8 and is a measure of the school's efficiency.

Exhibit 4.22.

(This item is displayed on page
143 in the print version)

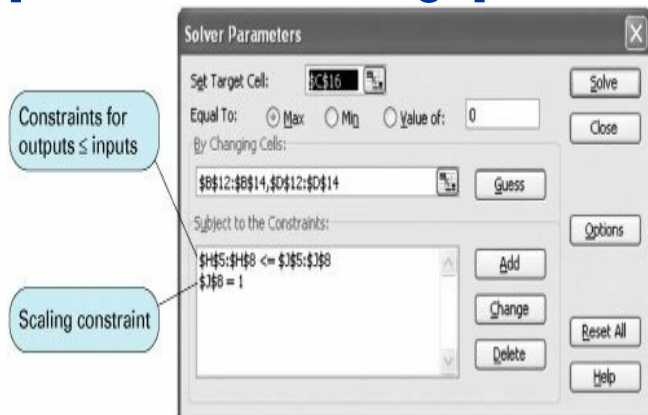
[\[View full size image\]](#)



Value of outputs, also in cell H8

Exhibit 4.23.

[\[View full size image\]](#)



The output and input values for each school are embedded in cells **H5:H8** and **J5:J8**, respectively. These values are

used to develop the model constraints shown in Solver as **H5:H8 ≤ J5:J8**. The scaling constraint is shown as **J8=1**, where the formula for the value of the inputs for the Delancey school, **=E8*D12+F8*D13+G8*D14**, is embedded in cell J8.

Solution Analysis

The only relevant solution value is the value of the objective function:

$$Z = 0.8582$$

Because this value is less than 1, the Delancey school is inefficient relative to the other schools. It is less efficient at converting its resources into outputs than the other schools. This means that a combination of efficient schools can achieve at least the same level of output as the Delancey school achieved with fewer input resources than required by the Delancey school. In retrospect, we can see that this is a logical result by looking back at the school's inputs and outputs.

While Delancey's input values are among the highest, its output test scores are among the lowest.

It is a simple process to access the efficiency of the other three schools in the town. For example, to ascertain the efficiency of the Alton school, make the value of its output in cell H5 the objective function in Solver, and make the formula for the value of the Alton school's inputs in cell I5 the scaling constraint, **I5=1**, in Solver. The efficiency of the other two schools can be

assessed similarly. Doing so indicates an objective function value of 1 for each of the three other schools, indicating that all three are efficient.

Summary

Generally, it is not feasible to attempt to "see" the whole formulation of the constraints and objective function at once, following the definition of the decision variables. A more prudent approach is to construct the objective function first (without direct concern for the constraints) and then to direct attention to each problem restriction and its corresponding model

constraint. This is a systematic approach to model formulation, in which steps are taken one at a time. In other words, it is important not to attempt to swallow the whole problem during the first reading.

[Page 145]

A systematic approach to model formulation is first to define decision variables, construct the objective function, and finally

*develop each
constraint
separately; don't try
to "see" it all at once.*

Formulating a linear programming model from a written problem statement is often difficult, but formulating a model of a "real" problem that has no written statement is even more difficult. The steps for model formulation described in this section are generally

followed; however, the problem must first be defined (i.e., a problem statement or some similar descriptive apparatus must be developed).

Developing such a statement can be a formidable task, requiring the assistance of many individuals and units within an organization.

Developing the parameter values that are presented as givens in the written problem statements of this chapter frequently requires extensive data collection efforts. The objective function and model

constraints can be very complex, requiring much time and effort to develop. Simply making sure that all the model constraints have been identified and no important problem restrictions have been omitted is difficult. Finally, the problems that one confronts in actual practice are typically much larger than those presented in this chapter. It is not uncommon for linear programming models of real problems to encompass hundreds of functional relationships and decision

variables. Unfortunately, it is not possible in a textbook to re-create a realistic problem environment with no written problem statement and a model of large dimensions. What is possible is to provide the fundamentals of linear programming model formulation and solution prerequisite to solving linear programming problems in actual practice.

References

See the references at the end of [Chapter 2](#).

Example Problem Solution

As a prelude to the problems, this section presents an example solution of the formulation and computer solution of a linear programming problem.

Problem Statement

Bark's Pet Food Company produces canned cat food called

Meow Chow and canned dog food called Bow Chow. The company produces the pet food from horse meat, ground fish, and a cereal additive. Each week the company has 600 pounds of horse meat, 800 pounds of ground fish, and 1,000 pounds of cereal additive available to produce both kinds of pet food. Meow Chow must be at least half fish, and Bow Chow must be at least half horse meat. The company has 2,250 16-ounce cans available each week. A can of Meow Chow earns \$0.80 in profit, and

a can of Bow Chow earns \$0.96 in profit. The company wants to know how many cans of Meow Chow and Bow Chow to produce each week in order to maximize profit.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve the model by using the computer

Solution

A. Model Formulation

Define the Decision Variables

This problem encompasses six decision variables, representing the amount of each ingredient i in pet food j :

x_{ij} = ounces of ingredient i in pet food j per week, where $i = h$ (horse meat), f (fish), and c (cereal), and $j =$

Step 1. m (Meow Chow) and b (Bow Chow)

[Page 146]

To determine the number of cans of each pet food produced per week, the total ounces of Meow Chow produced, $x_{hm} + x_{fm} + x_{cm}$, and the total amount of Bow Chow produced, $x_{hb} + x_{fb} + x_{cb}$, would each be divided by 16 ounces.

Formulate the Objective Function

The objective function is to maximize the total profit earned each week, which is determined by multiplying the number of cans of each pet food produced by the profit per can.

However, because the decision variables are defined in terms of ounces, they must be

Step 2. converted to equivalent cans by dividing by 16 ounces, as follows:

$$\text{maximize } Z = \frac{0.80}{16}(x_{hm} + x_{fm} + x_{cm}) + \frac{0.96}{16}(x_{hb} + x_{fb} + x_{cb})$$

or

$$\begin{aligned} \text{maximize } Z = \\ \$0.05(x_{hm} + x_{fm} + \\ x_{cm}) + 0.06(x_{hb} + x_{fb} \\ + x_{cb}) \end{aligned}$$

Formulate the Model Constraints

The first set of constraints represents the amount of each ingredient available each week. The problem provides these in terms of pounds of horse meat, fish, and cereal additives. Thus, because the decision variables are expressed as ounces, the ingredient amounts must be converted to

ounces by multiplying each pound by 16 ounces. This results in these three constraints:

$$x_{hm} + x_{hb} \leq 9,600 \text{ oz.}$$

of horse meat

$$x_{fm} + x_{fb} \leq 12,800 \text{ oz.}$$

of fish

$$x_{cm} + x_{cb} \leq 16,000 \text{ oz.}$$

of cereal additive

Next, there are two

recipe requirements specifying that at least half of Meow Chow be fish and at least half of Bow Chow be horse meat. The requirement for Meow Chow is formulated as

Step 3.

$$\frac{x_{fm}}{x_{hm} + x_{fm} + x_{cm}} \geq \frac{1}{2}$$

or

$$-x_{hm} + x_{fm} - x_{cm} \geq 0$$

The constraint for Bow Chow is developed similarly:

$$\frac{x_{hb}}{x_{hb} + x_{fb} + x_{cb}} \geq \frac{1}{2}$$

or

$$x_{hb} \geq x_{fb} + x_{cb}$$

Finally, the problem indicates that the company has 2,250 16-ounce cans available each week.

These cans must also be converted to ounces to conform to our decision variables, which results in the following constraint:

$$X_{hm} + X_{fm} + X_{cm} + X_{hb} + X_{fb} + X_{cb} \leq 36,000 \text{ oz.}$$

The Model Summary

The complete model is summarized as

maximize $Z = \$0.05x_{hm} + 0.05x_{fm} + 0.05x_{cm} + 0.06x_{hb} + 0.06x_{fb} + 0.06x_{cb}$
 subject to

$$x_{hm} + x_{hb} \leq 9,600$$

$$x_{fm} + x_{fb} \leq 12,800$$

$$x_{cm} + x_{cb} \leq 16,000$$

$$-x_{hm} + x_{fm} - x_{cm} \geq 0$$

[Page 147]

$$x_{hb} \quad x_{fb} \quad x_{cb} \geq 0$$

$$x_{hm} + x_{fm} + x_{cm} + x_{hb} + x_{fb} + x_{cb} \leq 36,000$$

$$x_{ij} \geq 0$$

B. Computer Solution

[\[View full size image\]](#)

Original Problem answers

Example Problem Solution

	x_{1m}	x_{2m}	x_{3m}	x_{4m}	x_{5m}	x_{6m}		RHS	Dual
Maximize	.05	.05	.05	.06	.06	.06			Max: 2500m +
Horse meat (oz)	1	0	0	1	0	0	≤	9,000	.02
Fish (oz)	0	1	0	0	1	0	≤	12,000	0
Cereal additive (oz)	0	0	1	0	0	1	≤	18,000	0
Meow Chow recipe	-1	1	-1	0	0	0	≥	0	0
Bow Chow recipe	0	0	0	1	-1	-1	≤	0	-.01
Cats (oz)	1	1	1	1	1	1	≤	36,000	.05
Solution =	0	8,400	8,400	9,600	4,400	5,200	Optimal Z =	1,992	$x_{1m} + x_{2m} +$

The computer solution for this model was generated by using the QM for Windows computer software package.

The solution is

$$x_{hm} = 0$$

$$x_{fm} = 8,400$$

$$x_{cm} = 8,400$$

$$x_{hb} = 9,600$$

Step 4.

$$x_{fb} = 4,400$$

$$x_{cb} = 5,200$$

$$Z = \$1,992$$

To determine the

number of cans of each pet food, we must sum the ingredient amounts for each pet food and divide by 16 ounces (the size of a can):

$$x_{hm} + x_{fm} + x_{cm} = 0 + 8,400 + 8,400 = 16,800 \text{ oz. of Meow Chow}$$

or

$$16,800 \div 16 = 1,050 \text{ cans of Meow Chow}$$

$$x_{hb} + x_{fb} + x_{cb} = 9,600 + 4,400 + 5,200 = 19,200 \text{ oz. of Bow Chow}$$

or

$$19,200 \div 16 = 1,200$$

cans of Bow Chow

Note that this model has multiple optimal solutions. An alternate optimal solution is $x_{fm} = 10,400$, $x_{cm} = 6,400$, $x_{hb} = 9,600$, and $x_{cb} = 9,600$. This converts to the same number of cans of each pet food; however, the ingredient mix per can

is different.

Problems

In the product mix example in this chapter, we considered operators who would reduce processing times. This would also increase the cost of each unit of product by the same amount (because an increase in cost is a linear change). Sensitivity analysis be evaluated using the same method? Should Quick-Screen be solved again? Should Quick-Screen be solved again?

In this problem, the profit per shirt is constant. The computer solution output shows that the profit per shirt is \$4.11. Quick-Screen decided to acquire extra T-shirts at an additional \$4.11 for each extra T-shirt. What is the maximum limit of T-shirts?

1. Quick-Screen decided to acquire extra T-shirts at an additional \$4.11 for each extra T-shirt. What is the maximum limit of T-shirts?

If Quick-Screen were to produce equal

company reformulate the linear program
new solution to this reformulated model

In the diet example in this chapter, what happens if we increase the minimum calorie requirement to 1000 calories?

2.

Increase the breakfast calorie requirement to 1000, establish upper limits on the servings of cereal and appetizing. Determine the solution

In the investment example in this chapter, what happens if we increase the requirement that the entire \$70,000 be invested in the amount available for investment?

3.

If the entire amount available for investment is increased by \$10,000 (to \$80,000), will the entire \$10,000 increase be used?

For the marketing example in this chapter, what happens if we increase the audience exposures to 1000?

4. If Biggs Department Store were to war of the three types of advertisements, reformulated? What would be the new

For the transportation example in this were to incur storage costs of \$9 at Ci would these storage costs be reflected problem, and what would the new solu

5. The Zephyr Television Company is cons new warehouse would have a supply o New York, \$9 to Dallas, and \$12 to De (ignoring the cost of leasing the wareh company will lease the new warehouse

If supply could be increased at any one would there be on the amount of the ir

For the blend example in this chapter, if grade of motor oil" were changed to e affect the optimal solution?

6. If the company could acquire more of c
What would be the effect on the total p

On their farm, the Friendly family grows three productsapple butter, applesauce, several local grocery stores, at craft fair Pumpkin Festival for 2 weeks in October their kitchen, their own labor time, and available, and it requires 3.5 hours to c cook 10 gallons of applesauce, and 2.8 of apple butter requires 1.2 hours of lal jelly requires 1.5 hours. The Friendly fa They produce about 6,500 apples each

7. 10-gallon batch of applesauce requires
After the products are canned, a batch a batch of applesauce will generate a s; generate sales revenue of \$155. The F butter, applesauce, and apple jelly to pr
-

1. Formulate a linear programming model
2. Solve the model by using the computer

1. If the Friendlys in [Proble](#) feed livestock, which the worth \$0.08 per apple in model and solution?

8. 2. Instead of feeding the let Friendlys are thinking about require 1.5 hours of cooking per batch, and it will sell Friendlys use all their apples their other three products

A hospital dietitian prepares breakfast r
of the dietitian's responsibility is to mak
A and B are met. At the same time, the
The main breakfast staples providing vi
vitamin requirements and vitamin contri

Vitamin Contrib

Vitamin	mg/Egg	mg/Bacon Strip
---------	--------	-------------------

9.

A	2	4
B	3	2

An egg costs \$0.04, a bacon strip cost
dietitian wants to know how much of e
daily vitamin requirements while minimi

1. Formulate a linear programming r
2. Solve the model by using the corr

Lakeside Boatworks is planning to man
recreational boatsa fishing (bass) boat,
selling price and variable cost for each t

Boat	Variable Cost
------	---------------

Bass	\$12,500
------	----------

Ski	8,500
-----	-------

10. Speed	13,700
-----------	--------

The company has incurred fixed costs of \$100,000 and begin production. Lakeside has also been required by the state in the region to provide a minimum of 70 boats. Alternatively, the company is unsure of the demand for boats. Production to no more than 120 of any type of boat. The number of boats that it must sell to break even.

1. Formulate a linear programming model.
2. Solve the model by using the corner-point method.

The Pyrotec Company produces three different types of fireworks. The products have the following resource requirements:

Resource

	Cost/Unit
Clock	\$7
Radio	10
Toaster	5

11.

The manufacturer has a daily production labor. Maximum daily customer demand
 Clocks sell for \$15, radios for \$20, and
 optimal product mix that will maximize

- 1.** Formulate a linear programming r
- 2.** Solve the model by using the cor

Betty Malloy, owner of the Eagle Tavern and she must determine how much beer to buy from Yodel, Shotz, and Rainwater. The cost per gallon follows:

Brand

Yodel

Shotz

12. Rainwater

The tavern has a budget of \$2,000 for beer. The cost per gallon for Yodel is at a rate of \$3.00 per gallon, Shotz at \$2.50 per gallon. Based on past football games, Betty has

be 400 gallons of Yodel, 500 gallons of the capacity to stock 1,000 gallons of wants to determine the number of galk maximize profit.

1. Formulate a linear programming r
2. Solve the model by using the cor

The Kalo Fertilizer Company produces t Growat plants in Fresno, California, and resources available to produce 5,000 p enough resources to produce 6,000 pc brand at each plant is as follows:

Product	Fresno
Super Two	\$2

13. Green Grow

2

The company has a daily budget of \$45,000. The company knows the maximum demand for Super Two and 7,000 pounds for Green Grow. Super Two and \$7 per pound for Green Grow. pounds of each brand of fertilizer to produce

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner point method.

Grafton Metalworks Company produces metal B, no more than 7% of metal C, according to the following specifications:

proportion of the four metals in each ore are provided in the following table:

Metal (%)					
Ore	A	B	C	D	
1	19	15	12	14	
2	43	10	25	7	
3	17	0	0	53	
4	20	12	0	18	
5	0	24	10	31	

14.

6 12 18 16 25

When the metals are processed and re-

The company wants to know the amount of each metal to use to minimize the cost per ton of the alloy.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner-point method.

The Roadnet Transport Company has eight trailer trucks from a bankrupt competitor that it purchased trucks at each of its shipping terminals. The company makes shipments from each of its terminals: Atlanta, and New York. Each truck is currently assigned to a terminal manager. Each terminal manager has each indicated the number of additional trucks that can be accommodated. St. Louis can accommodate 40 additional trucks, New York can accommodate 60 additional trucks, and

additional trucks. The company makes each warehouse to each terminal. The shipped, shipping costs, and transport i

15.

Warehouse	St. Louis	To
Charlotte	\$1,800	s
Memphis	1,000	
Louisville	1,400	

The company wants to know how mar

terminal) to maximize profit.

1. Formulate a linear programming r
2. Solve the model by using the cor

The Hickory Cabinet and Furniture Cor
in Greensboro, North Carolina. The plan
upholstery, and labor. The resource req
resources available weekly are as follow

Resource

	Wood (lb.)	Upho
--	------------	------

Sofa	7	
------	---	--

Table	5
-------	---

16. Chair	4
------------------	---

Total available resources	2,250
---------------------------	-------

The furniture is produced on a weekly basis. Each week, when it is shipped out. The warehouse has a limited capacity. Each sofa earns \$400 in profit, each table earns \$300. The manager wants to know how many pieces of each type of furniture to produce to maximize the total profit.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the graphical method.

Lawns Unlimited is a lawn care and maintenance company that installs new lawns as well as bare or damaged lawns. The company uses basic grass seed mixes it calls Home 1, Home 2, and Home 3. Home 1 is a mix of tall fescue, grass seed tall fescue, mustang fescue, and bermudagrass. Home 2 is a mix of tall fescue, grass seed tall fescue, mustang fescue, and bermudagrass. Home 3 is a mix of tall fescue, grass seed tall fescue, mustang fescue, and bermudagrass. The company also offers a variety of lawn care services including fertilization, aeration, and topdressing.

Mix

Home 1	No m
--------	------

	At lea
--	--------

Home 2	At lea
--------	--------

	At lea
--	--------

17.

	No m
--	------

Commercial 3

At least
70%

At least

The company believes it needs to have of Home 2 mix, and 2,400 pounds of C fescue costs the company \$1.70, a po bluegrass costs \$3.25. The company w grass seed to purchase to minimize co

- 1.** Formulate a linear programming r
- 2.** Solve this model by using the con

Alexandra Bergson has subdivided her 2 with three local farm families to operate plant three crops: corn, peas, and soyk

the capabilities of each local farmer. Plot
given in the following tables:

Plot

1

2

3

18.

Maximum Acre

Corn	900
Peas	700
Soybeans	1,000

Any of the three crops may be planted several restrictions on the farming operation. Further, to ensure that each potential and resources (which determine proportion of each plot to be under cultivation) each crop to plant on each plot to maximize

- 1.** Formulate a linear programming model
- 2.** Solve the model by using the corner

As a result of a recently passed bill, a corn

for programs and projects. It is up to the congressman to decide how to spend the money. The congressman has decided to spend the money because of their importance to his district: a water sanitation project, and a mobile library. He wants to spend the money in a manner that will please the voters and win the votes in the upcoming election. His statement shows that for every dollar spent for the various programs a

Program	Votes
Job training	0
Parks	0
Sanitation	0
Mobile library	0

In order also to satisfy several local infl obligated to observe the following guide

- None of the programs can receive
- The amount allocated to parks ca sanitation project and the mobile
- The amount allocated to job train sanitation project.

Any money not spent in the district will congressman wants to spend it all. The allocate to each program to maximize

- 1.** Formulate a linear programming r
- 2.** Solve the model by using the cor

Anna Broderick is the dietitian for the S to determine a nutritious lunch menu for guidelines for each lunch serving:

- Between 1,500 and 2,000 calories
 - At least 5 mg of iron
 - At least 20 but no more than 60 g of fat
 - At least 30 g of protein
 - At least 40 g of carbohydrates
 - No more than 30 mg of cholesterol
-

She selects the menu from seven basic contribution per pound and the cost as

		Calories (per lb.)	Iron (mg/lb.)	Protein (g/lb.)	Ca
20.	Chicken	520	4.4	17	
	Fish	500	3.3	85	
	Ground beef	860	0.3	82	
	Dried beans	600	3.4	10	
	Lettuce	50	0.5	6	
	Potatoes	460	2.2	10	

Milk (2%)	240	0.2	16
--------------	-----	-----	----

The dietitian wants to select a menu to total cost per serving.

- 1.** Formulate a linear programming r
- 2.** Solve the model by using the cor
- 3.** If a serving of each of the food ite
a half pound, what effect would th

The Midland Tool Shop has four heavy p
covers and housings for electronic cons
and are of different sizes. Currently the
contract calls for 400 units of product :
3. The time (in minutes) required for ea
follows:

Ma

Product	1	2
1	35	41
2	40	36
3	38	37

Machine 1 is available for 150 hours, machine 2 for 100 hours, machine 3 for 120 hours, and machine 4 for 250 hours. The production quantities for each product on each machine they are produced on, because the production cost per unit per machine for each product is such that

Product	1	2
1	\$7.8	\$7.8
2	6.7	8.9
3	8.4	8.1

The company wants to know how mar in order to maximize profit.

- 1. Formulate this problem as a linear
- 2. Solve the model by using the cor

The Cabin Creek Coal (CCC) Company and it supplies coal to four utility power coal from each mine to each plant, the demand at each plant are shown in the

Mine	Plant		
	1	2	3
1	\$7	\$9	\$10
2	9	7	8
3	11	14	5

Demand (tons)	110	160	90
------------------	-----	-----	----

- 22.** The cost of mining and processing coal and \$75 per ton at mine 3. The percent each mine is as follows:
-

Mine	% Ash
1	9
2	5
3	4

Each plant has different cleaning equipment and can process no more than 6% ash and 5% sulfur; plant 2 can process sulfur combined; plant 3 can have no more than 6% ash and sulfur combined. The company has to decide how much coal to produce at each mine and ship to each plant.

1. Formulate a linear programming model.
2. Solve this model by using the computer.

Ampco is a manufacturing company that produces parts from April through September. However, it can store the parts during this period, so it can produce in one month period. Following are Ampco's sales data.

Month	
-------	--

April	
-------	--

May

June

July

August

September

The rental agent Ampco is dealing with warehouse space. This schedule shows is. For example, if Ampco rents space for 100,000 square feet for 10 years, whereas if it rents the same space for 100,000 square feet for 10 years, it would be a different situation.

23.

Rental Period (months)

6

5

4

3

2

1

Ampco can rent any amount of warehouse space for any number of (whole) months. Ampco wants to determine a plan that will exactly meet its space needs each month.

- 1.** Formulate a linear programming model for Ampco's problem.
- 2.** Solve the model by using the computer.
- 3.** Suppose that Ampco were to relax its space requirements every month such that it would require 10% more space. How would this affect the optimal solution?

Brooks City has three consolidated high schools. The school board has partitioned the city into three districts, each with different high schools located in the central, west, and south districts, and the school board wants to determine the average distances from each school to the student population in each district are as follows:

	Distance (miles)	
	Central School	West School
North	8	11
24. South	12	9
East	9	16
West	8	
Central		8

The school board wants to determine the number of buses to assign to each school to minimize the total busin

1. Formulate a linear programming r
2. Solve the model by using the cor

1. In [Problem 24](#) the school board has decided that the east districts must be bused, the west, and central districts must all be bused. Formulate a linear programming model to reflect this and solve on the computer.

25.

2. The school board has further decided that the number of buses should be equal. Formulate this as a linear programming model and solve by using the computer.

The Southfork Feed Company makes a soybean meal, a soybean oil, and a vitamin supplement. T

of corn, 200 pounds of soybeans, and mix. The company has the following re

- At least 30% of the mix must be
- At least 20% of the mix must be
- The ratio of corn to oats cannot e
- The amount of oats cannot excee
- The mix must be at least 500 pou

26.

A pound of oats costs \$0.50; a pound pound of vitamin supplement, \$2.00. T pounds of each ingredient to put in the

- 1.** Formulate a linear programming r
- 2.** Solve the model by using the cor

The United Charities annual fund-raising
Donations are collected during the day
contact. The average donation resulting

Phone

Day \$2

Night 3

The charity group has enough donated
contacts during one day and night com
each type of interview are as follows:

27.

Phone (min

Day	6
-----	---

Night	5
-------	---

The charity has 20 volunteer hours available each night. The chairperson of the fund types of contacts to schedule in a 24-hour total donations.

- 1.** Formulate a linear programming model
- 2.** Solve the model by using the corner-point method

Ronald Thump is interested in expanding his business. He has determined three areas in which he might invest: (1) product development, (2) manufacturing operations, and (3) advertising and sales promotion program.

manner is expected to yield a return of in manufacturing operations improvement investment plus 30% (at the end of each year) and development would be for a 3-year period plus 50% (at the end of the 3-year period).

28. wishes to include the requirement that advertising and sales promotion program improvements, and at least \$50,000 of cash value of the initial \$500,000 investment at the beginning of the first year). Ronald wants to choose between the three alternatives, during each year.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner-point method.

Iggy Olweski, a professional football player, has started an insurance business. He plans to sell three types of insurance: health insurance, life insurance, and life insurance. The average annual premium for each type of insurance policy is as follows:

2. Solve the model by using the con

A publishing house publishes three wee
Surf's Up. Publication of one issue of ea
of production time and paper:

	Production (h
<i>Daily Life</i>	0.01
<i>Agriculture Today</i>	0.03
<i>Surf's Up</i>	0.02

30.

Each week the publisher has available 1 paper. Total circulation for all three mag company is to keep its advertisers. The for *Agriculture Today*, and \$1.50 for *St* that the maximum weekly demand for 2,000 issues; and for *Surf's Up*, 6,000 number of issues of each magazine to revenue.

1. Formulate a linear programming r
2. Solve the model by using the cor

The manager of a department store in amounts of advertising the store should radio station, television station, and ne describe their audiences. The television commercial, which costs \$15,000, wo breakdown of the audience is as follow

	Male
Senior	5,000
Young	5,000

The newspaper representative claims to reach 10,000 potential customers at a cost of \$4,000 per ad.

	Male
Senior	4,000
Young	2,000

The radio station representative says that the cost of 10 radio commercials, which costs \$6,000, is 1% of the total advertising cost as follows:

	Male
31. Senior	1,500
Young	4,500

The store has the following advertising

- Use at least twice as many radio

- Reach at least 100,000 customer
- Reach at least twice as many you
- Make sure that at least 30% of th

Available space limits the number of ne
optimal number of each type of advert

1. Formulate a linear programming r
2. Solve the model by using the cor
3. Suppose a second radio station a
its commercials, which cost \$7,50
demographic breakdown:

	Male
Senior	2,400

Young

4,000

If the store were to consider this station, would this affect the solution?

The Mill Mountain Coffee Shop blends coffee from three basic blends in 1-pound bags. Special blends different types of coffee to produce the specialty blends. The shop has the following blend recipe requirements.

Blend

Mix Requirements

Special

At least 40% Columbian,
30% mocha

Dark	At least 60% Brazilian, no more than 10% mild
------	---

32.

Regular	No more than 60% mild, at least 30% Brazilian
---------	---

The cost of Brazilian coffee is \$2.00 per pound, the cost of Columbian is \$2.90 per pound, the store has 110 pounds of Brazilian coffee, 70 pounds of Columbian coffee, and 150 pounds of mild coffee available per week. To make a blend it should prepare each week to maximize profit.

- 1.** Formulate a linear programming model.
- 2.** Solve this model by using the computer.

ToyZ is a large discount toy store in Valley Forge, PA. The store has a large inventory of toys and games. The store is planning to expand its inventory to include more toys and games. The store is planning to expand its inventory to include more toys and games.

the summer months that increase during summer and fall, the store must build up during the Christmas season. To purchase and build up inventory, the store's expenses are low, the store borrows money.

Following is the store's projected revenues (where revenues are received and bills

Month	Revenues
July	\$ 20,000
August	30,000
September	40,000
October	50,000

November

80,000

December

100,000

At the beginning of July, the store can't borrow more money at the current interest rate and must be paid back at the end of the year. The store is considering an early payment by paying back the loan early. The cost of borrowing is 5% interest per month.

Money borrowed on a monthly basis must be repaid at the end of the year. The store wants to borrow enough money to cover its needs at the lowest possible cost of borrowing.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve this model by using the computer.
- 3.** What would the effect be on the cost of borrowing if the interest rate for a 6-month loan from another bank were 4%?

The Skimmer Boat Company manufactures engines it specializes in marine engines. Skimmer June, and July:

Month

April

May

June

July

34.

Mar-gine usually manufactures and ships engines to customers who are due. However, from April through July, customers are not due, and it can manufacture only engines in January through July. Mar-gine has several alternative ways to produce engines. For example, it can produce up to 30 engines in January, February, and March, at a cost of \$50 per engine per month until the customer is due. For example, it could build an engine in January and ship it in February, or it could charge. Mar-gine can also manufacture engines on an overtime basis, with an additional cost of \$10 per engine. The least costly production schedule that meets the customer's demand is the one that

- 1.** Formulate a linear programming model for the problem.
- 2.** Solve this model by using the computer.
- 3.** If Mar-gine were able to increase the number of engines produced in March from 30 to 40 engines, what would be the change in the total cost?

The Donnor meat processing firm produces pork, and a cereal additive. The firm produces

meat. The company has the following a

Pounds/Day

Chicken	200
---------	-----

Beef	300
------	-----

Pork	150
------	-----

Cereal additive	400
-----------------	-----

Each type of wiener has certain ingredie

Specifications

35.

Regular	Not more than 10% b and pork combined
	Not less than 20% chicken
Beef	Not less than 75% be
All-meat	No cereal additive
	Not more than 50% b and pork combined

The firm wants to know how many poi

maximize profits.

1. Formulate a linear programming model
2. Solve the model by using the computer

Joe Henderson runs a small metal parts shop. He has a drill press, a lathe, and a grinder. Joe has three operators and three machines. However, each operator performs a different task. The shop has contracted to do a big job that requires the use of the various operators to perform the various tasks. The tasks are as follows:

Operator	Drill Press (min.)	Lathe (min.)
1	22	18

2

41

36.

3

25

Joe Henderson wants to assign one operator to each of three machines. The time for all three operators is minimized.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner point method.
- 3.** Joe's brother, Fred, has asked him to hire his sister-in-law, Kelly. Kelly can perform each of the three jobs. Should Joe hire his sister-in-law?

Green Valley Mills produces carpet at two plants. It ships carpet to two outlets in Chicago and Atlanta. The cost of shipping carpet from each of the two plants to the two warehouses is given below.

From

Chicago

St. Louis

\$40

37. Richmond

70

The plant at St. Louis can supply 250 tons of carpet per week. The plant at Chicago can supply 400 tons per week. The Chicago plant has a fixed cost of \$100,000. The outlet at Atlanta demands 350 tons of carpet. Determine the number of tons of carpet to ship from each plant to Atlanta to minimize the total shipping cost.

1. Formulate a linear programming model for this problem.
2. Solve the model by using the corner-point method.

38. Dr. Maureen Becker, the head administrator, wants to determine a schedule for nurses to maintain the demand for nurses throughout the day. During the day, the demand for nurses is divided into twelve 2-hour periods. The number of nurses required for each period is as follows: 12:00 A.M. to 2:00 A.M., 30 nurses; 2:00 A.M. to 4:00 A.M., 20 nurses; 4:00 A.M. to 6:00 A.M., 40 nurses; 6:00 A.M. to 8:00 A.M., 60 nurses; 8:00 A.M. to 10:00 A.M., 80 nurses; 10:00 A.M. to 12:00 P.M., 80 nurses. During the next four daytime periods, the demand for nurses decreases. Beginning with 12:00 P.M. and ending at 2:00 P.M., 60 nurses are required; from 2:00 P.M. to 4:00 P.M., 40 nurses are required; from 4:00 P.M. to 6:00 P.M., 20 nurses are required; and from 6:00 P.M. to 8:00 P.M., 10 nurses are required, respectively. A nurse reports for duty at the beginning of a period and works 8 consecutive hours. Dr. Becker wants to determine a nursing schedule that meets the requirements throughout the day while minimizing the number of nurses employed.
-

1. Formulate a linear programming model for this problem.
2. Solve this model by using the computer.

A manufacturer of bathroom fixtures produces a product that consists of three processes. Molding and smoothing can undergo each process in an hour is

Process

Molding

Smoothing

Painting

39.

(Note: The three processes are continuous. A product must be molded, smoothed or painted than have been molded, smoothed, and painted. The cost is \$5 for smoothing, and \$6.50 for painting. A total of 120 hours of labor is available.

bathtub requires 90 pounds of fiberglass available each week. Each company wants to know how many he profit.

1. Formulate a linear programming r
2. Solve the model by using the cor

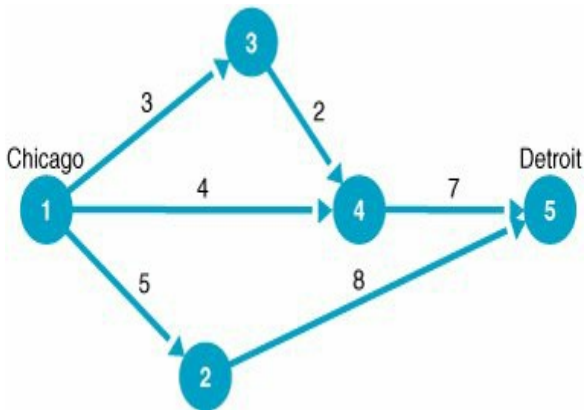
The admissions office at State University is planning the number of students entering freshman class. The university charges \$8,600 for an in-state student and \$19,000 for an out-of-state student. The university wants to maximize the money it receives from the freshman class subject to no more than 47% out-of-state students and at least 30% in-state students in its freshman class. The average SAT scores for last year's freshman class are the average SAT scores for last year's students in each college in the university. The average SAT scores for each college:

Average SAT S

40.	College	In-State	Out
	1. Architecture	1350	
	2. Arts and Sciences	1010	
	3. Agriculture	1020	
	4. Business	1090	
	5. Engineering	1360	
	6. Human Resources	1000	

- 1.** Formulate and solve a linear program for in-state students and out-of-state students that shows the optimal solution.
- 2.** If the solution in (a) does not achieve the goal, you might adjust the model to reflect the new information.

A manufacturing firm located in Chicago has two different routes available, as shown below.



Each circle in the network represents a junction. The number at the edge represents the cost of shipping 1 ton of product from junction to junction. Find the minimum cost of shipping 1 ton of product from Chicago to Detroit at the minimum cost.

- 1.** Formulate a linear programming model for this problem.
- 2.** Solve the model by using the corner point method.

A refinery blends four petroleum components to produce gasoline and diesel. The maximum quantities available are 1000 tons of gasoline and 800 tons of diesel. The maximum quantities available are 1000 tons of gasoline and 800 tons of diesel.

are as follows:

Component	Maximum Bar Available/Da
1	5,000
2	2,400
3	4,000
4	1,500

To ensure that each gasoline grade retail
put limits on the percentages of the co

selling prices for the various grades are

Grade

**Component
Specification**

42.

Regular

Not less than 40
1

Not more than 2
of 2

Not less than 30
3

Premium

Not less than 40
3

Diesel

Not more than 5

of 2

Not less than 10
1

The refinery wants to produce at least
Management wishes to determine the
maximize profit.

- 1.** Formulate a linear programming r
 - 2.** Solve the model by using the cor
-
-

The Cash and Carry Building Supply Co
three lengths:

Length

7 ft.

9 ft.

10 ft.

43.

The company has 25-foot standard-length boards that must be cut into the lengths needed. The company wishes to minimize the number of boards used. The company must therefore determine how to cut the boards to meet the requirements and minimize the number of boards used.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner-point method.

3. When a board is cut in a specific pattern, the loss of material is referred to as "trim loss." Reformulate the linear programming model so that the objective is to minimize the total number of boards used, and solve this model.

An investment firm has \$1 million to invest in real estate. The firm wishes to determine the optimal investment strategy to maximize the value at the end of 6 years.

Opportunities to invest in stocks and bonds are available over the next 6 years. Each dollar invested in stocks will return \$1.20 2 years later; the return can be immediately reinvested in bonds. Bonds will return \$1.40 3 years later; the return can be immediately reinvested in stocks.

Opportunities to invest in certificates of deposit are available at the end of the first and second years. Each dollar invested in a certificate of deposit will return \$1.05 at the end of the year.

44. Opportunities to invest in real estate will be available at the end of 1, 2, 3, and 4 years. Each dollar invested will return \$1.10 at the end of the year.

To minimize risk, the firm has decided that the amount invested in stocks cannot exceed 30% of the total amount invested. Investments must be in certificates of deposit for at least 2 years.

The firm's management wishes to determine alternatives that will maximize the amount of profit.

1. Formulate a linear programming model.
2. Solve the model by using the corner-point method.

The Jones, Jones, Smith, and Rodman companies are the only suppliers of the material. It will be able to buy and sell a certain quantity of the material at a certain price (c_i) and selling price (p_i) for each company.

	Montl	
	1	2
c_i	\$5	\$6
p_i	4	8

The firm's warehouse has a maximum capacity of 2,000 bushels in the first month, 2,000 bushels are in the warehouse at the end of the first month, and the amounts that should be bought and so that no storage costs are incurred and followed by purchases.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the corner-point method.

The production manager of Videotechnics Company indicates that in the upcoming 5-month production schedule, 2,000 recorders can be produced monthly on an overtime basis at a cost of \$20 per unit and 1,000 can be produced monthly on an overtime basis at a cost of \$15 for those produced in regular working hours and \$15 for those produced in overtime. The production costs per month are as follows:

Month

Co

1

2

3

4

5

46.

Inventory carrying costs are \$2 per rec
inventory carried over past the fifth mc
production that will minimize total prod

1. Formulate a linear programming model
2. Solve the model by using the corner-point method

The manager of the Ewing and Barnes store wants to assign the 10 employees to three departments in the store. The manager wants each of these departments to have at least four employees. Therefore, two departments will be assigned two employees and one department will be assigned six employees. Each employee has different daily sales each employee is expected to generate in each department.

Employee	Department	
	Lamps	Sports Equipment
1	\$130	275
2	275	\$130

47.

3

180

4

200

The manager wishes to know which en
maximize expected sales.

- 1.** Formulate a linear programming r
 - 2.** Solve the model by using the cor
 - 3.** Suppose that the department ma
department and to lay off the leas
programming model that reflects
-
-

Tidewater City Bank has four branches

input 1 = teller hours (100)

input 2 = space (100s ft.²)

input 3 = expenses (\$1,000s)

output₁ = deposits, withdrawals, and c

output₂ = loan applications

output₃ = new accounts (100s)

48.

The monthly output and input values fo

Outputs

Bank	1	2	3
A	76	125	12
B	82	105	8
C	69	98	9
D	72	117	14

Using data envelopment analysis (DEA)

Carillon Health Systems owns three hospitals. The hospitals are inefficient. The hospitals use (1) physician labor (1,000 hrs.), (2) nonphysician labor (1,000 hrs.), and (3) capital (100s) as inputs. The outputs are patient days (100s) for the elderly and (2) patient days (100s) for the under 65. The output and input values for the three hospitals are as follows:

Outputs

Hospital

1

2

3

A

9

5

18

B

6

8

9

49.

C

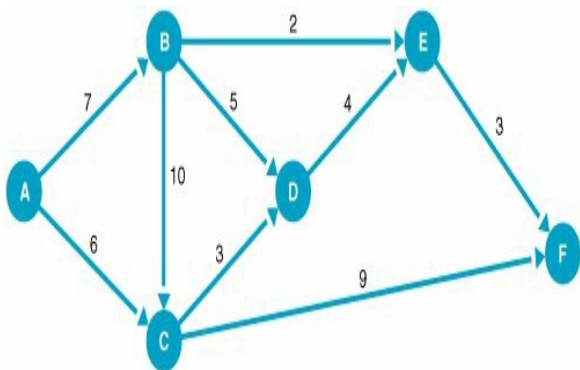
5

10

12

Using DEA, determine which hospitals a

Managers at the Transcontinent Shipping
tonnage of goods they can transport fr
cars on different rail routes linking these
the following diagram; all railroad cars ;



50.

The firm can transport a maximum amount. The maximum number of railroad cars shows the maximum tonnage that can be transported. Determine the maximum tonnage that can be transported.

1. Formulate a linear programming model.
2. Solve the model by using the corner-point method.

The law firm of Smith, Smith, Smith, and Smith, for the coming year. The firm has developed a schedule of casework it will need its new lawyers to handle.

Month	Casework (hr.)	Month
January	650	July

February	450	August
March	600	September
April	500	October
May	700	November
June	650	December

51.

Each new lawyer the firm hires is expected to work all year. All casework must be completed by the end of the year. The firm wants to know how many new lawyers

1. Formulate a linear programming model.
2. Solve this model by using the graphical method.

52. In [Problem 51](#) the optimal solution results in 10 lawyers being hired. Explain how you would change the number of lawyers hired with a whole (integer) number of lawyers between this new solution and the optimal solution.

The Goliath Tool and Machine Shop produces two types of subcomponents that are assembled to produce a final product. The subcomponents are manufactured in an operation that involves two machines. The time (in minutes per unit) for each machine to produce each subcomponent is given below.

		Production Time (minutes per unit)	
		Component 1	Component 2
Machine	Lathe	10	15
	Drill Press	15	10

53.

The shop splits the lathe workload evenly and the drill press workload evenly among the three presses. The quantities of components that will be produced are such that, on the average, no machine is idle while another machine operates.

The firm also wishes to produce a quarter of a million assembled products, without any particular restriction. The objective of the firm is to maximize the contribution.

- 1.** Formulate a linear programming model.
 - 2.** Solve the model by using the corner point method.
-

- 3.** The production policies established are not too restrictive. If the company were to produce more than a quarter of a million assembled products, the contribution would increase.

several viewer surveys that viewers do they focus on some segments more cl more attention to the local weather th watch the network news following the generated for commercials shown durin minute for local news and feature segre minute for sports, and \$1,000 per min

55. news is \$400 per minute, the cost for i is \$175 per minute, and for weather it's show for production costs. The station news and features must be at least 10 national news, sports, and weather mu more than 10 minutes. Commercial tir each of the four broadcast segment ty minutes of commercial time and broad sports, and weather to maximize adve

- 1.** Formulate a linear programming r
- 2.** Solve by using the computer.

The Douglas family raises cattle on thei which they grow ingredients for making

they sell in 16-ounce jars at local store chow-chow is \$2.25, and the profit for in each relish are cabbage, tomatoes, and 60% cabbage, 5% onions, and 10% tomatoes. The family has enough tin checking sales records for the past 5 years chow-chow than tomato relish. They want tomatoes, and 30 pounds of onions available many jars of relish to produce to maximize

- 56.** 10% onions. The family has enough tin checking sales records for the past 5 years chow-chow than tomato relish. They want tomatoes, and 30 pounds of onions available many jars of relish to produce to maximize
- 1.** Formulate a linear programming model.
 - 2.** Solve by using the computer.

The White Horse Apple Products Company produces applesauce and apple juice. It costs \$0.10 to produce a bottle of apple juice. The company has more than 60% of its output must be applesauce.

The company wants to meet but not exceed

manager estimates that the demand for additional 3 jars for each \$1 spent on an estimated to be 4,000 bottles, plus an apple juice. The company has \$16,000 and apple juice. Applesauce sells for \$1. The company wants to know how much advertising to spend on each to maximize

1. Formulate a linear programming model
2. Solve the model by using the corner-point method

Mazy's Department Store has decided to have a 24-hour operation. The store has divided the 24-hour day into six 4-hour periods. The personnel requirements for each period are given below.

Time

Midnight 4:00 A.M.

4:00 8:00 A.M.

8:00 Noon

58. Noon 4:00 P.M.

4:00 8:00 P.M.

8:00 Midnight

Personnel must report for work at the consecutive hours. The store manager to assign for each 4-hour segment to r

- 1.** Formulate a linear programming r
- 2.** Solve the model by using the cor

Venture Systems is a consulting firm that serves its clients. It has six available consultants. Consultants have different technical abilities, charge different hourly rates for their services, and are available for some projects more than others, and client suitability. The suitability of a consultant for a project is rated from 1 (the worst) to 5 (the best). The following table shows the suitability of each consultant for each project, as well as the hours available and maximum budget for each project.

Consultant	Hourly Rate	1	2	3
A	\$155	3	3	5
B	140	3	3	2

59.

C	165	2	1	3
D	300	1	3	1
E	270	3	1	1
F	150	4	5	3

Project Hours	500	240	400	4
---------------	-----	-----	-----	---

Contract Budget (\$1,000s)	100	80	120	9
-------------------------------	-----	----	-----	---

The company wants to know how many consultants to hire in order to best utilize the consultants' skills.

- 1.** Formulate a linear programming model.
- 2.** Solve the model by using the computer.
- 3.** If the company's objective is to maximize profit, and consultant compatibility, will the model be solved?

Great Northwoods Outfitters is a retail clothing and equipment. A phone station operators or temporary operators 8 hours experience and training, process more operators. However, temporary operators rate and they are not paid benefits. A full week, whereas a temporary operator (operator averages 1.1 defective orders 2.7 defective orders per week. The cost of staffing a station with of a station with part-time operators is forecasting techniques, the company has

week period, as follows:

60.

Week	Orders
1	19,500
2	21,000
3	25,600
4	27,200

The company does not want to hire or
the company wants a constant group of
company wants to determine how many
temporary operators to hire each week

costs.

1. Formulate a linear programming r
2. Solve this model by using the con

In [Problem 60](#) Great Northwoods Outfi
hiring a constant group of full-time ope
61. decided to hire and add full-time operat
it hires full-time operators, it will not dis
model to reflect this altered policy and

Blue Ridge Power and Light Company g
plants along the eastern seaboard in Vi
company purchases coal from six prod
Kentucky. Blue Ridge has fixed contract
producers:

Coal Producer	Tons	Co
---------------	------	----

ANCO	190,000
------	---------

Boone Creek	305,000
-------------	---------

Century	310,000
---------	---------

The power-producing capabilities of the to the quality of the coal. For example, per ton, while coal produced at Boone also purchases coal from three backup fixed contracts with these producers). more costly and lower grade:

Coal Producer	Available Tons	Cost
---------------	----------------	------

DACO	125,000	
------	---------	--

Eaton	95,000	
-------	--------	--

Franklin	190,000	
----------	---------	--

The demand for electricity at Blue Ridge requires approximately 10 million BTUs

62.	Power Plant	Electricity
-----	-------------	-------------

1. Afton		
----------	--	--

2. Surrey

3. Piedmont

4. Chesapeake

For example, the Afton plant must process 460,000 tons of coal annually, which translates to approximately 460,000 tons of coal processed annually.

Coal is primarily transported from the plant to the power plant. The cost of processing coal at each plant is different. The cost of processing costs for coal from each supplier is different.

Powe

Coal Producer	1. Afton	2. Surrey
ANCO	\$12.20	\$14.25
Boone Creek	10.75	13.70
Century	15.10	16.65
DACO	14.30	11.90
Eaton	12.65	9.35
Franklin	16.45	14.75

Formulate and solve a linear program
purchased and shipped from each supp

Valley United Soccer Club has 16 boys' to three town fields, which its teams pr enough to accommodate two teams a teams, while field 2 only has enough ro week, either on Monday and Wednesday Tuesday and Thursday from 3 P.M. to 5 of all the fields, so teams generally pref at field 3 because it can get crowded w to practice right after school, while the addition, some teams must practice lat work. Some teams also prefer specific Each team has been asked by the club and times, in priority order, and they ha

Team	1	
U11B	2, 35M	:
U11G	1, 35T	:
U12B	2, 35T	:
U12G	1, 35M	:
U13B	2, 35T	:
U13G	1, 35M	:
63. U14B	1, 57M	:

U14G	2, 35M	:
------	--------	---

U15B	1, 57T	:
------	--------	---

U15G	2, 57M	:
------	--------	---

U16B	1, 57T	:
------	--------	---

U16G	2, 57T	:
------	--------	---

U17B	2, 57M	:
------	--------	---

U17G	1, 57T	:
------	--------	---

U18B	2, 57M	:
------	--------	---

U18G	1, 57M	:
------	--------	---

For example, the under-11 boys' age group on Monday and Wednesday as its top priority, Wednesday as its second priority, and Saturday

Formulate and solve a linear programming problem and times, according to their priorities. top three selections? If not, how might you get to the best possible time and location? Can you assign teams to unacceptable locations? Can they only be at practice at 5 P.M.?

The city of Salem has four police stations

input 1 = number of police officers

input 2 = number of patrol vehicles

input 3 = space (100s ft.²)

output₁ = calls responded to (100s)

output₂ = traffic citations (100s)

output₃ = convictions

64.

The monthly output and input data for

	Outputs		
	1	2	3
Police Station			
A	12.7	3.6	35
B	14.2	4.9	42
C	13.8	5.2	56
D	15.1	4.2	39

Help the city council determine which o

USAir South Airlines operates a hub at Pittsburgh in the summer, the airline schedules 7 flights a day from Orlando to Pittsburgh, according to the following schedule.

Flight	Leave Pittsburgh	Arrive Orlando	Flight Number
1	6 A.M.	9 A.M.	A
2	8 A.M.	11 A.M.	B
3	9 A.M.	Noon	C
4	3 P.M.	6 P.M.	D
5	5 P.M.	8 P.M.	E

	6	7 P.M.	10 P.M.	F
65.	7	8 P.M.	11 P.M.	G
				H
				I
				J

The flight crews live in Pittsburgh or Orlando. A flight from Pittsburgh to Orlando and one flight from Orlando to Pittsburgh must return to its home city at the end of each day. For a flight from Orlando to Pittsburgh, it must then be scheduled for a flight from Pittsburgh to Orlando. A crew must have at least 1 hour of rest between flights. Some scheduling combinations are not possible. For example, a flight from Pittsburgh cannot return on flights A, B,

ferry one additional crew to a city in one of the cities. How many crews in that city.

The airline wants to schedule its crews to minimize the total ground time (i.e., the time the crew is on the ground) for a crew lengthens its workday, the airline. Formulate a linear programming problem and solve by using the computer. How much ground time will each crew experience?

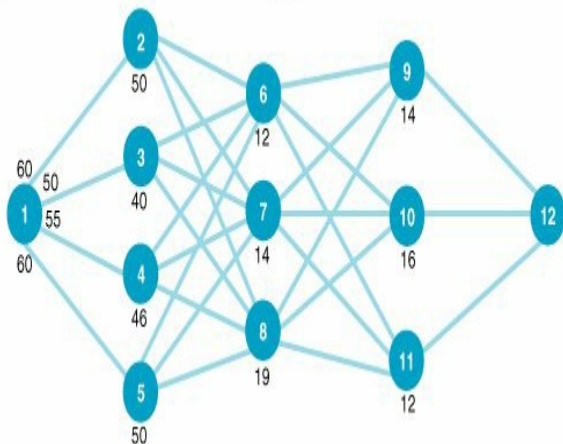
The National Cereal Company produces small pouches of four different cereals. Each cereal is produced at a different production facility, where the four different cereals are then sent to one of three distribution centers and shipped. The following diagram shows the production, packaging, and distribution.

Ingredients

Production

Packaging

Distribution

**66.**

Ingredients capacities (per 1,000 pouches) are shown at nodes 1, 2, 3, 4, and 5. For example, ingredients for 60,000 pouches are shown at node 1. The weekly production capacities (per 1,000 pouches) are shown at nodes 2, 3, 4, and 5. The weekly packaging capacities (per 1,000 pouches) are shown at nodes 6, 7, and 8. The weekly distribution capacities (per 1,000 pouches) are shown at nodes 9, 10, and 11. The weekly distribution costs (per 1,000 pouches) are shown at nodes 9, 10, and 11.

The various production, packaging, and distribution costs (per 1,000 pouches) are shown in the following table:

Facility	2	3	4	5	6	7
Unit cost	\$.17	.20	.18	.16	.26	.29

Weekly demand for the Light-Snak prod

Formulate and solve a linear program
produced at each facility to meet week

Valley Fruit Products Company has con
and New York to purchase apples that
Georgia, where they are processed into
gallons of apple juice. The juice is canne
truck to warehouses/distribution cente
costs per bushel from the farms to the
plants to the distribution centers are su

Farm	4. Indiana	5.
------	------------	----

1. Ohio	.41	
---------	-----	--

2. Pennsylvania	.37	
-----------------	-----	--

3. New York	.51	
-------------	-----	--

Plant Capacity	48,000	3
----------------	--------	---

67.

Distribution Costs		
Plant	6. Virginia	7. Iowa
4. Indiana	.22	
5. Georgia	.15	
Demand (gal.)	9,000	10,000

Formulate and solve a linear program that assigns the milk from the farms to the plants and from the plants to the markets so as to minimize the total shipping costs.

Case Problem

SUMMER SPORTS CAMP AT STATE UNIVERSITY

Mary Kelly is a scholarship soccer player at State University. During the summer, she works at a youth all-sports camp that several of the university's coaches operate. The sports camp runs for 8 weeks during July and August. Campers come for a 1-week period, during which time they

live in the State dormitories and use the State athletic fields and facilities. At the end of a week, a new group of kids comes in. Mary primarily serves as one of the camp soccer instructors. However, she has also been placed in charge of arranging for sheets for the beds the campers will sleep on in the dormitories. Mary has been instructed to develop a plan for purchasing and cleaning sheets each week of camp at the lowest possible cost.

Clean sheets are needed at the

beginning of each week, and the campers use the sheets all week. At the end of the week, the campers strip their beds and place the sheets in large bins. Mary must arrange either to purchase new sheets or to clean old sheets. A set of new sheets costs \$10. A local laundry has indicated that it will clean a set of sheets for \$4. Also, a couple of Mary's friends have asked her to let them clean some of the sheets. They have told her they will charge only \$2 for each set of sheets they clean. However, while the

laundry will provide cleaned sheets in a week, Mary's friends can only deliver cleaned sheets in 2 weeks. They are going to summer school and plan to launder the sheets at night at a neighborhood Laundromat.

The accompanying table lists the number of campers who have registered during each of the 8 weeks the camp will operate. Based on discussions with camp administrators from previous summers and on some old camp records and receipts, Mary estimates that each week

about 20% of the cleaned sheets that are returned will have to be discarded and replaced. The campers spill food and drinks on the sheets, and sometimes the stains do not come out during cleaning. Also, the campers occasionally tear the sheets, or the sheets get torn at the cleaners. In either case, when the sheets come back from the cleaners and are put on the beds, 20% are taken off and thrown away.

At the beginning of the summer, the camp has no sheets available, so initially

sheets must be purchased.
Sheets are thrown away at the
end of the summer.

Week	Registered Campers
1	115
2	210
3	250
4	230
5	260
6	300

7	250
8	190

Mary's major at State is management science, and she wants to develop a plan for purchasing and cleaning sheets by using linear programming. Help Mary formulate a linear programming model for this problem and solve it by using the computer.

Case Problem

SPRING GARDEN TOOLS

The Spring family has owned and operated a garden tool and implements manufacturing company since 1952. The company sells garden tools to distributors and also directly to hardware stores and home improvement discount chains. The Spring Company's four most popular small garden tools are a trowel, a hoe, a

rake, and a shovel. Each of these tools is made from durable steel and has a wooden handle. The Spring family prides itself on its high-quality tools.

The manufacturing process encompasses two stages. The first stage includes two operationsstamping out the metal tool heads and drilling screw holes in them. The completed tool heads then flow to the second stage, which includes an assembly operation where the handles are attached to the tool heads, a finishing

step, and packaging. The processing times per tool for each operation are provided in the following table:

[Page 177]

Tool (hr./unit)					Total Hours Available per Month
Operation	Trowel	Hoe	Rake	Shovel	
Stamping	0.04	0.17	0.06	0.12	500

Drilling	0.05	0.14		0.14	400
Assembly	0.06	0.13	0.05	0.10	600
Finishing	0.05	0.21	0.02	0.10	550
Packaging	0.03	0.15	0.04	0.15	500

The steel the company uses is ordered from an iron and steel works in Japan. The company has 10,000 square feet of sheet steel available each month. The metal required for each tool

and the monthly contracted production volume per tool are provided in the following table:

	Sheet Metal (ft.²)	Monthly Contracted Sales
Trowel	1.2	1,800
Hoe	1.6	1,400
Rake	2.1	1,600
Shovel	2.4	1,800

The primary reasons the company has survived and prospered are its ability always to meet customer demand on time and its high quality. As a result, the Spring Company will produce on an overtime basis in order to meet its sales requirements, and it also has a long-standing arrangement with a local tool and die company to manufacture its tool heads. The Spring Company feels comfortable subcontracting the first-stage

operations because it is easier to detect defects prior to assembly and finishing. For the same reason, the company will not subcontract for the entire tool because defects would be particularly hard to detect after the tool is finished and packaged. However, the company does have 100 hours of overtime available each month for each operation in both stages. The regular production and overtime costs per tool for both stages are provided in the following table:

	Stage 1		Stage 2	
	<i>Regular Cost</i>	<i>Overtime Cost</i>	<i>Regular Cost</i>	<i>Overtime Cost</i>
Trowel	\$ 6.00	\$ 6.20	\$3.00	\$3.10
Hoe	10.00	10.70	5.00	5.40
Rake	8.00	8.50	4.00	4.30
Shovel	10.00	10.70	5.00	5.40

The cost of subcontracting in

stage 1 adds 20% to the regular production cost.

The Spring Company wants to establish a production schedule for regular and overtime production in each stage and for the number of tool heads subcontracted, at the minimum cost. Formulate a linear programming model for this problem and solve the model using the computer. Which resources appear to be most critical in the production process?

Case Problems

Susan Wong's Personal Budgeting Model [\[1\]](#)

[1] This case was adapted from T. Lewis, "Personal Operations Research: Practicing OR on Ourselves," *Interfaces* 26, no. 5 (SeptemberOctober 1996): 3441.

After Susan Wong graduated from State University with a degree in management science, she went to work for a computer systems development

firm in the Washington, D.C., area. As a student at State, Susan paid her normal monthly living expenses for apartment rent, food, and entertainment out of a bank account set up by her parents. Each month they would deposit a specific amount of cash into Susan's account. Her parents also paid her gas, telephone, and bank credit card bills, which were sent directly to them. Susan never had to worry about things like health, car, homeowners', and life insurance; utilities; driver's and car licenses; magazine

subscriptions; and so on. Thus, while she was used to spending within a specific monthly budget in college, she was unprepared for the irregular monthly liabilities she encountered once she got a job and was on her own.

In some months Susan's bills would be modest and she would spend accordingly, only to be confronted the next month with a large insurance premium, or a bill for property taxes on her condominium, or a large credit card bill, or a bill for a magazine subscription, and so

on the next month. Such unexpected expenditures would result in months when she could not balance her checking account; she would have to pay her bills with her bank credit card and then pay off her accumulated debt in installments while incurring high interest charges. By the end of her first year out of school, she had hoped to have some money saved to begin an investment program, but instead she found herself in debt.

Frustrated by her predicament,

Susan decided to get her financial situation in order. First, she sold the condominium that her parents had helped her purchase and moved into a cheaper apartment. This gave her enough cash to clear her outstanding debts, with \$3,800 left over to start the new year with. Susan then decided to use some of the management science she had learned in college to help develop a budget. Specifically, she decided to develop a linear programming model to help her decide how much she should

put aside each month in short-term investments to meet the demands of irregular monthly liabilities and save some money.

[Page 178]

First, Susan went through all her financial records for the year and computed her expected monthly liabilities for the coming year, as shown in the following table:

Month	Bills	Month	Bills
-------	-------	-------	-------

January	\$2,750	July	\$3,050
February	2,860	August	2,300
March	2,335	September	1,975
April	2,120	October	1,670
May	1,205	November	2,710
June	1,600	December	2,980

Susan's after-taxes-and-benefits salary is \$29,400 per

year, which she receives in 12 equal monthly paychecks that are deposited directly into her bank account.

Susan has decided that she will invest any money she doesn't use to meet her liabilities each month in either 1-month, 3-month, or 7-month short-term investment vehicles rather than just leaving the money in an interest-bearing checking account. The yield on 1-month investments is 6% per year nominal; on 3-month investments, the yield is 8% per year nominal; and on a 7-

month investment, the yield is 12% per year nominal. As part of her investment strategy, any time one of the short-term investments comes due, she uses the principal as part of her budget, but she transfers any interest earned to another long-term investment (which she doesn't consider in her budgeting process). For example, if she has \$100 left over in January that she invests for 3 months, in April, when the investment matures, she uses the \$100 she originally invested in her

budget, but any interest on the \$100 is invested elsewhere. (Thus, the interest is not compounded over the course of the year.)

Susan wants to develop a linear programming model to maximize her investment return during the year so she can take that money and reinvest it at the end of the year in a longer-term investment program. However, she doesn't have to confine herself to short-term investments that will all mature by the end of the year; she can

continue to put money toward the end of the year in investments that won't mature until the following year. Her budgeting process will continue to the next year, so she can take out any surplus left over after December and reinvest it in a long-term program if she wants to.

- 1.** Help Susan develop a model for the year that will meet her irregular monthly financial obligations while achieving her investment objectives and solve the model.

2. If Susan decides she doesn't want to include all her original \$3,800 in her budget at the beginning of the year, but instead she wants to invest some of it directly in alternative longer-term investments, how much does she need to develop a feasible budget?

Case Problem

WALSH'S JUICE COMPANY

Walsh's Juice Company produces three products from unprocessed grape juice: bottled juice, frozen juice concentrate, and jelly. It purchases grape juice from three vineyards near the Great Lakes. The grapes are harvested at the vineyards and immediately converted into juice at plants at the vineyard sites and stored there in

refrigerated tanks. The juice is then transported to four different plants in Virginia, Michigan, Tennessee, and Indiana, where it is processed into bottled grape juice, frozen juice concentrate, and jelly. Vineyard output typically differs each month in the harvesting season, and the plants have different processing capacities.

In a particular month the vineyard in New York has 1,400 tons of unprocessed grape juice available, whereas the vineyard in Ohio has 1,700 tons and the

vineyard in Pennsylvania has 1,100 tons. The processing capacity per month is 1,200 tons of unprocessed juice at the plant in Virginia, 1,100 tons of juice at the plant in Michigan, 1,400 tons at the plant in Tennessee, and 1,400 tons at the plant in Indiana. The cost per ton of transporting unprocessed juice from the vineyards to the plant is as follows:

	Plant
--	-------

Vineyard	
----------	--

	Virginia Michigan Tennessee I
--	-------------------------------

New York	\$850	\$720	\$910
Pennsylvania	970	790	1,050
Ohio	900	830	780

The plants are different ages, have different equipment, and have different wage rates; thus, the cost of processing each product at each plant (\$/ton) differs, as follows:

Product	Plant			
	Virginia	Michigan	Tennessee	Indiana
Juice	\$2,100	\$2,350	\$2,200	\$1,950
Concentrate	4,100	4,300	3,950	3,900
Jelly	2,600	2,300	2,500	2,400

This month the company needs

to process a total of 1,200 tons of bottled juice, 900 tons of frozen concentrate, and 700 tons of jelly at the four plants combined. However, the production process for frozen concentrate results in some juice dehydration, and the process for jelly includes a cooking stage that evaporates water content. To process 1 ton of frozen concentrate requires 2 tons of unprocessed juice; 1 ton of jelly requires 1.5 tons of unprocessed juice; and 1 ton of bottled juice requires 1 ton of unprocessed juice.

Walsh's management wants to determine how many tons of grape juice to ship from each of the vineyards to each of the plants and the number of tons of each product to process at each plant. Thus, management needs a model that includes both the logistical aspects of this problem and the production processing aspects. It wants a solution that will minimize total costs, including the cost of transporting grape juice from the vineyards to the plants and the product processing costs. Help Walsh's

solve this problem by
formulating a linear
programming model and solve
it by using the computer.

Case Problems

The King's Landing Amusement Park

King's Landing is a large amusement theme park located in Virginia. The park hires high school and college students to work during the summer months of May, June, July, August, and September. The student employees operate virtually all the highly mechanized, computerized

rides; perform as entertainers; perform most of the custodial work during park hours; make up the workforce for restaurants, food services, retail shops, and stores; drive trams; and park cars. Park management has assessed the park's monthly needs based on previous summers' attendance at the park and the expected available workforce. Park attendance is relatively low in May, until public schools are out, and then it increases through June, July, and August, and decreases dramatically in

September, when schools reopen after Labor Day. The park is open 7 days a week through the summer, until September, when it cuts back to weekends only. Management estimates that it will require 22,000 hours of labor in each of the first 2 weeks of May, 25,000 hours during the third week of May, and 30,000 hours during the last week in May. During the first 2 weeks of June, it will require at least 35,000 hours of labor and 40,000 hours during the last 2 weeks in June. In July 45,000

hours will be required each week, and in August 45,000 hours will be needed each week. In September the park will need only 12,000 hours in the first week, 10,000 hours in each of the second and third weeks, and 8,000 hours the last week of the month.

The park hires new employees each week from the first week in May through August. A new employee mostly trains the first week by observing and helping more experienced employees; however, he or she works approximately 10 hours under

the supervision of an experienced employee. An employee is considered experienced after completing 1 week on the job. Experienced employees are considered part-time and are scheduled to work 30 hours per week in order to eliminate overtime and reduce the cost of benefits, and to give more students the opportunity to work. However, no one is ever laid off or will be scheduled for fewer (or more) than 30 hours, even if more employees are available than needed. Management believes

this is a necessary condition of employment because many of the student employees move to the area during the summer just to work in the park and live near the beach nearby. If these employees were sporadically laid off and were stuck with lease payments and other expenses, it would be bad public relations and hurt employment efforts in future summers. Although no one is laid off, 15% of all experienced employees quit each week for a variety of reasons, including homesickness, illness, and

other personal reasons, plus some are asked to leave because of very poor job performance.

Park management is able to start the first week in May with 700 experienced employees who worked in the park in previous summers and live in the area. These employees are generally able to work a lot of hours on the weekends and then some during the week; however, in May attendance is much heavier on the weekends, so most of the labor hours are needed then. The park expects

to have a pool of 1,500 available applicants to hire for the first week in May. After the first week, the pool is diminished by the number of new employees hired the previous week, but each week through June the park gets 200 new job applicants, which decreases to 100 new applicants each week for the rest of the summer. For example, the available applicant pool in the second week in May would be the previous week's pool, which in week 1 is 1,500, minus the

number of new employees hired in week 1 plus 200 new applicants. At the end of the last week in August, 75% of all the experienced employees will quit to go back to school, and the park will not hire any new employees in September. The park must operate in September, using experienced employees who live in the area, but the weekly attrition rate for these employees in September drops to 10%.

Formulate and solve a linear programming model to assist the park's management to plan

and schedule the number of new employees it hires each week in order to minimize the total number of new employees it must hire during the summer.

Chapter 5. Integer Programming

[Page 181]

In the linear programming models formulated and solved in the previous chapters, the implicit assumption was that solutions could be fractional or real numbers (i.e., non-integer). However, non-integer solutions are not always practical.

When only integer solutions

are practical or logical, it is sometimes assumed that noninteger solution values can be "rounded off" to the nearest feasible integer values. This method would cause little concern if, for example, $x_1 = 8,000.4$ nails were rounded off to 8,000 nails because nails cost only a few cents apiece. However, if we are considering the production of jet aircraft and $x_1 = 7.4$ jet airliners, rounding off could affect profit (or cost) by millions of dollars. In this case we need to solve the problem so that an *optimal*

integer solution is guaranteed.
In this chapter the different
forms of integer linear
programming models are
presented.

Integer Programming Models

There are three basic types of integer linear programming models: a total integer model, a 0-1 integer model, and a mixed integer model. In a **total integer model**, all the decision variables are required to have integer solution values. In a **0-1 integer model**, all the decision variables have integer values of zero or one. Finally, in a **mixed integer model**, some

of the decision variables (but not all) are required to have integer solutions. The following three examples demonstrate each of these types of integer programming models.

The three types of integer programming models are total, 01, and mixed.

A Total Integer Model

Example

The owner of a machine shop is planning to expand by purchasing some new machines presses and lathes. The owner has estimated that each press purchased will increase profit by \$100 per day and each lathe will increase profit by \$150 daily. The number of machines the owner can purchase is limited by the cost of the machines and the available floor space in the shop. The machine purchase prices and space requirements

are as follows:

Machine	Required Floor Space (ft. ²)	Purchase Price
Press	15	\$8,000
Lathe	30	4,000

*In a **total integer model**, all decision variables have integer solution*

values.

The owner has a budget of \$40,000 for purchasing machines and 200 square feet of available floor space. The owner wants to know how many of each type of machine to purchase to maximize the daily increase in profit.

The linear programming model for an integer programming problem is formulated in

exactly the same way as the linear programming examples in [Chapters 2, 3, and 4](#). The only difference is that in this problem, the decision variables are restricted to integer values because the owner cannot purchase a fraction, or portion, of a machine. The linear programming model follows:

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$\$8,000x_1 + 4,000x_2 \leq \$40,000$$

$$15x_1 + 30x_2 \leq 200 \text{ ft.}^2$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

where

x_1 = number of presses

x_2 = number of lathes

The decision variables in this model are restricted to whole machines. The fact that *both* decision variables, x_1 and x_2 , can assume any integer value greater than or equal to zero is what gives this model its designation as a total integer model.

A 01 Integer Model Example

A community council must decide which recreation facilities to construct in its community. Four new recreation facilities have been proposed: a swimming pool, a tennis center, an athletic field, and a gymnasium. The council wants to construct facilities that will maximize the expected daily usage by the residents of the community, subject to land and cost limitations. The expected daily usage and cost and land requirements for each facility follow:

Recreation Facility	Expected Usage (people/day)	Cost	Land Required (ac)
Swimming pool	300	\$35,000	10
Tennis center	90	10,000	3
Athletic field	400	25,000	15
Gymnasium	150	90,000	12

*In a **01** integer*

model, the solution values of the decision variables are zero or one.

The community has a \$120,000 construction budget and 12 acres of land. Because the swimming pool and tennis center must be built on the same part of the land parcel, however, only one of these two facilities can be constructed. The council wants to know

which of the recreation facilities to construct to maximize the expected daily usage. The model for this problem is formulated as follows:

$$\text{maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

subject to

$$\$35,000x_1 + 10,000x_2 + 25,000x_3 + 90,000x_4 \leq \$120,000$$

$$4x_1 + 2x_2 + 7x_3 + 3x_4 \leq 12 \text{ acres}$$

$$x_1 + x_2 \leq 1 \text{ facility}$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

where

x_1 = construction of a swimming pool

x_2 = construction of a tennis center

x_3 = construction of an athletic field

x_4 = construction of a gymnasium

In this model, the decision variables can have a solution value of either *zero* or *one*. If a facility is not selected for construction, the decision variable representing it will have a value of zero. If a facility is selected, its decision

variable will have a value of one.

The last constraint, $x_1 + x_2 \leq 1$, reflects the *contingency* that either the swimming pool (x_1) or the tennis center (x_2) can be constructed, but not both. In order for the sum of x_1 and x_2 to be less than or equal to one, either of the variables can have a value of one, or both variables can equal zero. This is also referred to as a *mutually exclusive constraint*.

If the community had specified

that either the swimming pool (x_1) or the tennis center (x_2) *must* be built, but not both, then the last constraint would become an equation, $x_1 + x_2 = 1$. This would result in a solution that would include $x_1 = 1$ or $x_2 = 1$, but both would not equal one (nor would both equal zero). In this manner, the model forces a choice between the two facilities. For this reason, it is often called a *multiple-choice constraint*.

A variation of the multiple-choice constraint can be used to formulate a situation where some specific number of facilities out of the total must be constructed. For example, if the community council had specified that exactly two of the four facilities must be built, this constraint would be formulated as

$$x_1 + x_2 + x_3 + x_4 = 2$$

If, alternatively, the council had specified that no more than two facilities must be constructed, the constraint would be

$$x_1 + x_2 + x_3 + x_4 \leq 2$$

Another type of 01 model constraint is a *conditional constraint*. In a conditional constraint, the construction of one facility would be conditional upon the construction of another.

Suppose, for example, that the pet project of the head of the community council is the swimming pool, and she also believes the tennis center is frivolous. The council head is very influential, so the rest of the council knows that the

tennis center has no chance of being selected if the pool is not selected first. However, even if the pool is selected, there is no guarantee that the tennis center will also be selected. Thus, the tennis center (x_2) is conditional upon construction of the swimming pool (x_1). This condition is formulated as

$$x_2 \leq x_1$$

Notice that the tennis center (x_2) cannot equal one (i.e., be selected) unless the pool (x_1) equals one. If the pool (x_1)

equals zero (i.e., it is not selected), then the tennis center (x_2) must also equal zero. However, this condition does allow the pool (x_1) to equal one and be selected and the tennis center to equal zero and not be selected.

A variation of this type of conditional constraint is the *corequisite constraint*, wherein if one facility is constructed, the other one would also be constructed and vice versa. For example, suppose the council worked out a political deal

among themselves, wherein if the pool is accepted, the tennis center must also be selected and vice versa. This constraint is written as

$$x_2 = x_1$$

This constraint makes x_1 and x_2 equal the same value, either zero or one.

A Mixed Integer Model Example

Nancy Smith has \$250,000 to

invest in three alternative investments: condominiums, land, and municipal bonds. She wants to invest in the alternatives that will result in the greatest return on investment after 1 year.

*In a **mixed integer model**, some solution values for decision variables are integers and others can be nonintegers.*

Each condominium costs \$50,000 and will return a profit of \$9,000 if sold at the end of 1 year; each acre of land costs \$12,000 and will return a profit of \$1,500 at the end of 1 year; and each municipal bond costs \$8,000 and will result in a return of \$1,000 if sold at the end of 1 year. In addition, there are only 4 condominiums, 15 acres of land, and 20 municipal bonds available for purchase.

[Page 184]

The linear programming model

for this problem is formulated as follows:

$$\text{maximize } Z = \$9,000x_1 + 1,500x_2 + 1,000x_3$$

subject to

$$\$50,000x_1 + 12,000x_2 + 8,000x_3 \leq \$250,000$$

$$x_1 \leq 4 \text{ condominiums}$$

$$x_2 \leq 15 \text{ acres}$$

$$x_3 \leq 20 \text{ bonds}$$

$$x_2 \geq 0$$

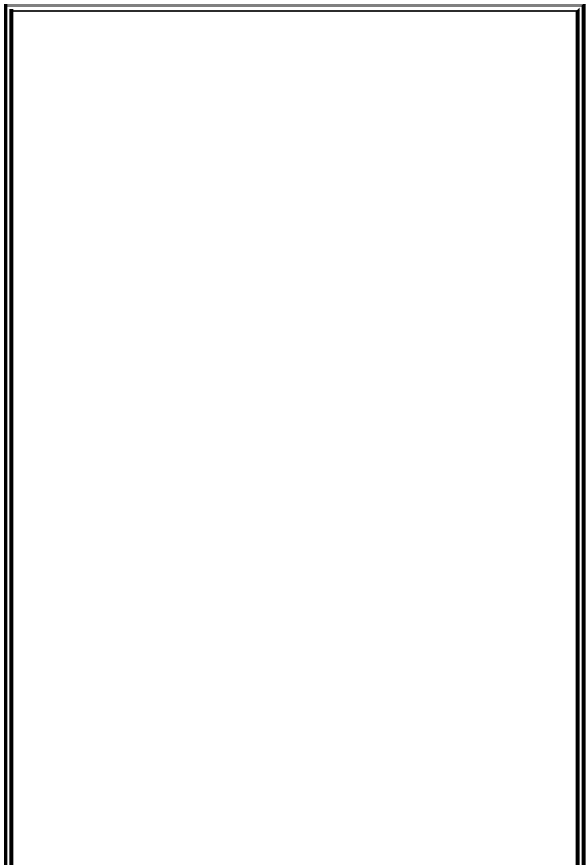
$$x_1, x_3 \geq 0 \text{ and integer}$$

where

x_1 = condominiums purchased

x_2 = acres of land purchased

x_3 = bonds purchased



Management Science

Application: Allocating Operating Room Time at Toronto's Mount Sinai Hospital

Mount Sinai Hospital in Toronto has 14 operating rooms that serve 5 surgical departments, including surgery, gynecology, ophthalmology, otolaryngology, and oral surgery. The largest of these departments, surgery, consists of 5 subareas, including orthopedics, general surgery, plastic surgery, vascular surgery, and urology. The hospital has 397.50 operating room hours available per week, and it employs an integer programming model to develop a master surgical schedule

that allocates operating room time to its 5 primary surgical departments. A daily operating room schedule includes the actual cases to be performed, start and end times, attending physicians, and anesthetists, and it is similar to a production schedule in a manufacturing plant. Operating room times are assigned as blocks; whenever possible, only one department is assigned to an operating room on a single day, thus constituting a block. It is possible to split blocks into morning and afternoon sessions and to alternate the same block between departments in different weeks, although the hospital attempts to minimize these types of allocations and keep the schedule as consistent as possible from week to week.

In the integer programming model,

the decision variables, x_{ijk} , equal the number of time blocks of operating room i , assigned to department j , on day k . The objective is to minimize the sum of the differences, or shortfalls, between the assigned operating room times and target times, represented as penalty functions, for all departments. A primary constraint is that the sum of all times allocated to a department, j , over all days, k , be (as nearly as possible) equal to a target number of hours. Additional constraints include daily and weekly bounds on the number of rooms assigned to a department, limits on the amount of under- or overallocation of time to any particular department, and daily room availability. The model solution provides an allocation of time blocks to departments that minimizes the shortage of time to each. Since its

implementation, the model has reduced the clerical time for producing a schedule from days (using a manual approach) to 1 or 2 hours, and has resulted in roughly \$20,000 per year in savings derived from the reduced time required by the operating room manager for schedule development. In addition, the model has provided hospital management with greater flexibility, increased ability to explore creative scheduling options, and generally improved quality schedules. However, perhaps the model's greatest benefit has been to provide unbiased, equitable schedules through a consistent, identifiable process, thus reducing conflict between departments and surgeons.



Source: J. T. Blake and J. Donald,
"Mount Sinai Hospital Uses Integer
Programming to Allocate Operating
Room Time," *Interfaces* 32, no. 2
(March/April 2002): 6373.

Notice that in this model the solution values for condominiums (x_1) and municipal bonds (x_3) must be integers. It is not possible to invest in a fraction of a condominium or to purchase part of a bond. However, it is possible to purchase less than an acre of land (i.e., a portion of an acre). Thus, two of the decision variables (x_1 and x_3) are restricted to integer values, whereas the other variable (x_4) can take on any real value greater than or equal to zero.

Integer Programming Graphical Solution

It may seem logical that an easy way to solve integer programming problems is to *round off* fractional solution values to integer values.

However, that can result in less than optimal (i.e., *suboptimal*) results. This outcome can be seen by using graphical analysis. As an example, consider the total integer model for the machine shop

formulated in the previous section:

maximize $Z = 100x_1 + 150x_2$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

First, we will use Excel to solve this model as a regular linear programming model without the integer requirements, as shown in [Exhibit 5.1](#)

Exhibit 5.1.

[\[View full size image\]](#)

Microsoft Excel - Exhibit5.1.xls

File Edit View Insert Format Tools Data Window Help

Take a screenshot for help

Formulas: =SUM(B3:D3)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Machine Shop Example														
2															
3	Machines		Presses	Lathes											
4	Profit per machine		100	150	Resources										
5	Resource constraints				Usage		Available	Left over							
6	purchase price (\$)		8000	4000	40000	<=	40000	0							
7	floor space (sq. ft.)		15	30	200	<=	200	0							
8															
9	Purchases														
10	Presses =		2.22												
11	Lathes =		5.56												
12	Profit =		1095.56												
13															

Notice that this model results in a noninteger solution of 2.22 presses and 5.56 lathes, or $x_1 = 2.22$ and $x_2 = 5.56$. Because the solution values must be

integers, let us first round off these two values to the *closest integer values*, which would be $x_1 = 2$ and $x_2 = 6$. However, if we substitute these values into the second model constraint, we find that this integer solution violates the constraint and thus is infeasible:

$$\begin{aligned} 15x_1 + 30x_2 &\leq 200 \\ 15(2) + 30(6) &\leq 200 \\ 210 &\nless 200 \end{aligned}$$

Rounding noninteger solution values up to the nearest integer value can result in an

infeasible solution.

In a model in which the constraints are all $\leq$ (and the constraint coefficients are positive), a feasible solution is always ensured by *rounding down*. Thus, a feasible integer solution for this problem is

A feasible solution is ensured by rounding down noninteger solution values.

$$x_1 = 2$$

$$x_2 = 5$$

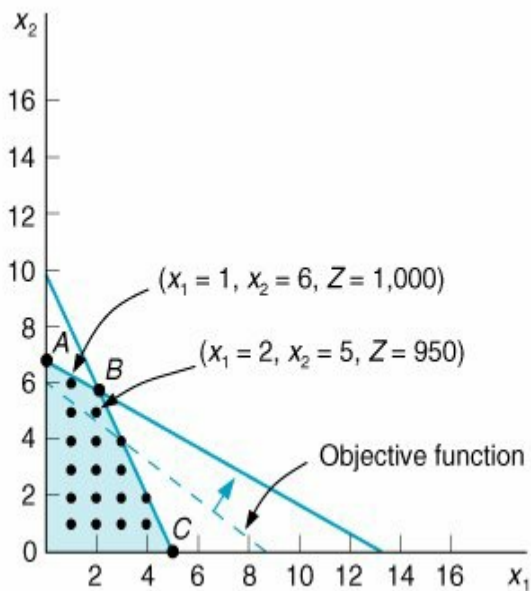
$$Z = \$950$$

[Page 186]

However, one of the difficulties of simply rounding down non-integer values is that another integer solution may result in a higher profit (i.e., in this problem, there may be an integer solution that will result in a profit higher than \$950). In order to determine whether a better integer solution exists,

let us analyze the graph of this model, which is shown in [Figure 5.1](#).

Figure 5.1. Feasible solution space with integer solution points



In [Figure 5.1](#) the dots indicate integer solution points, and the point $x_1 = 2, x_2 = 5$ is the rounded-down solution. Notice that as the objective function edge moves outward through the feasible solution space, there is an *integer* solution point that yields a greater profit than the rounded-down solution. This solution point is $x_1 = 1, x_2 = 6$. At this point, $Z = \$1,000$, which is \$50 more profit per day for the machine shop than is realized by the rounded-down integer solution.

A rounded-down integer solution can result in a less than optimal (suboptimal) solution.

This graphical analysis explicitly demonstrates the error that can result from solving an integer programming problem by simply rounding down. In the machine shop example, the optimal integer solution is $x_1 =$

1, $x_2 = 6$, instead of the rounded-down solution, $x_1 = 2$, $x_2 = 5$ (which is often called the *suboptimal* solution or result). Because erroneous results are caused by rounding down regular solutions, a more direct approach for solving integer problems is required.

The traditional approach for solving integer programming problems is the branch and bound method. It is a mathematical solution approach that can be applied to a number of different types of

problems. The branch and bound method is based on the principle that the total set of feasible solutions (such as the feasible area in [Figure 5.1](#)) can be partitioned into smaller subsets of solutions. These smaller subsets can then be evaluated systematically until the best solution is found. The branch and bound method is a tedious and often complex mathematical process.

Fortunately, both Excel and QM for Windows have the capability to solve integer programming problems, and thus we will rely

on the computer to solve the different types of integer models demonstrated in this chapter. However, for the interested reader, the CD that accompanies this text includes a supplementary module, "Integer Programming: The Branch and Bound Method," that demonstrates the branch and bound method in detail.

Computer Solution of Integer Programming Problems with Excel and QM for Windows

Integer programming problems can be solved using Excel spreadsheets and QM for Windows. In this section we demonstrate both of these computer solution approaches, using the examples for 01, total, and mixed integer programming problems

developed in the previous sections.

Solution of the 01 Model with Excel

Recall our recreational facilities example, formulated on page 182:

$$\text{maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

subject to

$$\$35,000x_1 + 10,000x_2 + 25,000x_3 + 90,000x_4 \leq \$120,000$$

$$4x_1 + 2x_2 + 7x_3 + 3x_4 \leq 12 \text{ acres}$$

$$x_1 + x_2 \leq 1 \text{ facility}$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

where

x_1 = construction of a swimming pool

x_2 = construction of a tennis center

x_3 = construction of an athletic field

x_4 = construction of a gymnasium

Z = total expected usage (people) per day

[Exhibit 5.2](#) shows this example

model set up in Excel spreadsheet format. The decision variables for the facilities are in cells **C12:C15**, and the objective function (Z) is embedded in cell C16. The objective function is shown on the formula bar at the top of the spreadsheet. The model constraints are contained in cells G7, G8, and G9. For example, cell G7 includes the cost constraint,

$$= \mathbf{C7 * C12 + D7 * C13 + E7 * C14 + F}$$

and the available budget of \$120,000 is contained in cell I7. Thus, the cost constraint

will be written in Solver as **G7≤I7**.

Exhibit 5.2.

[\[View full size image\]](#)

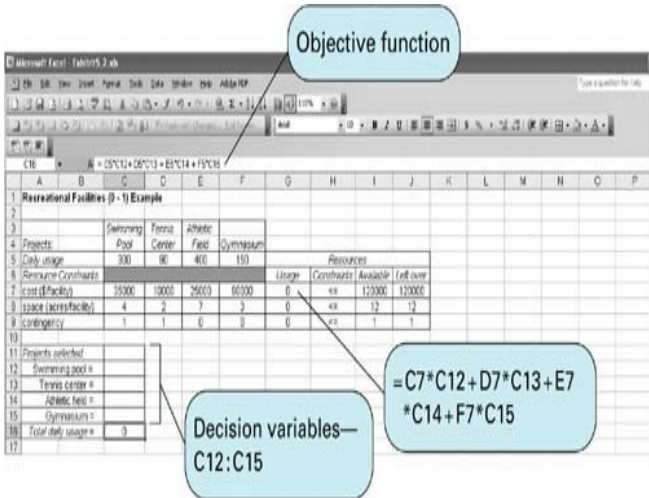


Exhibit 5.3 shows the Solver screen for our spreadsheet example. Notice that we have established the 01 condition of

our variables by constraining the decision variable cells to be less than or equal to one (i.e., **C12:C15≤1**).

[Page 188]

Exhibit 5.3.

[\[View full size image\]](#)

Solver Parameters

Set Target Cell:

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

☐
☐
☐

Restricts variables, C12:C15,
to integer and 0–1 values.

Time Out: For Ralph E.

Gomory

In 1958 Ralph Gomory was the first individual to develop a systematic (algorithmic) approach for solving linear integer programming problems. His "cutting plane method" is an algebraic approach based on the systematic addition of new constraints (or cuts), which are satisfied by an integer solution but not by a continuous variable solution. An alternative to an algebraic approach like Gomory's is an enumerative approach, which is a means of searching through all possible integer solutions (but in an intelligent manner) to limit the extent of the search. One such search method is the branch and bound technique, introduced in 1960 by A.

H. Land and A. C. Doig of the London School of Economics and Political Science.

Next, we constrain the decision variables to be integer (i.e., **C12:C15=integer**). The addition of this latter constraint is shown in [Exhibit 5.4](#). These two constraints, along with the nonnegativity constraint (**C12:C15≥0**), establish a condition wherein if the decision variables are nonnegative, less than or equal

to one, and integer, then they must be zero or one. The remaining constraints are the same as you would have for a normal linear programming model. The spreadsheet solution to our example is obtained by clicking on the "Solve" button in the Solver window and is shown in [Exhibit 5.5](#).

Exhibit 5.4.

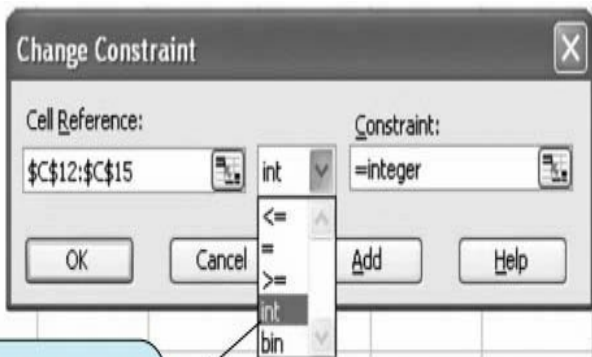


Exhibit 5.5.

(This item is displayed on page 189 in the print version)

[\[View full size image\]](#)

Microsoft Excel - Exhibit5.2.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

Formulas: =SUM(C12:D15)+E15*14+F15*15

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Recreational Facilities (0 - 1) Example																
2																	
3			Swimming	Tennis	Athletic												
4	Project:		Pool	Center	Field	Gymnasium											
5	Daily usage		300	90	400	150											
6	Resource Constraints						Usage	Constraints	Available	Left over							
7	cost (\$/facility)		35000	10000	25000	90000	80000	<=	120000	80000							
8	space (acres/facility)		4	2	7	3	7	<=	12	5							
9	contingency		1	1	0	0	1	<=	1	0							
10																	
11	Project selected																
12	Swimming pool =		1														
13	Tennis center =		0														
14	Athletic field =		1														
15	Gymnasium =		0														
16	Total daily usage =		700														
17																	

**Solution of the 01 Model
with QM for Windows**

The 01 integer programming problem is solved in the "Integer and Mixed Integer" module of QM for Windows. We will demonstrate this module by using our example for selecting recreational facilities that we just solved with Excel.

[Page 189]

[Exhibit 5.6](#) shows the data input screen for our recreational facilities example problem. At the bottom of the screen, when you click on "Variable Type" for a variable, a

menu will be displayed from which you can indicate if the variable is 01, integer, or real. In the case of this example, all the variables should be designated 01. The model is solved by clicking on the "Solve" button at the top of this screen, which results in the solution screen shown in [Exhibit 5.7](#).

Exhibit 5.6.

[\[View full size image\]](#)

Click to Solve.

File Edit View Styles Format Tools Help

And

Objective
☒ Maximize
☐ Minimize

Maximum number of iterations

Maximum level (depth) in procedure

Instruction
 Enter the type of variable. This is a drop-down box. Click on the arrow and then choose the appropriate one. You MUST use the drop-down box.

Revised Simplex Example

	x1	x2	x3	x4			Right-hand side
Maximize	30x	50	40	75x			Max: 30x1 + 50x2 + 40x3 + 75x4
Subject to	25,000	10,000	25,000	80,000	=	1,000,000	25,000x1 + 10,000x2 + 25,000x3 + 80,000x4 = 1,000,000
Space (sqm)	4	2	7	3	=	12	4x1 + 2x2 + 7x3 + 3x4 = 12
Facility complexity	1	3	8	8	=	3	x1 + 3x2 + 8x3 + 8x4 = 3
Variable type	0/1	0/1	0/1	0/1			

Integer
 Real
 0/1

Variable type

Click on "0/1" to make variables 0 or 1.

Exhibit 5.7.

Integer & Mixed Integer Programming Results



Recreational Facilities Example Solution

Variable	Type	Value
X1	0/1	1
X2	0/1	0
X3	0/1	1
X4	Integer	0
Solution value		700

[Page 190]

Management Science

Application: College of Business Class Scheduling at Ohio University

The College of Business at Ohio University in Athens, Ohio, encompasses four academic departments Accounting, Finance, Management Information Systems, and Marketing with approximately 65 to 75 instructors. Each academic term, the college departments offer 110 to 130 sections of different courses. The college's home building is Copeland Hall, which includes 14 to 16 classrooms. Prior to 1998 the college used a manual process to schedule courses and instructors into classrooms at different times during

the day. An associate dean would allocate the classrooms to the different departments, and the department chair would assign courses to instructors and schedule the course sections to the rooms, sometimes taking into account instructor preferences. If a chairperson did not use all of the department's allocated classrooms, they were shared with other departments, and if a department's available classrooms were not sufficient to accommodate all its courses, they were assigned to classrooms outside the building.

While this process was workable, it was not optimal; and instructors' preferences weren't often considered, and they often taught outside the building because course sections couldn't be matched with classrooms of sufficient size. For

example, instructors might have wished to only teach certain courses at certain times and on certain days, and they might have wanted to avoid back-to-back classes.

Since 1998 the college has used an integer programming model to develop course schedules. The model assigns instructors to courses in classrooms during specific time slots, based on instructor preference forms developed within the departments.

The typical integer programming model for course scheduling in a semester includes more than 2,500 variables and almost 2,000 constraints. The model has improved the use of classroom space, instructors have to teach fewer courses outside Copeland Hall (despite increasing enrollments), instructors are more satisfied with their schedules, and the time to

develop schedules has been cut in half.



Source: C. H. Martin, "Ohio University's College of Business Uses Integer Programming to Schedule Classes," *Interfaces* 34, no. 6 (November/December 2004): 460-465.

The solution shown in the QM for Windows output is

$x_1 = 1$ swimming pool

$x_2 = 0$ tennis center

$x_3 = 1$ athletic field

$x_4 = 0$ gymnasium

$Z = 700$ people per day
expected usage

Solution of the Total Integer Model with Excel

We will demonstrate the Excel solution of a total integer programming problem for the machine shop example we solved previously using the branch and bound method. Recall that this model was formulated as

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq \$40,000$$

$$15x_1 + 30x_2 \leq 200 \text{ ft.}^2$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

[Page 191]

where

x_1 = number of presses

x_2 = number of lathes

The machine shop example set up in spreadsheet format is shown in [Exhibit 5.8](#). This is the same basic format as a linear programming model. The essential difference between solving a regular linear programming model and an integer programming model is that we must designate the cells representing the decision variables as being "integer." We accomplish this by adding a

constraint within Solver that establishes **B10:B11** as integers, as shown in [Exhibit 5.10](#). The complete Solver, with all constraints, is shown in [Exhibit 5.9](#). Clicking on the "Solve" button will generate the spreadsheet solution, as shown in [Exhibit 5.11](#).

Exhibit 5.8.

[\[View full size image\]](#)

Objective function

Microsoft Excel - Exhibit5.9.xls

File Edit View Insert Format Tools Data Window Help

Formulas: =SUM(B10:B11)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Machine Shop Example														
2															
3	Machines		Press	Lathe											
4	Profit per machine		100	150		Resources									
5	Resource constraints				Usage		Available	Left over							
6	purchase price (\$)	8000	4000	0	<=	40000	40000								
7	floor space (sq. ft.)	15	30	0	<=	200	200								
8															
9	Purchases														
10	Presses =														
11	Lathes =														
12	Profit =	0.00													
13															

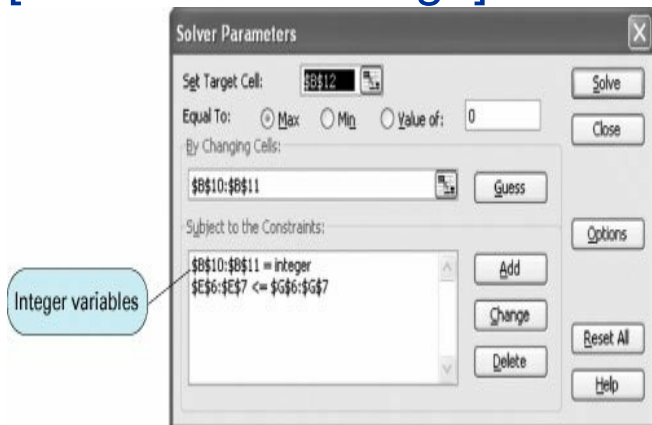
Slack, =G6-E6

=C6*B10+D6*B11

Decision variables—
B10:B11

Exhibit 5.9.

[\[View full size image\]](#)



[Page 192]

Exhibit 5.10.

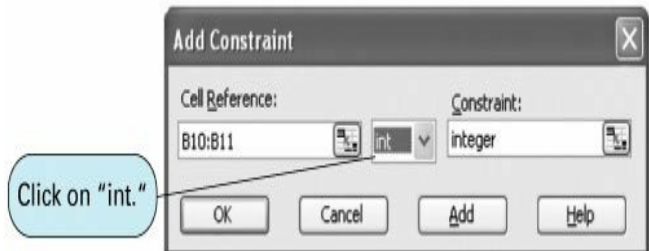


Exhibit 5.11.

[\[View full size image\]](#)

will be demonstrated by using the example for investments formulated earlier in the chapter:

$$\text{maximize } Z = \$9,000x_1 + 1,500x_2 + 1,000x_3$$

subject to

$$\$50,000x_1 + 12,000x_2 + 8,000x_3 \leq \$250,000$$

$$x_1 \leq 4 \text{ condominiums}$$

$$x_2 \leq 15 \text{ acres}$$

$$x_3 \leq 20 \text{ bonds}$$

$$x_2 \geq 0$$

$$x_1, x_3 \geq 0 \text{ and integer}$$

where

x_1 = condominiums purchased

x_2 = acres of land purchased

x_3 = bonds purchased

where

x_1 = condominiums purchased

x_2 = acres of land purchased

x_3 = bonds purchased

The Excel spreadsheet solution is shown in [Exhibit 5.12](#), and the Solver window is shown in [Exhibit 5.13](#). The decision variables are contained in cells **B8:B10**. The objective function formula is in cell B11 and is also displayed on the formula bar at the top of the spreadsheet. Notice that we did not include the constraint

values for the availability of each type of investment (i.e., 4 condos, 15 acres of land, and 20 bonds) in the spreadsheet setup. Instead, it was easier just to enter these constraints directly into Solver, as shown in [Exhibit 5.13](#). Notice in Solver that we have designated B8 and B10 as integers, but we have not designated B9 as an integer, reflecting the fact that x_1 and x_3 are integers, whereas x_2 is a real variable.

Exhibit 5.12.

[\[View full size image\]](#)

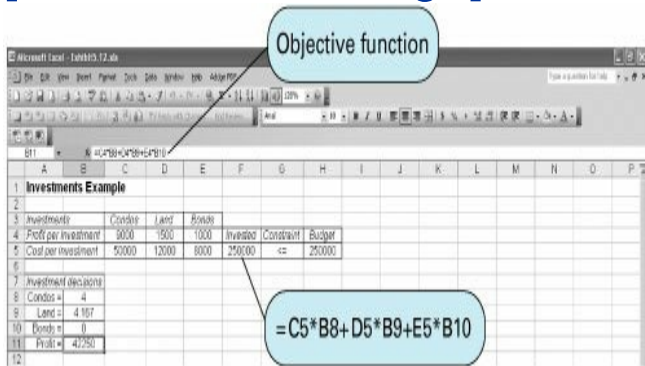
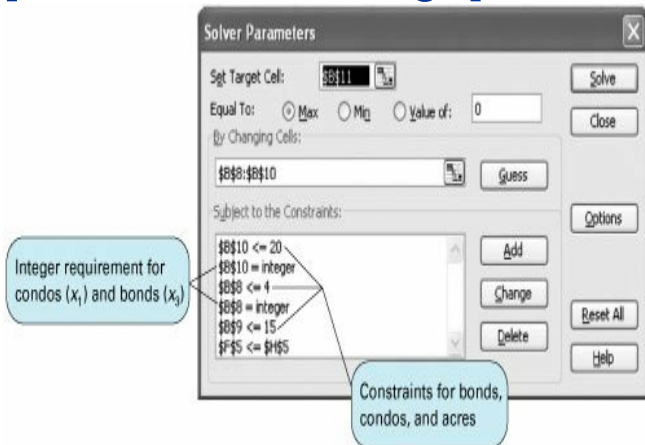


Exhibit 5.13.

[\[View full size image\]](#)

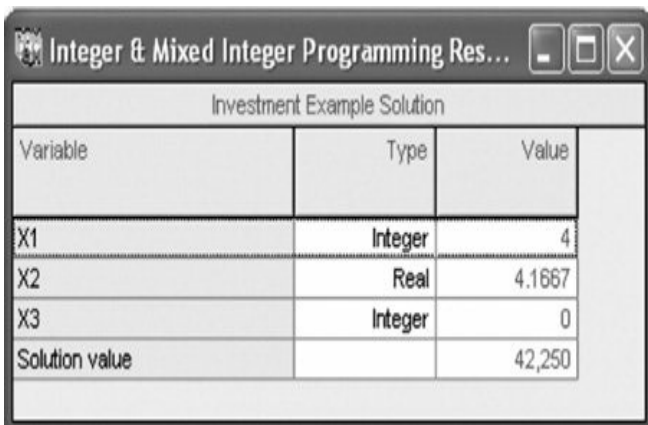


Solution of the Mixed Integer Model with QM

for Windows

The "Integer and Mixed Integer" module of QM for Windows is used to solve our investment example. When the problem data are input, we must designate the variable types for each decision variable. In this case, x_1 and x_3 are entered as integer variables, and x_2 is entered as a real variable. The QM for Windows input screen for our investment example is shown in [Exhibit 5.14](#), and the

Exhibit 5.15.



The screenshot shows a software window titled "Integer & Mixed Integer Programming Res...". Inside the window, there is a table titled "Investment Example Solution". The table has three columns: "Variable", "Type", and "Value". The rows of the table are as follows:

Variable	Type	Value
X1	Integer	4
X2	Real	4.1667
X3	Integer	0
Solution value		42,250

01 Integer Programming Modeling Examples

Some of the most interesting and useful applications of integer programming involve 01 variables. In these applications the variables allow for the selection of an item (or activity) where a value of one indicates that the item is selected and a value of zero indicates that it is not. For example, a decision variable might represent the purchase

of a building; if the value of the variable is one, the building is purchased. If the value is zero, the building is not purchased. We will demonstrate three of the most popular applications of 01 integer programming: a capital budgeting problem, a fixed charge and facility location problem, and a set covering problem.

A Capital Budgeting Example

The University Bookstore at

Tech is considering several expansion projects, including developing a store Web site for online retail and catalog purchases, buying an off-campus warehouse and its subsequent expansion, developing a clothing and gift department specializing in university logo apparel, opening a computer department carrying both hardware and software products, and creating a banking pavilion of three automated teller machines outside the store. Some of the

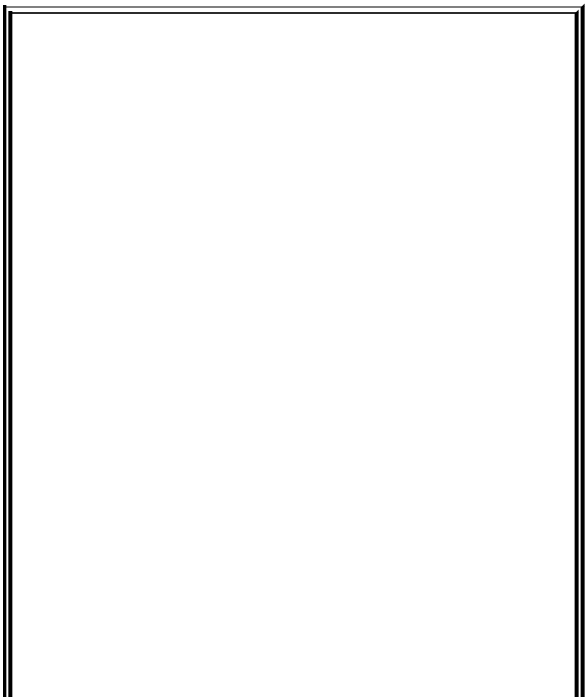
projects will be developed over a 2-year period and some over a 3-year period, as funds permit. The net present value costs per year and the projected net present value of returns for a 5-year period for each of the projects are shown in the following table:

[Page 195]

**Project
Costs/Year
(\$1,000s)**

NPV

Project	Return (\$1,000s)	1	2	3
1. Web site	\$120	\$55	\$40	\$25
2. Warehouse	85	45	35	20
3. Clothing department	105	60	25	
4. Computer department	140	50	35	30
5. ATMs	70	30	30	
Available funds per year		\$150	\$110	\$60



Management Science

Application: Optimal

Assignment of Gymnasts to Events

(This item is displayed on page 194 in
the print version)

An integer programming model was developed to assign members of a women's gymnastics team to the four events conducted at a typical NCAA meet: vault, uneven bars, balance beam, and floor exercises. Each team can enter up to six gymnasts in each event, and the top five scores among these entrants contribute to the team score. At least four of the entrants must participate in all four events. These

conditions formed the model constraints; the objective was to maximize the team's overall expected score. The model was tested at Utah State University and allowed officials to analyze the effects of changing conditions, such as improved performance or injuries, on the team score; to indicate to a team member why she was not selected for a particular event; and to eliminate the time the coach spent manually evaluating different team combinations.



Source: P. Ellis and R. Corn, "Using Bivalent Integer Programming to Select Teams for Intercollegiate Women's Gymnastics Competition," *Interfaces* 14, no. 3 (May/June 1984): 4146.

In addition, the store does not have enough space available to create both a computer department and a clothing department. The bookstore director wants to know which projects to select to maximize returns.

This problem requires a 01 integer programming model formulation where

x_1 = selection of Web site project

x_2 = selection of warehouse project

x_3 = selection of clothing department project

x_4 = selection of computer department project

x_5 = selection of ATM project

$x_i = 1$ if project i is selected; 0
if project i is not selected

maximize $Z = \$120x_1 + 85x_2 + 105x_3 + 140x_4 + 70x_5$
subject to

$$55x_1 + 45x_2 + 60x_3 + 50x_4 + 30x_5 \leq 150$$

$$40x_1 + 35x_2 + 25x_3 + 35x_4 + 30x_5 \leq 110$$

$$25x_1 + 20x_2 + 30x_4 \leq 60$$

$$x_3 + x_4 \leq 1$$

$$x_i = 0 \text{ or } 1$$

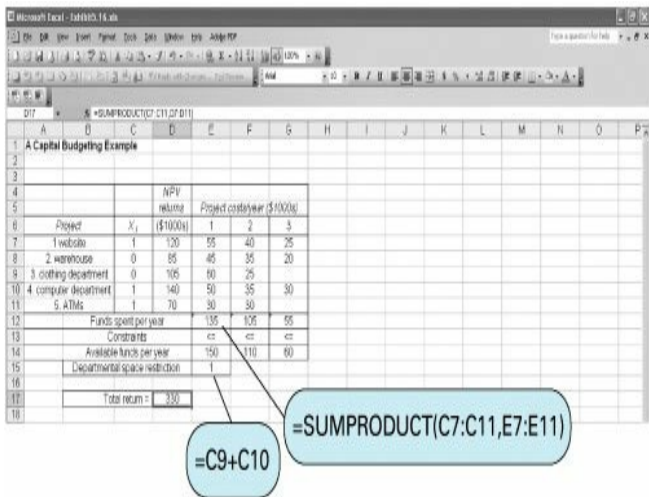
[Exhibit 5.16](#) shows this capital budgeting example set up in Excel spreadsheet format. The decision variables for the projects are in cells **C7:C11**, and the objective function (Z) is in cell D17. The objective function is shown on the

formula bar at the top of the spreadsheet. The model constraints are included in cells **E12:G12**. For example, cell E12 contains the budget constraint for year 1, **=SUMPRODUCT (C7:C11, E7:E11)**, and the available budget is in cell E14. Thus, this budget constraint will be written in Solver as **E12≤E14**. The mutually exclusive constraint, $x_3 + x_4 \leq 1$, is included in cell E15 as **=C9+C10**.

Exhibit 5.16.

(This item is displayed on page 196 in the print version)

[\[View full size image\]](#)

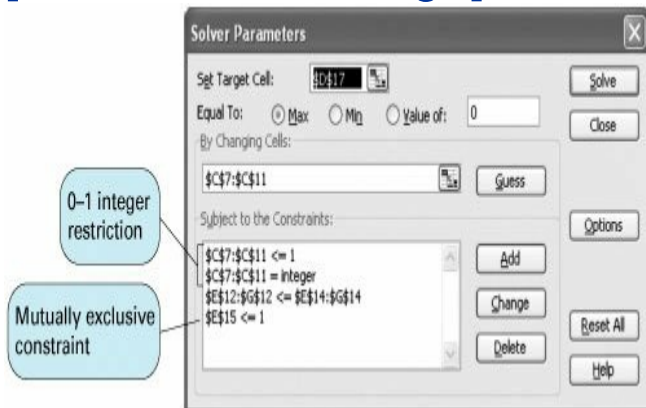


[Exhibit 5.17](#) shows the Solver screen for the capital budgeting example. The 01 variable restriction has been established by the constraints **C7:C11** ≤ 1 and **C7:C11** = integer. The mutually exclusive constraint is shown as **E15** ≤ 1.

[Page 196]

Exhibit 5.17.

[\[View full size image\]](#)



The model solution is

$$x_1 = 1 \text{ (Web site)}$$

$x_4 = 1$ (computer department)

$x_5 = 1$ (ATMs)

$Z = \$330,000$

A Fixed Charge and Facility Location Example

Frijo-Lane Food Products own farms in the Southwest and Midwest, where it grows and harvests potatoes. It then ships these potatoes to three processing plants in Atlanta, Baton Rouge, and Chicago,

where different varieties of potato products, including chips, are produced. Recently, the company has experienced a growth in its product demand, so it wants to buy one or more new farms to produce more potato products.

[Page 197]

The company is considering six new farms with the following annual fixed costs and projected harvest:

Projected

Farm	Fixed Annual Costs (\$1,000s)	Annual Harvest (thousands of tons)
1	\$405	11.2
2	390	10.5
3	450	12.8
4	368	9.3
5	520	10.8
6	465	9.6

The company currently has the following additional available production capacity (tons) at its three plants, which it wants to utilize:

Plant	Available Capacity (thousands of tons)
A	12
B	10
C	14

The shipping costs (\$) per ton from the farms being considered for purchase to the plants are as follows:

Plant (shipping costs, \$/ton)			
Farm	A	B	C
1	18	15	12
2	13	10	17
3	16	14	18

4	19	15	16
5	17	19	12
6	14	16	12

The company wants to know which of the six farms it should purchase to meet available production capacity at the minimum total cost, including annual fixed costs and shipping costs.

This problem is formulated as a 01 integer programming model because the selection of the farms requires 01 decision variables:

$y_i = 0$ if farm i is not selected,
and 1 if farm i is selected,

where

$i = 1, 2, 3, 4, 5, 6$

The variables for the amounts shipped from each prospective farm to each plant are non-integer, defined as follows:

x_{ij} = potatoes (tons, 1,000s)
shipped from farm i to plant j ,

where

$i = 1, 2, 3, 4, 5, 6$ and $j = A, B, C$

[Page 198]

The objective function (Z)
combines shipping costs and
the annual fixed costs, where Z
= \$1,000s:

$$\begin{aligned} \text{minimize } Z = & 18x_{1A} + 15x_{1B} + 12x_{1C} + 13x_{2A} + 10x_{2B} + 17x_{2C} + 16x_{3A} + 14x_{3B} + \\ & 18x_{3C} + 19x_{4A} + 15x_{4B} + 16x_{4C} + 17x_{5A} + 19x_{5B} + 12x_{5C} + 14x_{6A} + \\ & 16x_{6B} + 12x_{6C} + 405y_1 + 390y_2 + 450y_3 + 368y_4 + 520y_5 + 465y_6 \end{aligned}$$

The constraints for production capacity require that potatoes be shipped (i.e., x_{ij} variables have positive values) only from a farm if it is selected; for example, for farm 1,

$$x_{1A} + x_{1B} + x_{1C} \leq 11.2y_1$$

In this constraint, if $y_1 = 1$, then the 11.2 thousand tons of potatoes are available at farm 1 for shipment to one or more of the plants. Similar constraints are developed for each of the five other farms. Constraints are also necessary

for the production capacity to be filled at each plant.

The complete model is formulated as

$$\begin{aligned} \text{minimize } Z = & 18x_{1A} + 15x_{1B} + 12x_{1C} + 13x_{2A} + 10x_{2B} + 17x_{2C} + 16x_{3A} + 14x_{3B} + \\ & 18x_{3C} + 19x_{4A} + 15x_{4B} + 16x_{4C} + 17x_{5A} + 19x_{5B} + 12x_{5C} + 14x_{6A} + \\ & 16x_{6B} + 12x_{6C} + 405y_1 + 390y_2 + 450y_3 + 368y_4 + 520y_5 + 465y_6 \end{aligned}$$

subject to

$$x_{1A} + x_{1B} + x_{1C} - 11.2y_1 \leq 0$$

$$x_{2A} + x_{2B} + x_{2C} - 10.5y_2 \leq 0$$

$$x_{3A} + x_{3B} + x_{3C} - 12.8y_3 \leq 0$$

$$x_{4A} + x_{4B} + x_{4C} - 9.3y_4 \leq 0$$

$$x_{5A} + x_{5B} + x_{5C} - 10.8y_5 \leq 0$$

$$x_{6A} + x_{6B} + x_{6C} - 9.6y_6 \leq 0$$

$$x_{1A} + x_{2A} + x_{3A} + x_{4A} + x_{5A} + x_{6A} = 12$$

$$x_{1B} + x_{2B} + x_{3B} + x_{4B} + x_{5B} + x_{6B} = 10$$

$$x_{1C} + x_{2C} + x_{3C} + x_{4C} + x_{5C} + x_{6C} = 14$$

$$x_{ij} \geq 0$$

$$y_i = 0 \text{ or } 1$$

[Exhibit 5.18](#) shows this example model set up in Excel spreadsheet format. The decision variables for the shipments between farms and plants, x_{ij} , are in cells **C5:E10**. The 01 decision variables for

farm selection, y_i , are in cells

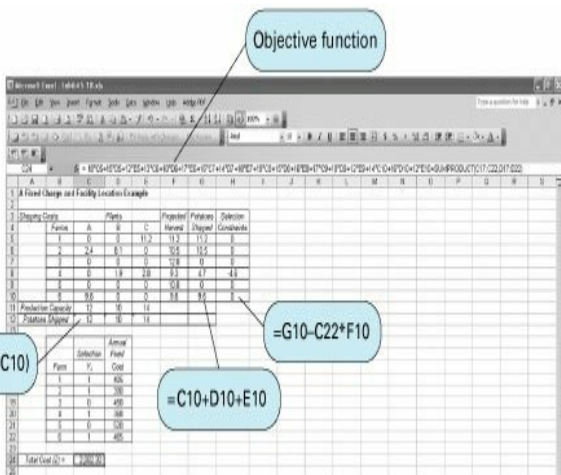
C17:C22. The objective function (Z) is embedded in cell C24 and is shown on the formula bar at the top of the spreadsheet. The model constraints for potato availability at the farms are embedded in cells **H5:H10**. For example, the total amount shipped from farm 1, $x_{1A} + x_{1B} + x_{1C}$, is in cell G5 as **=C5+D5+E5**. The constraint formulation is in cell H5 as **=G5C17*F5**. The constraints for potatoes shipped from the plants are in cells **C12:E12**. For

example, the constraint formula in C12 is **=SUM(C5:C10)**.

Exhibit 5.18.

(This item is displayed on page 199 in the print version)

[\[View full size image\]](#)



[Exhibit 5.19](#) shows the Solver screen for this example. The 01 variable restriction for the y_i variables is established by the

constraints **C17:C22** ≤ 1 and **C17:C22** = integer. The farm harvest (potato availability) constraints are shown as **H5:H10** ≤ 0, and the plant production capacity constraints are shown as **C12:E12** = **C11:E11**.

Exhibit 5.19.

(This item is displayed on page 199 in the print version)

[\[View full size image\]](#)

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-
-

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

Callouts:

- Plant capacity constraints (points to \$C\$5:\$E\$10)
- 0-1 integer restriction (points to \$C\$17:\$C\$22)
- Harvest constraints (points to \$H\$5:\$H\$10)

The model solution is

$$x_{1C} = 11,200 \text{ tons}$$

$$x_{2A} = 2,400 \text{ tons}$$

$$x_{2B} = 8,100 \text{ tons}$$

$$x_{4B} = 1,900 \text{ tons}$$

$$x_{4C} = 2,800 \text{ tons}$$

$$x_{6A} = 9,600 \text{ tons}$$

$$y_1 = 1 \text{ (farm 1)}$$

$$y_2 = 1 \text{ (farm 2)}$$

$$y_4 = 1 \text{ (farm 4)}$$

$$y_6 = 1 \text{ (farm 6)}$$

$$Z = \$2,082,300$$

[Page 199]

A Set Covering Example

American Parcel Service (APS) has determined that it needs to add several new package distribution hubs to service cities east of the Mississippi River. The company wants to

construct the minimum set of new hubs in the following 12 cities so that there is a hub within 300 miles of each city (i.e., every city is covered by a hub):

[Page 200]

<i>City</i>	<i>Cities Within 300 Miles</i>
1. Atlanta	Atlanta, Charlotte, Nashville
2. Boston	Boston, New York
	Atlanta, Charlotte,

3. Charlotte Richmond

4. Cincinnati Cincinnati, Detroit,
Indianapolis, Nashville,
Pittsburgh

5. Detroit Cincinnati, Detroit,
Indianapolis, Milwaukee,
Pittsburgh

6. Indianapolis Cincinnati, Detroit,
Indianapolis, Milwaukee,
Nashville, St. Louis

7. Milwaukee Detroit, Indianapolis,
Milwaukee

Atlanta, Cincinnati,

- | | |
|----------------|---|
| 8. Nashville | Indianapolis, Nashville, St. Louis |
| 9. New York | Boston, New York, Richmond |
| 10. Pittsburgh | Cincinnati, Detroit, Pittsburgh, Richmond |
| 11. Richmond | Charlotte, New York, Pittsburgh, Richmond |
| 12. St. Louis | Indianapolis, Nashville, St. Louis |

This problem requires a 01 integer programming model in which the decision variables are the available cities:

$$x_i = \text{city } i, i = 1 \text{ to } 12$$

where

$$x_i = 0$$

if city i is not selected as a hub
and

$$x_i = 1$$

if city i is selected

The objective function (Z) is to minimize the number of hubs:

$$\begin{aligned} \text{minimize } Z = & x_1 + x_2 + x_3 + x_4 \\ & + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \\ & + x_{11} + x_{12} \end{aligned}$$



Management Science

Application: Managing

Prototype Vehicle Test Fleets at Ford

Ford Motor Company, with 350,000 employees in 111 manufacturing plants in 38 countries, is the second-largest manufacturer of cars and trucks in the world. The product design process at Ford is a time-consuming and extremely expensive process, and designs must be painstakingly verified. For the verification of some designs, a prototype vehicle must be constructed, at a cost routinely exceeding \$250,000. Prototypes are costly because most of the parts and assembly processes are unique, and

the tools that would eventually be used in mass production do not exist yet. Complex vehicle development programs often require more than 100, and sometimes more than 200, vehicle prototypes. Because the cost of developing prototype vehicles depends solely on the number built, the goal is to build as few prototypes as possible to perform the required testing.

A group of students and faculty associated with the 3-year engineering master's program at Wayne State University developed the prototype optimization model (POM) to reduce the number of prototype vehicles Ford needed to verify its product designs. Initially a group of students in this program (who were also engineering managers at Ford) noticed a conceptual similarity between a class

set covering example and the problem of developing a prototype fleet to cover a set of required tests. These students undertook the prototype-planning problem as their team project required in the master's program. This group developed a hypothetical data set and model, and students in the next graduating class picked up where they left off to eventually develop POM.

The model's objective was to determine the minimum number of prototype vehicles required to complete every verification test by a specific deadline. This requires that prototype vehicles be shared among design groups, thereby increasing the usage of vehicles and reducing the time they sit idle, waiting for testing. In the first step in POM, a set covering problem is formulated to determine the minimum number of

unique vehicle configurations needed to cover all the tests. A list of every realistic vehicle configuration, called the building combination matrix (BCM), is developed. An actual design could initially include more than 1,000 possible vehicle configurations. POM was first used for the European Transit vehicle program. It began with 38,880 possible combinations, and the model identified a set of 27 unique vehicle configurations required to cover all the tests, offering a cost savings of \$12 million. Ford has since used POM to budget, plan, and manage more than 10 complex vehicle design programs, including those for Taurus, Windstar, Explorer, and Ranger. The overall costs for prototype vehicles were reduced by more than \$250 million from 1995 to 2000.



Source: K. Chelst, J. Sidelko, A. Przebienda, J. Lockledge, and D. Mihailidis, "Rightsizing and Management of Prototype Vehicle Testing at Ford Motor Company," *Interfaces* 31, no. 1 (January/February 2001): 91107.

The model constraints establish the set covering requirement (i.e., that each city be within 300 miles of a hub). For example, Atlanta covers itself, Charlotte, and Nashville:

$$\text{Atlanta: } x_1 + x_3 + x_8 \geq 1$$

In this constraint, at least one of the variables must equal 1 in order for Atlanta to be assured of a hub within 300 miles. Similar constraints are developed for the other 11 cities.

The complete 01 integer

programming model is summarized as follows:

minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12}$

subject to

Atlanta: $x_1 + x_3 + x_8 \geq 1$

Boston: $x_2 + x_9 \geq 1$

Charlotte: $x_1 + x_3 + x_{11} \geq 1$

Cincinnati: $x_4 + x_5 + x_6 + x_8 + x_{10} \geq 1$

Detroit: $x_4 + x_5 + x_6 + x_7 + x_{10} \geq 1$

Indianapolis: $x_4 + x_5 + x_6 + x_7 + x_8 + x_{12} \geq 1$

Milwaukee: $x_5 + x_6 + x_7 \geq 1$

Nashville: $x_1 + x_4 + x_6 + x_8 + x_{12} \geq 1$

New York: $x_2 + x_9 + x_{11} \geq 1$

Pittsburgh: $x_4 + x_5 + x_{10} + x_{11} \geq 1$

Richmond: $x_3 + x_9 + x_{10} + x_{11} \geq 1$

St. Louis: $x_6 + x_8 + x_{12} \geq 1$

$x_i = 0 \text{ or } 1$

The Excel spreadsheet for this

model is shown in [Exhibit 5.20](#). The decision variables for the cities and hub sites are in cells **B20:M20**, and the objective function (Z) is in cell B22. The formula for the objective function embedded in B22 is the sum of the cities selected as hubs, **=SUM (B20:M20)**. The model constraints are in cells **N7:N18**. Notice that the values of 1 in cells B7, D7, and I7 indicate the cities covered by a possible hub at Atlanta.

Exhibit 5.20.

variable restriction has been established by the constraints **B20:M20** ≤ 1 and **B20:M20** = **integer**. Notice that the formulas in cells **N7:N18** are set ≥ 1 (i.e., **N7:N18** ≥ 1) to complete the "set covering" constraints in the model.

[Page 202]

Exhibit 5.21.

[\[View full size image\]](#)

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Guess

Subject to the Constraints:

Add

Change

Delete

Solve

Close

Options

Reset All

Help

City constraints
set ≤ 1

Management Science

Application: Planning Next-Day Air Shipments at UPS

UPS is the leading package-delivery company in the world, each day delivering more than 13 million packages to 8 million customers in more than 200 countries. UPS Airlines, the ninth-largest commercial airline in the United States, with more than 345 aircraft, is the key component that enables the company to provide expedited (express) delivery service. The airline's next-day service delivers more than 1 million packages each night and annually generates over \$5 billion in revenue.

In the UPS air network, trucks

transport originating packages to ground centers and from ground centers to more than 100 airports. Each plane carries its packages directly to one of 6 regional hubs (Columbia, SC, Dallas, TX, Hartford, CT, Ontario, CA, Philadelphia, PA, and Rockford, IL) or 1 all-points hub (in Louisville, KY), and it may stop at an airport along the way to pick up additional packages. Packages are sorted at the hubs and loaded onto aircraft for outbound delivery to a destination airport. At the destination airport, workers transfer the packages to trucks that move them to ground centers, where they are scanned, sorted, and loaded onto smaller trucks for delivery to their final destinations. Each type of aircraft has a maximum flying range, effective speed, landing restrictions at different locations, and cargo

capacity. The maximum number of airports a plane can visit on a pickup or delivery route (not including a hub) is two.

Planning a shipping network of this size with 7 hubs, more than 100 airports, more than 2,000 combinations of airport origin and destination pairs, and a huge volume of packages is a very large, complex problem. In 2000 the company implemented VOLCANO (Volume, Location, and Aircraft Network Optimizer) to determine aircraft routes, fleet assignments, and package routing to ensure overnight delivery at a minimum cost.

VOLCANO is based on an integer programming set covering model formulation. This system saved the company more than \$87 million between 2000 and 2002 by ensuring that it would use the most efficient

routes with the fewest planes. UPS estimated that it would result in savings of more than \$189 million over the next decade.



Source: A. P. Armacost, C. Barnhart, K. A. Ware, and A. M. Wilson, "UPS Optimizes Its Air Network," *Interfaces* 34, no. 1 (January/February 2004): 1525.

The model solution is

$$\begin{aligned}x_2 \text{ (Boston)} &= 1 \\x_3 \text{ (Charlotte)} &= 1 \\x_5 \text{ (Detroit)} &= 1 \\x_{12} \text{ (St. Louis)} &= 1 \\Z &= 4 \text{ hubs}\end{aligned}$$

Notice that there is only one city covered by two hubs, Indianapolis (i.e., 2 in cell N12), which is overlapped by both Detroit and St. Louis. Also, there are multiple optimal solutions to this problem (i.e.,

different mixes of four hubs).

Summary

In this chapter we found that simply rounding off non-integer simplex solution values for models requiring integer solutions is not always appropriate. Rounding can often lead to suboptimal results. Therefore, direct solution approaches using the computer are needed to solve the three forms of linear integer programming models. total integer models, 01

integer models, and mixed integer models.

Having analyzed integer problems and the computer techniques for solving them, we have now covered most of the basic forms of linear programming models and solution techniques. Two exceptions are the transportation model and a problem that contains more than one objective, goal programming. These special cases of linear programming are presented in the next two chapters.

References

Baumol, W. J. *Economic Theory and Operations Analysis*, 4th ed. Upper Saddle River, NJ: Prentice Hall, 1977.

Dantzig, G. B. "On the Significance of Solving Linear Programming Problems with Some Integer Variables." *Econometrica* 28 (1960): 3044.

Gomory, R. E. "An Algorithm for Integer Solutions to Linear Programs." In *Recent Advances in Mathematical Programming*, edited by R. L. Graves and P. Wolfe. New York: McGraw-Hill, 1963.

Lawler, E. L., and Wood, D. W. "Branch and Bound Methods A Survey." *Operations Research* 14 (1966): 699-719.

Little, J. D. C., et al. "An Algorithm for the Traveling Salesman Problem." *Operations Research* 11 (1963): 972-89.

McMillan, C., Jr. *Mathematical Programming*. New York: John Wiley & Sons, 1970.

Mitten, L. G. "Branch-and-Bound Methods: General Formulation and Properties." *Operations Research* 18

(1970): 2434.

Plane, D. R., and McMillan, C., Jr. *Discrete Optimization*. Upper Saddle River, NJ: Prentice Hall, 1971.

Wagner, H. M. *Principles of Operations Research*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1975.

Example Problem Solution

The following example problem demonstrates the model formulation and solution of a total integer programming problem.

The Problem Statement

A textbook publishing company has developed two new sales regions and is planning to

transfer some of its existing sales force into these two regions. The company has 10 salespeople available for the transfer. Because of the different geographic configurations and the location of schools in each region, the average annual expenses for a salesperson differ in the two regions; the average is \$10,000 per salesperson in region 1 and \$7,000 per salesperson in region 2. The total annual expense budget for the new regions is \$72,000. It is estimated that a salesperson

in region 1 will generate an average of \$85,000 in sales each year and a salesperson in region 2 will generate \$60,000 annually in sales. The company wants to know how many salespeople to transfer into each region to maximize increased sales.

[Page 204]

Formulate this problem and solve it by using QM for Windows.

Solution

Step 1: Formulate the Integer Programming Model

maximize $Z = \$85,000x_1 + 60,000x_2$

subject to

$$x_1 + x_2 \leq 10 \text{ salespeople}$$

$$\$10,000x_1 + 7,000x_2 \leq \$72,000 \text{ expense budget}$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

Step 2: Solve the Model Using QM for Windows

[\[View full size image\]](#)

Chapter5-Example Solution

	X1	X2		RHS	Equation form
Maximize	85,000	60,000			Max: $85000X_1 + 60000X_2$
Salespeople	1	1	$\leq$	10	$X_1 + X_2 \leq 10$
Expense budget (\$)	10,000	7,000	$\leq$	72,000	$10000X_1 + 7000X_2 \leq 72000$
Variable type	Integer	Integer			
Solution->	3	6	Optimal Z->	615,000	

Problems

Consider the following linear program

$$\text{maximize } Z = 5x_1 + 4x_2$$

subject to

1.

$$3x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

Demonstrate the graphical solution of t

Solve the following linear programming

$$\text{minimize } Z = 3x_1 + 6x_2$$

2.

subject to

$$7x_1 + 3x_2 \geq 40$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

A tailor makes wool tweed sport coats and gets a shipment of 150 square yards of wool each month to make coats and slacks, and has 130 hours of labor to make them each month. A coat requires 3 yards of wool and 10 hours to make, and a pair of slacks requires 2 yards of wool and 4 hours to make. The tailor knows that each coat he makes earns \$40 from each sale and each pair of slacks earns \$30. How many coats and pairs of slacks should he make each month to know how many coats and pairs of slacks to make for maximum profit.

3.

[Page

1. Formulate an integer linear programming model for this problem.
2. Determine the integer solution to the problem using a computer. Compare this solution with the continuous solution and indicate whether the integer solution has been optimal.

A jeweler and her apprentice make silver and gold chains. It takes 3 hours to make a silver chain and 2 hours to make a gold chain. The jeweler has 18 hours of labor available each day. The jeweler has 10 ounces of silver and 8 ounces of gold available each day. The jeweler knows that each silver chain she makes earns \$10 from each sale and each gold chain earns \$15. How many silver and gold chains should she make each day to know how many silver and gold chains to make for maximum profit.

Each week they have 80 hours of labor available. It requires 8 hours of labor and 10 ounces of gold for a ring and 10 hours of labor and 6 ounces of gold for a pin. Each pin also contains a small gem of silver and the jeweler can use no more than six per week. A pin earns \$100 and a ring

4. necklace earns \$100. The jeweler wants to decide how many of each item to make each week to maximize profit.

- 1.** Formulate an integer programming model for this problem.
- 2.** Solve this model by using the computer. Compare the solution with the solution without integer requirements. How much profit would the rounded-down solution would lose?

A glassblower makes glass decanters and glass trays. Each item requires 1 pound of glass, and the glassblower has 10 pounds of glass available each week. A glass decanter requires 2 hours of labor and a glass tray requires only 1 hour of labor. The glassblower has 16 hours a week. The profit from a decanter is \$20 and from a tray is \$10. The glassblower wants to decide how many decanters (x_1) and trays (x_2) that he should produce each week to maximize his profit.

5. The glassblower wants to decide how many decanters (x_1) and trays (x_2) that he should produce each week to maximize his profit.

1. Formulate an integer programming model.
2. Solve this model by using the computer.

The Livewright Medical Supplies Company wants to assign 12 salespeople to three regions. The Southern region has 5 salespeople, the Eastern region has 4 salespeople, and the Midwestern region has 3 salespeople. A salesperson in the South earns \$600 per month for the company, a salesperson in the East earns \$450 per month, and a salesperson in the Midwest earns \$375 per month. The company has a budget of \$50,000 available for expenses for all 12 salespeople.

6. The company has a budget of \$50,000 available for expenses for all 12 salespeople. The company has average expenses of \$80 per day, average expenses of \$70 per day, and average daily expenses of \$50. The company wants to assign the number of salespeople to each region.

1. Formulate an integer programming model.
2. Solve this model by using the computer.

Helen Holmes makes pottery by hand in her home. She has 40 hours available each week to make bowls and plates. She can make 10 bowls per week or 20 plates per week. She wants to maximize her profit. The profit for a bowl is \$10 and the profit for a plate is \$15.

hours of labor, and a vase requires 2 hours of special clay to make a bowl and 5 pounds of special clay to make a vase. She is able to acquire 35 pounds of clay per week for \$50 and her vases for \$40. She wants to make each week to maximize her revenue.

1. Formulate an integer programming model.
2. Solve this model by using the computer. Compare the solution with the solution without integer requirements. What would the rounded-down solution would be?

Lauren Moore has sold her business for \$1,000,000 and is planning to invest in condominium units (which she intends to lease to a farmer). She estimates that each unit will return of \$8,000 for each condominium unit. A condominium unit costs \$70,000. A condominium will cost her \$1,000 for maintenance and upkeep. She has budgeted for these annual expenses. Let x be the number of condominiums and y be the number of acres of land to invest in.

1. Formulate a mixed integer program
2. Solve this model by using the computer

The owner of the Consolidated Machine Shop can purchase a lathe, a press, a grinder, or any combination of the following 0-1 integer linear programming model to determine which of the three machines (lathe, press, grinder) should be purchased in order to maximize profit.

9. maximize $Z = 1,000x_1 + 700x_2 + 800x_3$ (profit, \$)
 subject to

$$5,000x_1 + 6,000x_2 + 4,000x_3 \leq 10,000$$
 (cost, \$)

$$x_1, x_2, x_3 = 0 \text{ or } 1$$

Solve this model by using the computer.

Solve the following mixed integer linear programming model using the computer:

maximize $Z = 5x_1 + 6x_2 + 4x_3$

10. subject to

$$5x_1 + 3x_2 + 6x_3 \leq 20$$

$$x_1 + 3x_2 \leq 12$$

$$x_1, x_3 \geq 0$$

$$x_2 \geq 0 \text{ and integer}$$

Northwoods Backpackers is a retail cat specializes in outdoor clothing and cam are taken each day by a large pool of c whom are permanent and some temp process an average of 76 orders per d operator can process an average of 53 averages at least 600 orders per day. 7 workstations. A permanent operator p

11. errors each day, whereas a temporary with errors daily. The store wants to lin permanent operator is paid \$81 per da temporary operator is paid \$50 per da the number of permanent and tempora costs.

Formulate an integer programming model by using the computer.

Consider the following linear programming

$$\text{maximize } Z = 20x_1 + 30x_2 + 10x_3 + 40x_4$$

subject to

12.

$$2x_1 + 4x_2 + 3x_3 + 7x_4 \leq 10$$

$$10x_1 + 7x_2 + 20x_3 + 15x_4 \leq 40$$

$$x_1 + 10x_2 + x_3 \leq 10$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

Solve this problem by using the computer.

In the example problem solution for this textbook company was attempting to assign representatives to each of two sales offices. The company has now decided that if any sales representative is assigned to a sales office, a sales office must be established there. The cost of establishing a sales office is \$18,000. This altered problem is an example of a fixed-charge problem.

13. programming problem known as a "fixed

1. Reformulate the integer programming condition.
2. Solve this new problem by using the

[Page 20]

The Texas Consolidated Electronics Corporation has initiated a research and development program encompassing five different projects. The company is constrained first by the number of available management scientists (10) and second by the amount available for R&D projects (\$300,000). In addition, project 5 must also be selected (but not necessarily all of its resource requirements and the estimated

Project	Expense (\$1,000s)	Management Scientists Required
---------	-----------------------	--------------------------------------

14.

1	\$60	7
2	110	9
3	53	8
4	47	4
5	92	7
6	85	6
7	73	8
8	65	5

Formulate the integer programming model by using the computer.

Mazy's Department Store has decided to operate on a 24-hour basis. The store manager has divided the day into 6 four-hour periods and has determined the personnel requirements for each period:

Time	Personnel Needed
Midnight-4:00 A.M.	90
4:00-8:00 A.M.	215
8:00 A.M.-Noon	250

15.

Noon4:00 P.M.	65
4:008:00 P.M.	300
8:00 P.M.Midnight	125

Store personnel must report to work a time periods and must work for 8 cons manager wants to know the minimum to each 4-hour segment to minimize th Formulate and solve this problem.

The athletic boosters club for Beaconvi raising drive to purchase uniforms for a improve facilities. Donations will be solic telephone and personal contact. The bc

local college students to donate their time
average donation from each type of contact
volunteer to solicit each type of donation

Average Donation (\$)

	Phone	Personal
Day	\$16	\$33
Night	17	37

16.

The boosters club has gotten several boosters to
donate gasoline and cars for the college
maximum of 575 personal contacts da

The college students will donate a total 43 hours at night during the drive.

[Page 20]

The president of the boosters club wants types of donor contacts to schedule donations. Formulate and solve an integer problem. What is the difference in the total donations between the integer and non-integer solutions to this problem?

The Metropolitan Arts Council (MAC) will have a season of plays, concerts, and ballets. The season costs \$25,000 and will supposedly reach 50,000 people. The breakdown of the audience is as follows:

Age	Male	Female
18-34	12,000	20,000

<35	7,000	14,000
-----	-------	--------

A newspaper ad costs \$7,000, and the will reach an audience of 30,000 poten as follows:

Age	Male	Female
-----	------	--------

≥ 35	12,000	8,000
-----------	--------	-------

<35	6,000	4,000
-----	-------	-------

17.

A radio ad costs \$9,000, and it is estimated to reach 41,000 arts customers, with the following breakdown:

Age	Male	Female
≥ 35	7,000	11,000
< 35	10,000	13,000

The arts council has established several goals for the radio ad campaign: to reach at least 200,000 potential arts customers; to reach people who are more likely to buy tickets than the general population; to reach at least 1.5 times as many people as the radio ad costs; The council also believes women are more likely to buy tickets and make purchases to the arts than men, so it wants the campaign to be at least 60% female.

1. Formulate and solve an integer programming model to determine the number of ads of each type that will result in minimum cost.
2. Solve this model without integer requirements and compare the results.

Juan Hernandez, a Cuban athlete who has traveled to Europe frequently, is allowed to return to his home country with consumer items not generally available in Cuba. The total weight carried in a duffel bag, cannot exceed a certain limit. When he is in Cuba, he sells the items at highly inflated prices. The most popular items in Cuba are denim jeans, rock groups. The weight and profit (in dollars) for each item follows:

Item	Weight(lb.)	Profit(\$)
Denim jeans	2	\$90

CD players	3	150
Compact discs	1	30

[Page 20]

Juan wants to determine the combination of items to put in his duffel bag to maximize his profit. This is an example of an integer programming problem known as the knapsack problem. Formulate and solve this problem.

The Avalon Floor Cleaner Company is trying to decide how many salespeople it should allocate to its three regions: the East, the South, and the West. The company has 100 salespeople to allocate to the three regions. The annual average sales per salesperson in each region is as follows:

Region**Units per
Salesperson**

East

25,000

Midwest

18,000

West

31,000

19.

Because travel distances, costs of living in the three regions, the annual cost of having a salesperson in the East, \$11,000 in the Midwest, and \$13,000 in the West, and the company has \$700,000 budgeted for advertising exposure for its product, the company must have at least 10 salespeople. The company must allocate to each region a number of salespeople so that the total units sold. Formulate an integer program

and solve it by using the computer.

During the war with Iraq in 1991, the Toyota Motor Company developed a lightweight, all-terrain vehicle code-named Land Cruiser. The company is now planning to sell the vehicle in 10 plants that manufacture the vehicle and 10 distribution centers. The company is unsure of public reaction and is considering reducing its fixed operating costs by closing plants, even though it would incur an increase in transportation costs. The relevant costs for the problem are the transportation costs. The transportation costs are per thousand vehicles. For example, the cost of shipping 1,000 vehicles from plant C is \$32,000.

From Plant	Transportation Costs (\$1,000s) to Warehouse			
	A	B	C	D
1	\$56	\$21	\$32	\$65

20.

2	18	46	7	35
3	12	71	41	52
4	30	24	61	28
5	45	50	26	31

Annual Demand	6,000	14,000	8,000	10,000
--------------------------	-------	--------	-------	--------

Formulate and solve an integer program to assist the company in determining which plant should be closed and the number of units shipped from each plant to each warehouse.

The Otter Creek Winery produces three white, and a red. The winery has 30,000 pounds of grapes to produce wine this season. A cask of blue requires 375 pounds of grapes, a cask of white requires 375 pounds of grapes, and a cask of red requires 410 pounds. The winery has enough storage space to store 67 casks of wine. The winery has 14 hours of labor available. It requires 14 hours to produce a cask of white, and 18 hours to produce a cask of red. From records of previous years' sales, the winery estimates that it can sell at least twice as much blush as red and at least as much white as blush. The profit for a cask of blush is \$8,700, and the profit for a cask of white is \$8,700, and the profit for a cask of red is \$8,700. The winery wants to know the number of casks of each color to produce. Formulate and solve an integer programming problem.

Brenda Last, an undergraduate business student, is attempting to determine her course schedule. She is considering seven 3-credit-hour courses.

following table. Also included are the average hours that students expect to have to devote to each course (based on information from other students) and the average hours that teachers devote to each course, based on an analysis of the time that teachers in each course:

Course	Average Hours Per Week
Management I	5
Principles of Accounting	10
Corporate Finance	8
Quantitative Methods	12

22.	Marketing Management	7
	C Programming	10
	English Literature	8

An A in a course earns 4 quality credits, a C earns 2 quality credits, a D earns no quality credits per hour. Brenda will provide at least a 2.0 grade point average as a time student, which she must do to complete her degree. She must take at least 12 credit hours. Principles of Finance, Quantitative Methods, and Computer Applications in computing and mathematics, and Brenda must take more than two of these courses. To remain eligible for graduation, she needs to take at least one of the following: Management I, Principles of Accounting

Brenda wants to develop a course schedule. The number of hours she has to work each week is 10.

1. Formulate a 01 integer programming model for Brenda's problem.
2. Solve this model by using the computer. How many hours Brenda should expect to work each week and her minimum grade point average?

The artisans at Jewelry Junction in Phoenix make jewelry during a 2-month period for the store. They make bracelets, necklaces, and pins. Each bracelet requires 10 hours of gold and 17 hours of labor, each necklace requires 15 hours of gold and 10 hours of labor, and each pin requires 20 hours of gold and 5 hours of labor. Jewelry Junction has available 320 hours of labor. A bracelet sells for \$1,200, a necklace for \$900, and a pin for \$790. The store wants to know how many of each item to produce to maximize revenue.

23.

1. Formulate an integer programming model for Jewelry Junction's problem.
2. Solve this model by using the computer. How many of each item should the store produce with the solution with the integer constraint?

whether the rounded-down solution

- Harry and Melissa Jacobson produce handcrafted furniture at a workshop on their farm. They have obtained 1000 board feet of birch from a neighbor and are planning to make tables and ladder-back chairs during the summer. Each table requires 30 hours of labor, each chair requires 15 hours. They have a total of 480 hours of labor available. Each table requires 40 board feet of wood to make, and a chair requires 20 board feet. A table earns the couple \$575 in profit, a chair earns \$230. Most people who buy a table also want a chair. The Jacobsons must sell every table that is produced, at least for the summer. Although additional chairs can also be sold, the Jacobsons must solve an integer programming model to determine the number of tables and chairs the Jacobsons should make.
- 24.**

[Page 21]

The Jacobsons in [Problem 24](#) have been a successful furniture furnishings company that needs a wood supplier. A new company has asked the Jacobsons to provide wood for their new

- 25.** stools, and the Jacobsons would realize that a stool will require 4 board feet of wood. Formulate and solve an integer program to help the Jacobsons determine whether they should

The Skimmer Boat Company manufactures fiberglass recreational boats: a bass fishing boat, a speedboat, and a ski boat. The profit for a bass boat is \$12,000, and the profit for a speedboat is \$15,000. The company believes it will sell more bass boats than speedboats combined but no more than twice as many speedboats as bass boats in its production model, and bass boats and speedboats must be produced in equal numbers.

- 26.** The company has production capacity for 100 ski-type boats; however, a bass boat requires 200 square feet of fiberglass, a speedboat requires 300 square feet, and a ski boat requires 100 square feet of fiberglass. In addition, only 100 square feet of fiberglass are installed in the bass boats and speedboats are available. The company wants to produce boats of each type to produce to maximize profit. Formulate an integer programming model for this problem.

The Reliance Manufacturing Company produces a part that can be produced by any of the company's five work centers. The company can produce the part entirely on any of its five work centers or on multiple computerized machines. The cost of production varies from work center to work center, all of which are different because they use different machines and operators at different times. Each work center has a single operator, and each operator has a different output. Operators have different skill levels, resulting in different average daily output and product quality. The following table shows the average daily output and average number of defects per part for each of the company's five operators who are capable of producing the part:

Average Daily Output			
Operator	A	B	C
1	18	20	21
2	19	15	22

3	20	20	17
4	24	21	16
5	22	19	21

27.

Average Number of Defects per Machine			
Operator	A	B	C
1	0.3	0.9	0.6

2	0.8	0.5	1.1
3	1.1	1.3	0.6
4	1.2	0.8	0.6
5	1.0	0.9	1.0

[Page 21]

The company wants to determine which machine to maximize daily output and to less than 4%.

- 1.** Formulate a 01 integer program
- 2.** Solve this model by using the con

29. hour. Corsouth wants to know the number of temporary employees it should use, plus the cost it should arrange for. Formulate and solve the problem.

Rowntown Cab Company has 70 drivers who work 8-hour shifts. However, the demand for cabs varies dramatically according to the time of day. Between midnight and 4:00 A.M. the demand for the calls that are received have the same pattern as the people are going to the airport at that time for long-distance trips. It is estimated that the demand during that period. The largest fares received in the morning. Thus, the drivers who start their shift at 4:00 A.M. to 8:00 A.M. average \$500 in fares, while those who start at 8:00 A.M. average \$420. Drivers who start at 8:00 P.M. average \$300, while drivers who start at 4:00 P.M. average \$210 in fares during their 8-hour shift.

To retain customers and acquire new ones, the company must maintain a high customer service level. To do so, it

30.

number of drivers it needs working during each segment: 10 from midnight to 4:00 A.M., 15 from 4:00 A.M. to 8:00 A.M., 25 from 8:00 A.M. to noon, 20 from noon to 4:00 P.M., and 18 from 4:00 P.M. to midnight.

- 1.** Formulate and solve an integer programming model to schedule its drivers.
- 2.** If Rowntown has a maximum of 60 drivers working the late shift from midnight to 8:00 A.M., reflect this complication and solve the model.
- 3.** All the drivers like to work the day shift from 8:00 A.M. to 4:00 P.M., so the company has decided to limit the number who work this 8-hour shift to 20. reflect this restriction and solve it.

Globex Investment Capital Corporation has provided the following estimated returns (in millions of dollars) over the next 3 years:

**Year Sold (estimated |
\$1,000,000s)**

Company

1

2

1

\$14

\$18

2

9

11

3

18

23

4

16

21

5

12

16

6

21

23

31.

To generate operating funds, the company has a net worth of assets in year 1, \$25 million in year 2, and \$30 million in year 3. Globex wants to develop a plan for the next 3 years to maximize return.

Formulate an integer programming model for this problem by using the computer.

The Heartland Distribution Company is a company that has a contract with a grocery store chain in Southeast cities. The company wants to build warehouses/distribution centers in some of these cities to serve the stores in those cities plus all the other cities that don't have distribution centers. The company effectively service all stores within a 30-mile radius. The company wants to limit its fixed annual costs to \$10 million. The company wants to build the minimum number of

The following table shows the cities with the projected fixed annual charge for a

City	Annual Fixed Charge (\$1,000s)
1. Atlanta	\$276
2. Charlotte	253
3. Cincinnati	394
4. Cleveland	408

5. Indianapolis	282
-----------------	-----

6. Louisville	365
---------------	-----

7. Nashville	268
--------------	-----

8. Pittsburgh	323
---------------	-----

9. Richmond	385
-------------	-----

10. St. Louis	298
---------------	-----

1. Formulate an integer programming model to solve it by using the computer.
2. What is the solution if the cost coefficients are changed to the original model formulation? What

[Page 21]

The Kreeger Grocery Store chain has been a successful store chain. However, it now has too many stores in each other in certain cities. In Roanoke, Virginia, the company does not want any stores closer than 2 miles apart. The following are the monthly revenues (\$1,000s) for each store. The stores showing the general proximity of the stores to each other are circled:

Store	Monthly Revenue (\$1,000s)
-------	-------------------------------

1	\$127
---	-------

2	83
---	----

3	165
---	-----

4	96
---	----

5	112
---	-----

6	88
---	----

<u>33.</u>	7	135
------------	---	-----

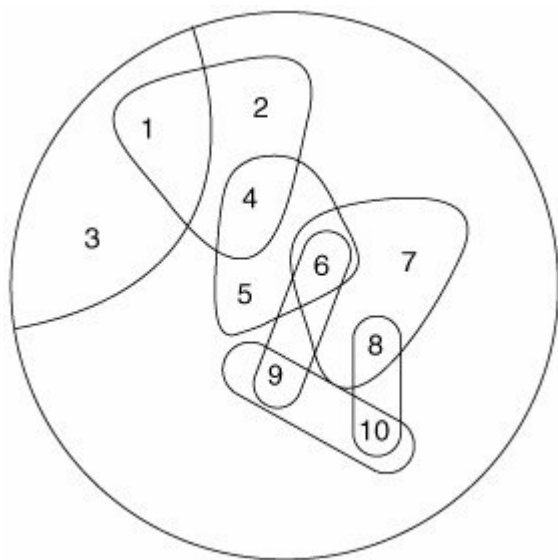
8	141
---	-----

9

117

10

94



Develop and solve an integer program
stores the Kreeger chain should keep o

A youth soccer club has contracted with an agency, to broker hotel rooms for out-of-town teams for the club's Labor Day weekend. The club has 12 teams it needs to arrange room for. The following tables show the number of rooms needed by each team, the number of rooms available at each hotel, and the maximum room rate each team is willing to pay.

Team	Max Rate	Room Needed
1. Arsenal	\$70	18
2. United	75	18
3. Wildcats	60	20

4. Rage	80	10
5. Rapids	110	10
6. Storm	90	10
7. Tigers	70	10
8. Stars	80	10
9. Comets	80	20
10. Hurricanes	65	10
11. Strikers	90	20

12. Bees	100	14
----------	-----	----

Hotel	Room Rate	Room Available
-------	-----------	----------------

34.	A. Holiday	\$90	4
	B. Roadside	75	20
	C. Bates	55	38
	D. Hampson	95	21

E. Tilton	100	20
F. Marks	80	30
G. Bayside	70	30
H. Harriott	80	50

[Page

- 1.** All of a team's rooms must be at model and develop a solution for as many teams as possible, according to the
- 2.** The travel agency has requested that you list the hotels it would prefer to stay at, including price, location, and facilities. The t

the following table:

							Total
Hotel Priority	1	2	3	4	5	6	
1	C	B	C	F	D	F	
2	G	G		H	E	A	
3		C		B	A	H	

Determine a revised hotel room allocation while reflecting their preferences to the

The Tech Software School contracts to the use of various software products and programs are all delivered on 3 consecutive daily durations, depending on the software week (including weekends), the school hires different companies for training sessions. The companies can send their employees only on certain days, as shown in the following table: training requirements per day:

Hours			
Company	First Day	Second Day	Third Day
A	4	6	3

35.

B	3	2	5
C	7	1	1
D	1	3	6
E	8	5	2
F	5	5	5
G	6	3	3
H	4	6	6

The school has 20 staff-hours per day sessions.

1. Develop a *feasible* training schedule.
2. Develop a *feasible* training schedule that minimizes the weekend training hours (days 6 and 7).

Each day Seacoast Food Services makes a delivery of its supplies in the metro Atlanta area. The truck starts at its warehouse, makes a delivery to one of the restaurants, and then returns to the warehouse. The mileage between each of the restaurants, 2, 3, 4, and 5,

36.	Location	1	2	3
	1		10	1
	2	10		1
	3	15	12	
	4	20	16	1
	5	40	24	2

Formulate and solve an integer program to find the shortest route (or tour) the truck should take to

each restaurant once, and return to the total distance traveled.

HTM Realty has a storefront rental property in a small college town next to the Tech campus that encompasses four business spaces that it currently has. HTM has a long-term tenant in each space, all of which are running out at approximately the same time. HTM wants to increase the rent for each space, and the current tenants have indicated that they will stay on at a slightly higher rental price that HTM wants. HTM has successfully negotiated and has gotten commitments from four new tenants. However, HTM realizes that there is a very small chance that current tenants will remain in business for the next year lease, while there is a much smaller chance that new tenants will be able to stay in business and fulfill their lease. The following table shows the current and proposed rent for each space, their monthly payments for a 3-year lease, and the probability that each will remain in business:

Space	Current Rent	Proposed Rent	Current Monthly Payment	Proposed Monthly Payment	Probability of Remaining in Business
-------	--------------	---------------	-------------------------	--------------------------	--------------------------------------

Tenant Payment

37.

A	Doughnut Shop	\$1,200	.90
B	Jewelry Store	800	.90
C	Music Store	1,000	.95
D	Gift & Card Shop	1,100	.95

HTM wants to lease each space to business. The table shows the expected lease payments. It also wants

businesses staying in business. Formulate a linear programming model for this problem.

Case Problem

PM COMPUTER SERVICES

Paulette Smith and Maureen Becker are seniors in engineering and business, respectively, at State University. They have set up a company, PM Computer Services, to assemble and sell their own brand of personal computers. They buy component parts on the open market from a variety of

sources in the United States and overseas, and they assemble their computers, mostly at night, in their three-bedroom apartment. They sell their computers primarily to departments at State University and to other students. They hire other students to perform the assembly operations and to test and package the computers. In addition to managing the operations, Paulette and Maureen help with all other tasks, including sales and accounting.

They pay the students who work for them \$8 per hour for a 40-hour week, or \$1,280 per month. They hire students on a monthly basis, and their delivery schedule is also on a monthly (i.e., end-of-the-month) basis. PM currently has five employees. PM Computers has determined that each of its employees is able to produce 12.7 computers, on average, per month. When the monthly demand for its computers exceeds its regular production capacity, PM employs limited overtime. Each computer

produced on an overtime basis adds \$12 to the labor cost of a computer. A PM employee can produce 0.6 computers per month on an overtime basis.

[Page 217]

Paulette and Maureen have received the following computer orders for the next 6 months:

Month i	Computer Orders
-----------------------------	------------------------

1

63

2	74
3	95
4	57
5	68
6	86

In the past, PM has met its demand strictly from regular and overtime production. To meet demand in some months

when it did not have sufficient regular and overtime production, the company would plan ahead and produce computers in previous months with available capacity. However, Paulette and Maureen's apartment was completely filled with components and workspace, so they could not store completed computers. Instead, they leased warehouse space in town to store their completed computers for delivery in future months. They had to transport the computers across town to

the warehouse and pay for all handling; also, the warehouse had to be climate controlled. The cost of holding a computer in storage at the warehouse is \$15 per month.

Paulette and Maureen are considering an alternative production strategy wherein they would hire new workers on a monthly basis as needed and fire workers when they are not needed. They estimate the cost of hiring new workers to be \$200, primarily for related paperwork and training. The cost of firing a worker is \$320,

or approximately 1 week's wages. They may want to rehire some of the workers they fire at a later date, so they want them to leave with a good feeling about PM.

Determine a planning schedule for PM Computer Services, indicating the number of employees working each month, including the number hired and the number fired, the number of computers produced each month in both regular time and overtime, and the number of computers carried over in inventory each month.

There should be no inventory left over after month 6. Provide integer solution values for these different variables.

Compare this solution with the one you would obtain without integer restrictions.

Case Problems

The Tennessee Pterodactyls

The Tennessee Pterodactyls is a new professional basketball franchise in Nashville. The team's general manager, Jerry East, and coach, Phil Riley, are trying to develop a roster of players. They drafted seven players from a pool to which the other teams in the league each contributed two players. However, the general manager

and coach perceive these acquisitions to be no more than role players. They believe that the nucleus of their new team must come from the free agents who are currently available on the market. The team is well under the salary cap, and the owner has made \$50 million per year available to them to sign players. The coach and general manager have put together the following list of 12 free agents, with important statistics for each, including their rumored asking price in terms of annual salary:

		Per-Game Averages		
Player	Position	Points	Rebounds	Assists
1. Mack Madonna	Backcourt	14.7	4.4	9.3
2. Darrell Boards	Frontcourt	12.6	10.6	2.1
3. Silk Curry	Backcourt	13.5	8.7	1.7
4. Ramon Dion	Backcourt	27.1	7.1	4.5
5. Joe				

Eastcoast	Backcourt	18.1	7.5	5.1
6. Abdul Famous	Frontcourt	22.8	9.5	2.4
7. Hiram Grant	Frontcourt	9.3	12.2	3.5
8. Antoine Roadman	Front court	10.2	12.6	1.8
9. Fred Westcoast	Frontcourt	16.9	2.5	11.4
10. Magic Jordan	Backcourt	28.5	6.5	1.3

11. Barry Bird	Front court	24.8	8.6	6.9
12. Grant Hall	Frontcourt	11.3	12.5	3.2

[Page 218]

Jerry and Phil want to sign five free agents. They would like the group they sign to average at least 80 points per game (16 points per player), pull down an average of 40 rebounds per game (8 per player), dish out

an average of 25 assists, and have averaged 190 minutes (38 minutes per player) per game in the past. Furthermore, they do not want to sign more than two front court and three back court players. Their immediate objective is to identify the players who as a group would meet their objectives at the minimum cost.

- 1.** Formulate an integer linear programming model to help the general manager and coach determine which players they should sign

and solve it by using the computer.

- 2.** Is the money provided by the owner sufficient to sign the group of players identified in (A)? If not, reformulate the model so that the available funds are a constraint and the objective is to maximize the average points of the group.

Case Problem

NEW OFFICES AT ATLANTIC MANAGEMENT SYSTEMS

Atlantic Management Systems is a consulting firm that specializes in developing computerized decision support systems for computer manufacturing companies. The firm currently has offices in Chicago, Charlotte, Pittsburgh, and Houston. It is considering opening new offices in one or

more cities including Atlanta, Boston, Washington, DC, St. Louis, Miami, Denver, and Detroit and it has \$14 million available for this purpose. Because of the highly specialized nature of its high-tech consulting work, the firm must necessarily staff any new offices with a minimum number of its employees from its existing offices. However, it has a limited number of employees available to transfer to any new offices. In addition, the cost of transferring employees depends on the city they are

leaving and the city to which they might move.

Following are the costs for opening a new office in each of the prospective cities and the start-up staffing needs at each office:

Prospective Office	Setup Cost (\$1,000,000s)	Staffing Needs (employees)
1. Atlanta	\$1.7	9
2. Boston	3.6	14

3. Denver	2.1	8
4. Detroit	2.5	12
5. Miami	3.1	11
6. St. Louis	2.7	7
7. Washington, DC	4.1	18

The numbers of employees available for transfer from each of the current offices are as

follows:

Existing Office	Available Employees
Chicago	24
Charlotte	19
Pittsburgh	16
Houston	21

The costs (in thousands of

dollars) of transferring an employee from an existing office to a new office differ according to housing costs and moving expenses plus cost of living adjustments. They are as follows:

Prospective New Offices

Current Office	1. <i>Atlanta</i>	2. <i>Boston</i>	3. <i>Denver</i>	4. <i>Detroit</i>	5. <i>Minneapolis</i>
1. Chicago	\$19	\$32	\$27	\$14	\$

2.	14	47	31	28	3
Charlotte					
3.					
Pittsburgh	16	39	26	23	3
4.					
Houston	22	26	21	18	2

[Page 219]

The firm has ranked the possible new offices according to their profit potential, with Washington, DC, being the best

(i.e., greatest potential), as follows:

New Office	Rank
Washington, DC	1
Miami	2
Atlanta	3
Boston	4
Denver	5
St. Louis	6

In addition, the firm wants at least one new office in the Midwest (i.e., Detroit and/or St. Louis) and one new office in the Southeast (i.e., Atlanta or Miami).

Formulate and solve an integer programming model to help Atlantic Systems determine how many new offices it should open, where they should be

located, and how to transfer employees.

Case Problems

Scheduling Television Advertising Slots at the United Broadcast Network

The United Broadcast Network (UBN) sells to advertisers commercial advertising slots on its television shows. The network announces its new fall television schedule during the previous spring and shortly thereafter begins selling its inventory of advertising slots to

its customers. Long-standing priority customers receive the first opportunities to purchase advertising slots. The new fall season begins in the third week of September. Advertising slots are mostly 15 seconds and 30 seconds in duration. The network develops a detailed sales plan for the year, typically on a monthly, bimonthly, 6-week, or quarterly basis. Advertisers generally have certain shows in mind that they prefer to advertise on in order to reach a certain demographic audience that they want to

market their product to, as well as a specific advertising budget. The network sells advertising slots at rates based on a performance score related to the popularity of a show, primarily determined by demographics and audience size.

Nanocom, a business software development firm, wants to purchase 30- and 15-second advertising slots on several shows this coming fall.

Nanocom wants to reach an older, more mature, upper-income audience that is likely

to include many high-tech businesspeople. It has an advertising budget of \$600,000, and it has informed the UBN advertising staff that it prefers to purchase ads on the following shows: *Bayside*, *Newsline*, *The Hour*, *Cops and Lawyers*, *The Judge*, *Friday Night Football*, and *ER Doctor*. *Newsline* and *The Hour* are news magazine shows, whereas the others are adult dramas, except for *Friday Night Football*, which is professional football. Nanocom wants at least 50% of the total number of advertising

slots it purchases to be on *Newsline*, *The Hour*, and *Friday Night Football*. Nanocom would like UBN to develop a sales plan covering the 6-week period beginning with the third week in October through November (including the November sweeps).

As indicated, UBN bases its sales plans on performance scores for the different shows. The primary objective of both UBN and the advertiser is to develop a sales plan that will achieve the highest total performance score. The

performance scores are based on several factors, including how well the show matches the desired audience demographics, the audience strength of the show, the historical ratings for the time slot, the competing shows in the same time slot, and the performance of adjacent shows. The network sets its advertising rates based on these performance scores. Both the performance scores and the rates are then multiplied by a weighting factor that is based on the week of the show and, because total

viewers vary according to the week of the year, the total audience expected during that week. The following table shows the costs, performance scores, and available inventory of advertising slots for 15- and 30-second commercials for each show:

[Page 220]

Show	Commercial Length (seconds)	Cost (\$1,000s)	Performance Score
<i>Bayside</i>	30	\$50.0	1

	15	25.0	
<i>Newsline</i>	30	41.0	1
	15	20.5	1
<i>The Hour</i>	30	36.0	
	15	18.0	
<i>Cops and Lawyers</i>	30	45.0	1
	15	27.5	

<i>The Judge</i>	30	52.0	1
	15	26.0	
<i>Friday Night Football</i>	30	25.0	
	15	12.5	
<i>ER Doctor</i>	30	46.0	1
	15	23.0	

The network advises, and Nanocom agrees, that there should be a minimum and maximum number of advertising slots during each of the 6 weeks and that Nanocom should have at most only one ad slot (either 15 or 30 seconds) per show per week. The following table shows the weighting factor each week and the minimum and maximum numbers of slots per week:

October

November

Week	<i>3</i>	<i>4</i>	<i>1</i>	<i>2</i>	<i>3</i>	<i>4</i>
Weight	1.1	1.2	1.2	1.4	1.4	1.6
Minimum	1	1	2	2	2	3
Maximum	4	5	5	5	5	5

Assist UBN in developing a sales plan for the 6week period to maximize the total performance score for Nanocom.

Case Problem

THE DRAPERSON PARKS AND RECREATION DEPARTMENT'S FORMATION OF GIRLS' BASKETBALL TEAMS

Each fall the Draperston Parks and Recreation Department holds a series of tryouts for its boys' and girls' youth basketball leagues. All leagues are formed by age group, each usually encompassing a couple of ages. One such grouping is the 12- to 13-year-old girls'

league. This is a particularly competitive age group because the girls are all trying to improve their skills in hopes of making the high school junior varsity team in the immediate future and the high school varsity in several years. Some of the girls even hope to make the high school varsity next year, when they are freshmen. This is also the first age group in which not all girls who try out are placed on teams; some are cut. In the younger age groups, all players are assigned to teams. However, at this age

group, four teams are formed, each with only seven players. This policy was agreed upon by a committee consisting of the Recreation Department director, a group of parents of past players, and the high school boys' and girls' basketball coaching staff. The small roster size allows each player to get a lot of playing time and makes the games more competitive. It also makes the teams more competitive in tournaments with teams from other towns and cities. This makes the girls very competitive, and it makes

their parents even more competitive. Sandy Duncan, the Recreation Department's director of team sports, knows from past experience that parents do a lot of jockeying and politicking during tryouts. This has given rise to parental protests regarding the tryout process and about how players are selected and assigned to teams. Sandy and the player evaluators have frequently been accused of showing favoritism and stupidity.

This year, in an effort to blunt some of this criticism, Sandy

has devised a tryout process that will hopefully remove her (and any other person) from direct involvement with the player selection and team formation process. First, she conducted tryouts for all 36 players who were trying out over two weekends that included four sessions each weekend: one on Friday evening, two on Saturday, and one on Sunday afternoon. She formed an evaluation team of several coaches from other towns and some of the current players and assistant coaches

from the nearby State University women's basketball team. Each player was assigned a tryout number and evaluated for different skills and ability, and at the end of the tryouts, each player was given an overall score from 1 to 10, with 10 being the highest, as follows:

[Page 221]

Player Score		Player Score		Player Score	
--------------	--	--------------	--	--------------	--

1	7	13	10	25	8
---	---	----	----	----	---

2	6	14	9	26	4
3	9	15	9	27	9
4	5	16	4	28	9
5	7	17	5	29	3
6	8	18	5	30	3
7	3	19	6	31	3
8	3	20	3	32	6
9	4	21	2	33	6

10	6	22	1	34	7
11	4	23	7	35	8
12	8	24	5	36	10

Sandy took a course in management science while she was a student at State University, and she wants to develop an integer linear programming model to use these evaluation scores to select the players and assign them to the four teams in a fair

and equitable manner. She wants to select the best players, and in order to have competitive, equally matched teams, she wants each team to have an average overall player evaluation score between 6 and 7. As she sat down one evening to work on this problem, she quickly realized that it would require a relatively complex integer programming model. Help Sandy formulate and solve an integer programming model for this problem to select the best players, assign seven players to each team, and

achieve the average team evaluation score Sandy wants for each team. Compare the teams formed by your model and determine whether you think they are competitive (i.e., whether the model has achieved Sandy's objectives). Also, determine whether any deserving players were unfairly cut by the model.

Chapter 6.

Transportation, Transshipment, and Assignment Problems

[Page 223]

In this chapter, we examine three special types of linear programming model formulation *transportation*, *transshipment*, and *assignment problems*. They are part of a

larger class of linear programming problems known as *network flow problems*. We are considering these problems in a separate chapter because they represent a popular group of linear programming applications.

These problems have special mathematical characteristics that have enabled management scientists to develop very efficient, unique mathematical solution approaches to them. These solution approaches are variations of the traditional simplex solution procedure.

Like the simplex method, we have placed these detailed manual, mathematical solution procedures called the *transportation method* and *assignment method* on the CD that accompanies this text. As in previous chapters, we will focus on model formulation and solution by using the computer, specifically by using Excel and QM for Windows.

The Transportation Model

The **transportation model** is formulated for a class of problems with the following unique characteristics: (1) A product is *transported* from a number of sources to a number of destinations at the minimum possible cost; and (2) each source is able to supply a fixed number of units of the product, and each destination has a fixed demand for the product. Although the general

transportation model can be applied to a wide variety of problems, it is this particular application to the transportation of goods that is most familiar and from which the problem draws its name.

*In a **transportation problem**, items are allocated from sources to destinations at a minimum cost.*

The following example demonstrates the formulation of the transportation model. Wheat is harvested in the Midwest and stored in grain elevators in three different cities: Kansas City, Omaha, and Des Moines. These grain elevators supply three flour mills, located in Chicago, St. Louis, and Cincinnati. Grain is shipped to the mills in railroad cars, each car capable of holding 1 ton of wheat. Each grain elevator is able to supply the following number of tons (i.e., railroad cars) of wheat to

the mills on a monthly basis:

<i>Grain Elevator</i>	<i>Supply</i>
1. Kansas City	150
2. Omaha	175
3. Des Moines	275
Total	600 tons

Each mill demands the following number of tons of wheat per month:

<i>Mill</i>	<i>Demand</i>
A. Chicago	200
B. St. Louis	100
C. Cincinnati	300
Total	600 tons

The cost of transporting 1 ton of wheat from each grain elevator (source) to each mill (destination) differs, according to the distance and rail system. (For example, the cost of shipping 1 ton of wheat from the grain elevator at Omaha to the mill at Chicago is \$7.) These costs are shown in the following table:

[Page 224]

Mill

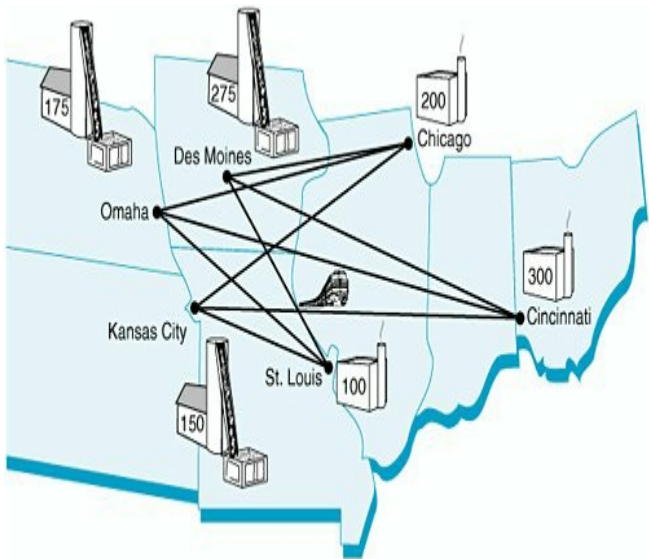
Grain Elevator	A. CHICAGO	B. ST. LOUIS	C. CINCINNATI
1. Kansas City	\$6	\$8	\$10
2. Omaha	7	11	11
3. Des Moines	4	5	12

The problem is to determine how many tons of wheat to transport from each grain elevator to each mill on a

monthly basis to minimize the total cost of transportation. A diagram of the different transportation routes, with supply and demand, is given in [Figure 6.1](#).

The linear programming model for a transportation problem has constraints for supply at each source and demand at each destination.

Figure 6.1. Network of transportation routes for wheat shipments



The linear programming model for this problem is formulated as follows:

minimize $Z = \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$
subject to

$$x_{1A} + x_{1B} + x_{1C} = 150$$

$$x_{2A} + x_{2B} + x_{2C} = 175$$

$$x_{3A} + x_{3B} + x_{3C} = 275$$

$$x_{1A} + x_{2A} + x_{3A} = 200$$

$$x_{1B} + x_{2B} + x_{3B} = 100$$

$$x_{1C} + x_{2C} + x_{3C} = 300$$

$$x_{ij} \geq 0$$

In this model the decision variables, x_{ij} , represent the number of tons of wheat transported from each grain elevator, i (where $i = 1, 2, 3$), to each mill, j (where $j = A, B, C$). The objective function represents the total transportation cost for each

route. Each term in the objective function reflects the cost of the tonnage transported for one route. For example, if 20 tons are transported from elevator 1 to mill A, the cost (\$6) is multiplied by x_{1A} (= 20), which equals \$120.

[Page 225]

The first three constraints in the linear programming model represent the supply at each elevator; the last three constraints represent the demand at each mill. As an

example, consider the first supply constraint, $x_{1A} + x_{1B} + x_{1C} = 150$. This constraint represents the tons of wheat transported from Kansas City to all three mills: Chicago (x_{1A}), St. Louis (x_{1B}), and Cincinnati (x_{1C}). The amount transported from Kansas City is limited to the 150 tons available. Note that this constraint (as well as all others) is an equation (=) rather than a $\leq$ inequality because all the tons of wheat available will be needed to meet the *total demand* of 600 tons. In other words, the three

mills demand 600 total tons, which is the exact amount that can be supplied by the three grain elevators. Thus, all that *can* be supplied *will be*, in order to meet demand. This type of model, in which supply exactly equals demand, is referred to as a **balanced transportation model**.

*In a **balanced transportation model** in which supply equals demand, all constraints are*

equalities.

Realistically, however, an **unbalanced problem**, in which supply exceeds demand or demand exceeds supply, is a more likely occurrence. In our wheat transportation example, if the demand at Cincinnati is increased from 300 tons to 350 tons, a situation is created in which total demand is 650 tons and total supply is 600 tons. This would result in the

following change in our linear programming model of this problem:

minimize $Z = 6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C}$
subject to

$$x_{1A} + x_{1B} + x_{1C} = 150$$

$$x_{2A} + x_{2B} + x_{2C} = 175$$

$$x_{3A} + x_{3B} + x_{3C} = 275$$

$$x_{1A} + x_{2A} + x_{3A} \leq 200$$

$$x_{1B} + x_{2B} + x_{3B} \leq 100$$

$$x_{1C} + x_{2C} + x_{3C} \leq 350$$

$$x_{ij} \geq 0$$

One of the demand constraints will not be met because there is not enough total supply to meet total demand. If, instead, supply exceeds demand, then the supply constraints would be

$\leq$.

Sometimes one or more of the routes in the transportation model may be *prohibited*. That is, units cannot be transported from a particular source to a particular destination. When this situation occurs, we must make sure that the variable representing that route does not have a value in the optimal solution. This can be accomplished by assigning a very large relative cost as the coefficient of this prohibited variable in the objective function. For example, in our

wheat-shipping example, if the route from Kansas City to Chicago is prohibited (perhaps because of a rail strike), the variable x_{1A} is given a coefficient of 100 instead of 6 in the objective function, so x_{1A} will equal zero in the optimal solution because of its high relative cost. Alternatively the prohibited variable can be deleted from the model formulation.



Time Out: For Frank L. Hitchcock and Tjalling C. Koopmans

Several years before George Dantzig formalized the linear programming technique, in 1941 Frank L. Hitchcock formulated the transportation problem as a method for supplying a product from several factories to a number of cities, given varying freight rates. In 1947 T. C. Koopmans independently formulated the same type of problem. Koopmans, an American economist originally from the Netherlands and a professor at Chicago and Yale, was awarded the Nobel Prize in 1975.

[Page 226]

Management Science

Application: Transportation

Models at Nu-kote

International

Nu-kote International, located near Nashville, Tennessee, is the largest independent manufacturer and distributor of supplies for home and office printing equipment, including ink-jet, laser, and toner cartridges; ribbons; and thermal fax supplies. It produces more than 2,000 products for use in more than 30,000 different types of imaging devices, and it serves more than 5,000 customers (e.g., commercial dealers and retail stores) around the world.

Nu-kote's transportation network

includes five plants and four warehouses in China; Chatsworth, California; Rochester, New York; Connellsville, Pennsylvania; and Franklin, Tennessee. Almost 80% of Nu-kote's products are manufactured in its Chatsworth plant or imported from China through the port of Long Beach. Nu-kote originally shipped most of its orders to its customers from its Franklin warehouse, which is within 500 miles of 50% of the U.S. population and within 1,000 miles of 80% of its customers. The other three U.S. warehouses were used for storing materials for manufacturing and for storing finished goods before shipping them to Franklin.

Nu-kote developed an Excel spreadsheet linear programming model to determine the optimal (least-cost) shipments of items from its vendors and plants to its

warehouses and from its warehouses to its customers. Different linear programming models were tested for different warehouse configurations; they included between 5,000 and 9,700 variables and 2,450 constraints. Decision variables were defined for the units of each product shipped from each vendor and plant to each warehouse and from each warehouse to each customer; constraints included capacities and demand at each facility. Nu-kote used the models to determine that it was more cost-effective to ship products to customers from two warehouses than from one, so it expanded its Chatsworth warehouse to ship finished products to customers from it, in addition to shipping from its Franklin warehouse. This new two-warehouse system reduced shipping times to its customers and saved

approximately \$1 million in annual transportation and inventory costs.



Source: L. J. LeBlanc, J. A. Hill, G. W. Greenwell, and A. O. Czesnat, "Nukote's Spreadsheet Linear-Programming Models for Optimizing Transportation," *Interfaces* 34, no. 2 (March/April 2004): 139-146.

Computer Solution of a Transportation Problem

Because a transportation problem is formulated as a linear programming model, it can be solved with Excel and using the linear programming module of QM for Windows that we introduced in previous chapters. However, management science packages such as QM for Windows also have transportation modules that enable problems to be

input in a special transportation tableau format. We will first demonstrate solution of the wheat shipping problem solved earlier in this chapter, this time by using Excel spreadsheets.

Computer Solution with Excel

We will first demonstrate how to solve a transportation problem by using Excel. A transportation problem must be solved in Excel as a linear programming model, using

Solver, as demonstrated in [Chapters 3](#) and [4](#). [Exhibit 6.1](#) shows a spreadsheet setup to solve our wheat-shipping example. Notice that the objective function formula for total cost is contained in cell C10 and is shown on the formula bar at the top of the spreadsheet. It is constructed as a SUMPRODUCT of the decision variables in cells **C5:E7** and a cost array in cells **K5:M7**. Cells **G5:G7** in column G and cells **C9:E9** in row 9, labeled "Grain Shipped," contain the constraint formulas

for supply and demand, respectively. For example, the constraint formula in cell G7 for the grain shipped from Des Moines to the three mills is **=C7+D7+E7**, and the right-hand-side quantity value of 275 available tons is in cell F7.

[Page 227]

Exhibit 6.1.

[\[View full size image\]](#)

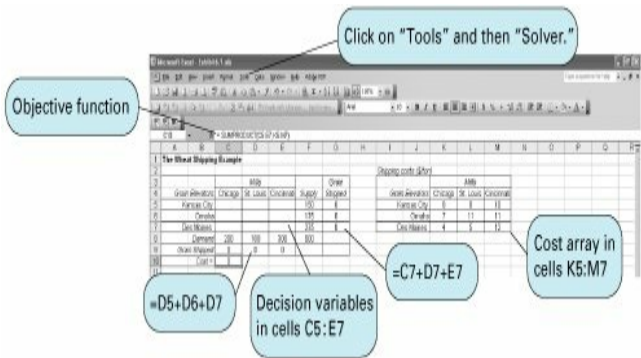
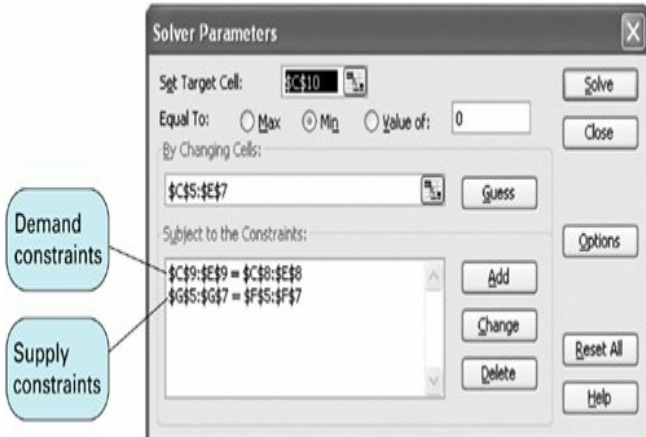


Exhibit 6.2 shows the Solver Parameters screen for our example. Cell C10, containing the objective function formula, is minimized. The constraint formula **C9:E9=C8:E8** includes all three demand constraints,

and the constraint formula **G5:G7=F5:F7** includes all three supply constraints. Before solving this problem, remember to click on "Options" and then, on the resulting screen, click on "Assume Linear Model" to invoke the linear programming solution approach and click on "Assume Non-negative."

Exhibit 6.2.

[\[View full size image\]](#)



[Exhibit 6.3](#) shows the optimal solution, with the amounts shipped from sources to destinations and the total cost. [Figure 6.2](#) shows a network

diagram of the optimal shipments.

[Page 228]

Exhibit 6.3.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 6.3.xls

File Edit View Insert Format Tools Data Window Help 100%

Type a question for help

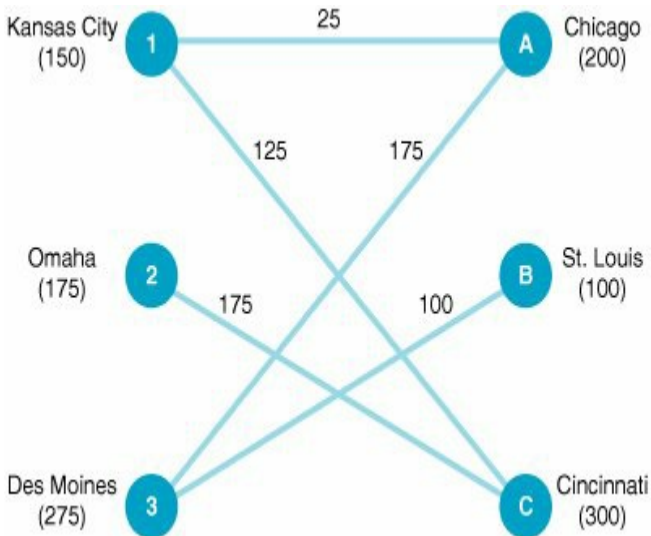
Formulas with Dependencies: A1:100 x 10

fx = SUMPRODUCT(C5:E7:H5:H7)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	The Wheat Shipping Example																	
2																		
3					Miles													
4		Grain Elevators	Chicago	St. Louis	Cincinnati	Supply	Shipped											
5		Kansas City	25	0	125	150	150											
6		Omaha	0	0	175	175	175											
7		Des Moines	175	100	0	275	275											
8		Demand	200	100	300	600												
9		Grain Shipped	200	100	300													
10		Cost =	4625															

		Chicago	St. Louis	Cincinnati
Grain Elevators				
Kansas City	6	8	10	
Omaha	7	11	11	
Des Moines	4	5	12	

Figure 6.2.
Transportation
network solution for
wheat shipping
example



**Computer Solution with
Excel QM**

Excel QM, which we introduced in [Chapter 1](#), includes a spreadsheet macro for transportation problems. Recall from [Chapter 1](#) that when Excel QM is activated, "QM" is shown on the menu bar at the top of the spreadsheet. Clicking on "QM" and then selecting "Transportation" will result in the Spreadsheet Initialization window appearing, as shown in [Exhibit 6.4](#). In this window we set the number of sources (i.e., origins) and destinations to 3, select "Minimize" as the objective, and type in the title

for our example. To exit this window, we click on "OK," and the spreadsheet shown in [Exhibit 6.5](#) is displayed. The transportation problem is completely set up as shown, with all the necessary formulas already in the cells. However, initially the data values in cells **B10:E13** are empty; the spreadsheet in [Exhibit 6.5](#) includes the values that we have typed in.

Exhibit 6.4.

Spreadsheet Initialization



Title:

Enter the number of origins



Name for origin

(Use A for A, B, C ... or a for a, b, c ...)

Enter the number of destinations



Name for destination

Objective

☐ Maximize

☒ Minimize

Use Default Settings

Help

Cancel

OK

Exhibit 6.5.

[\[View full size image\]](#)

1. Click on "QM" to access the macro menu.

The screenshot shows a Microsoft Excel spreadsheet titled "Exhibit6.5.xls". The spreadsheet is divided into two main sections: "Data" and "Shipments".

Data Section:

	M11	M12	M13	Supply
Grain Elevator 1	0	8	10	150
Grain Elevator 2	7	11	11	175
Grain Elevator 3	4	5	12	275
Demand	250	150	350	600/600

Shipments Section:

	M11	M12	M13	Row Totals
Grain Elevator 1				0
Grain Elevator 2				0
Grain Elevator 3				0
Column Totals	0	0	0	0/0

Callouts:

- Callout 1:** Points to the "QM" menu item in the Excel menu bar.
- Callout 2:** Points to the "Shipments" table, stating: "2. Enter data values for problems; initially this array is blank."
- Callout 3:** Points to the "Tools, Solver, and Solve" menu path, stating: "3. Click on 'Tools,' 'Solver,' and then 'Solve.'"

Formulas and Totals:

- Formula bar: `=SUMPRODUCT(B9:D12,B17:D17,0:0)`
- Cell B22: `Total Cost` with a value of `0`.

To solve the problem, we follow the instructions in the box superimposed on the spreadsheet in [Exhibit 6.5](#) click on "Tools" at the top of the spreadsheet, then click on "Solver," and when Solver appears, click on "Solve." We have not shown the Solver window here; however, it is very similar to the one shown in [Exhibit 6.2](#). It already includes all the decision variables and constraints required to solve the problem;

thus, all that is needed is to click on "Solve." The spreadsheet with the solution is shown in [Exhibit 6.6](#). You will notice that although the total cost is the same as in the solution we obtained with Excel in [Exhibit 6.3](#), the decision variable values in cells **B17:D19** are different. This is because this problem has multiple optimal solutions, and this is an alternate solution to the one we got previously.

Exhibit 6.6.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 5.xls

File Edit View Insert Format Tools Data Window Help 100% 10

Formulas: =SUM(B3:D3)*B7/D3

Wheat Shipping Example

Transportation

Enter the transportation costs, supplies and demands in the shaded area. Then go to **TOOLS, SOLVER, SOLVE** on the menu bar at the top. **SOLVER** is not a menu option in the Tools menu then go to **TOOLS, ADD-INS**.

Data

COSTS	M11	M12	M13	Supply
Grain Elevator 1	8	9	10	150
Grain Elevator 2	7	11	11	175
Grain Elevator 3	4	5	12	275
Demand	200	100	300	600/600

Shipments

Shipments	M11	M12	M13	Row Total
Grain Elevator 1	0	0	150	150
Grain Elevator 2	25	0	150	175
Grain Elevator 3	175	100	0	275
Column Total	200	100	300	600/600

Total Cost: 4525

QM for Windows Solution

To access the transportation module for QM for Windows, click on "Module" at the top of the screen and then click on "Transportation." Once you are in the transportation module, click on "File" and then "New" to input the problem data. QM for Windows allows for any of three initial solution methods: northwest corner, minimum cell cost, or VAM to be selected. These are the three starting solution procedures used in the mathematical procedure for solving transportation problems.

Because these techniques are not included in the text (but are included on the accompanying CD), it makes no difference which starting procedure we use. [Exhibit 6.7](#) shows the input data for our wheat shipping example.

[Page 230]

Exhibit 6.7.

[\[View full size image\]](#)

File Edit View Window Format Tools Help

100%

0.2 0.00

Destination
☒ Maximize
☐ Minimize

Starting method
 Any starting method

Iteration
 The cell cannot be changed

The Network Shipping Example

	Chicago	St. Louis	Cincinnati	Supply
Kansas City	8	8	12	100
Omaha	7	11	11	125
San Antonio	4	6	12	200
DEMAND	200	100	200	

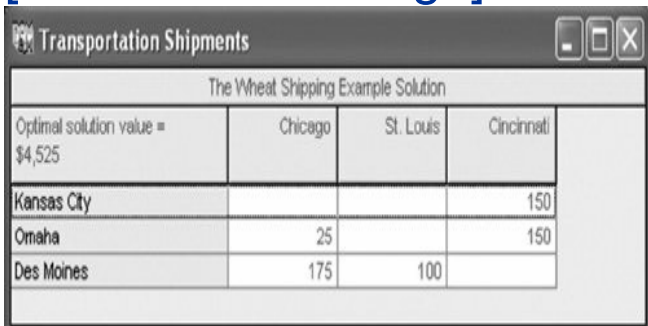
Use any starting method.

Once the data are input, clicking on "Solve" at the top of the screen will generate the solution, as shown in [Exhibits 6.8](#) and [6.9](#). QM for Windows will indicate if multiple optimal solutions exist, but it will not identify alternate solutions. This is the same optimal

solution that we achieved earlier in our Excel QM solution. [Exhibits 6.8](#) and [6.9](#) show the solution results in two different tables, or formats. [Exhibit 6.8](#) shows the shipments in each cell (or decision variable) plus total cost, while [Exhibit 6.9](#) lists the individual shipments from each source to destination and their costs.

Exhibit 6.8.

[\[View full size image\]](#)



The screenshot shows a software window titled "Transportation Shipments" with standard Windows window controls (minimize, maximize, close) in the top right corner. The window displays a table titled "The Wheat Shipping Example Solution". The table has four columns: "Optimal solution value =", "Chicago", "St. Louis", and "Cincinnati". The first row shows the optimal solution value as "\$4,525". The subsequent rows show the shipping quantities for three cities: Kansas City, Omaha, and Des Moines. The quantities are: Kansas City (150 to Cincinnati), Omaha (25 to Chicago, 150 to Cincinnati), and Des Moines (175 to Chicago, 100 to St. Louis).

The Wheat Shipping Example Solution			
Optimal solution value =	Chicago	St. Louis	Cincinnati
\$4,525			
Kansas City			150
Omaha	25		150
Des Moines	175	100	

Exhibit 6.9.

[\[View full size image\]](#)

Shipping list



The Wheat Shipping Example Solution

From	To	Shipment	Cost per unit	Shipment cost
Kansas City	Cincinnati	150	10	1,500
Omaha	Chicago	25	7	175
Omaha	Cincinnati	150	11	1,650
Des Moines	Chicago	175	4	700
Des Moines	St. Louis	100	5	500

The Transshipment Model

The **transshipment model** is an extension of the transportation model in which intermediate transshipment points are added between the sources and destinations. An example of a transshipment point is a distribution center or warehouse located between plants and stores. In a transshipment problem, items may be transported from sources through transshipment

points on to destinations, from one source to another, from one transshipment point to another, from one destination to another, or directly from sources to destinations, or some combination of these alternatives.

[Page 231]

*The **transshipment model** includes intermediate points between sources and destinations.*

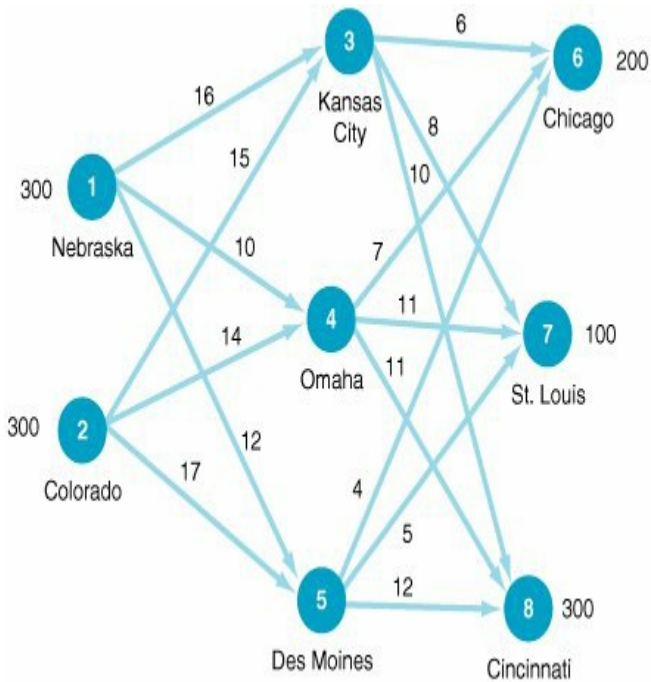
We will expand our wheat shipping example to demonstrate the formulation of a transshipment model. Wheat is harvested at farms in Nebraska and Colorado before being shipped to the three grain elevators in Kansas City, Omaha, and Des Moines, which are now transshipment points. The amount of wheat harvested at each farm is 300 tons. The wheat is then shipped to the mills in Chicago, St. Louis, and Cincinnati. The shipping costs

from the grain elevators to the mills remain the same, and the shipping costs from the farms to the grain elevators are as follows:

Farm	Grain Elevator		
	3. KANSAS CITY	4. OMAHA	5. DES MOINES
1. Nebraska	\$16	\$10	\$12
2. Colorado	15	14	17

The basic structure of this model is shown in the graphical network in [Figure 6.3](#).

Figure 6.3. Network of transshipment routes



As with the transportation problem, a linear programming model is developed with supply and demand constraints. The available supply constraints for the farms in Nebraska and Colorado are

$$x_{13} + x_{14} + x_{15} = 300$$

$$x_{23} + x_{24} + x_{25} = 300$$

The demand constraints at the Chicago, St. Louis, and Cincinnati mills are

$$x_{36} + x_{46} + x_{56} = 200$$

$$x_{37} + x_{47} + x_{57} = 100$$

$$x_{38} + x_{48} + x_{58} = 300$$

Next we must develop constraints for the grain elevators (i.e., transshipment points) at Kansas City, Omaha, and Des Moines. To develop these constraints we follow the principle that at each transshipment point, the amount of grain shipped *in* must also be shipped *out*. For example, the amount of grain shipped into Kansas City is

$$x_{13} + x_{23}$$

and the amount shipped out is

$$x_{36} + x_{37} + x_{38}$$

Thus, because whatever is shipped in must also be shipped out, these two amounts must equal each other:

$$x_{13} + x_{23} = x_{36} + x_{37} + x_{38}$$

or

$$x_{13} + x_{23} - x_{36} - x_{37} - x_{38} = 0$$

The transshipment constraints for Omaha and Des Moines are constructed similarly:

$$x_{14} + x_{24} - x_{46} - x_{47} - x_{48} = 0$$

$$x_{15} + x_{25} - x_{56} - x_{57} - x_{58} = 0$$

The complete linear programming model, including the objective function, is summarized as follows:

$$\begin{array}{ll} \text{minimize} & \$16x_{13} + 10x_{14} + 12x_{15} + \\ Z & 15x_{23} + 14x_{24} + 17x_{25} + \\ & 6x_{36} + 8x_{37} + 10x_{38} \end{array}$$

$$+ 7x_{46} + 11x_{47} + 11x_{48} + \\ 4x_{56} + 5x_{57} + 12x_{58}$$

subject
to

$$x_{13} + x_{14} + x_{15} = 300$$

$$x_{23} + x_{24} + x_{25} = 300$$

$$x_{36} + x_{46} + x_{56} = 200$$

$$x_{37} + x_{47} + x_{57} = 100$$

$$x_{38} + x_{48} + x_{58} = 300$$

$$x_{13} + x_{23} - x_{36} - x_{37} - x_{38} = 0$$

$$x_{14} + x_{24} - x_{46} - x_{47} - x_{48} = 0$$

$$x_{15} + x_{25} - x_{56} - x_{57} - x_{58} = 0$$

$$x_{ij} \geq 0$$

Computer Solution with Excel

Because the transshipment model is formulated as a linear programming model, it can be solved with either Excel or QM for Windows. Here we will demonstrate its solution with Excel.

[Exhibit 6.10](#) shows the spreadsheet solution and [Exhibit 6.11](#) the solver for our wheat shipping transshipment example. A network diagram of the optimal solution is shown in

Figure 6.4. The spreadsheet is similar to the original spreadsheet for the regular transportation problem in Exhibit 6.1 except that there are two tables of variables one for shipping from the farms to the grain elevators and one for shipping grain from the elevators to the mills. Thus, the decision variables (i.e., the amounts shipped from sources to destinations) are in cells **B6:D7** and **C13:E15**. The constraint for the amount of grain shipped from the farm in Nebraska to the three grain

elevators (i.e., the supply constraint for Nebraska) in cell F6 is **=SUM(B6:D6)**, which sums cells **B6+C6+D6**. The amount of grain shipped to Kansas City from the farms in cell B8 is **=SUM(B6:B7)**. Similar constraints are developed for the shipments from the grain elevators to the mills.

[Page 233]

Exhibit 6.10.

[View full size image]

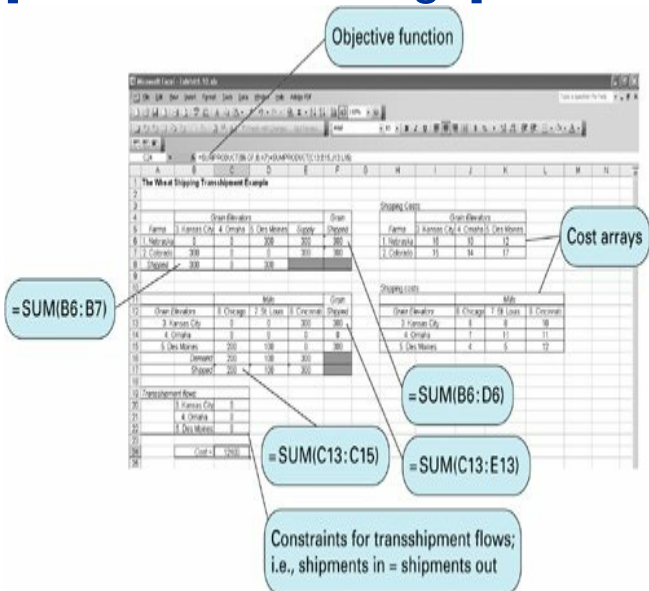


Exhibit 6.11.

[\[View full size image\]](#)

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-

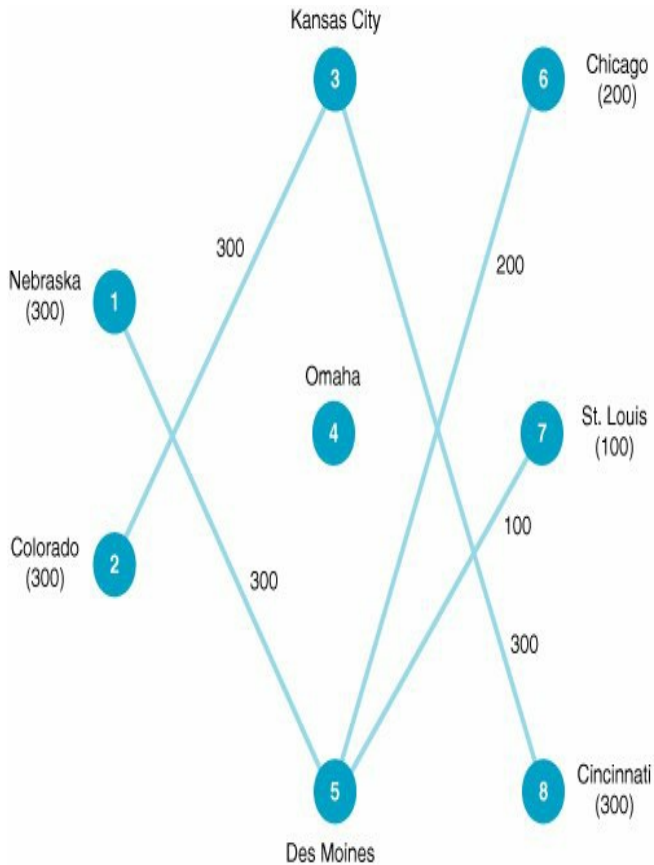
Transshipment constraints in cells C20:C22

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

Two cost arrays have been

developed for the shipping costs in cells **I6:K7** and cells **J13:L15**, which are then multiplied by the variables in cells **B6:D7** and **C13:E15** and added together. The objective function,
=SUMPRODUCT(B6:D7,I6:K7
is shown on the toolbar at the top of [Exhibit 6.10](#).
Constructing the objective function with cost arrays like this is a little easier than typing in all the variables and costs in a single objective function when there are a lot of variables and costs.

**Figure 6.4.
Transshipment
network solution for
wheat shipping
example**



The Assignment Model

The **assignment model** is a special form of a linear programming model that is similar to the transportation model. There are differences, however. In the assignment model, the supply at each source and the demand at each destination are each limited to one unit.

*An **assignment model** is for a special*

*form of
transportation
problem in which all
supply and demand
values equal one.*

The following example will demonstrate the assignment model. The Atlantic Coast Conference (ACC) has four basketball games on a particular night. The conference office wants to assign four teams of officials to

the four games in a way that will minimize the total distance traveled by the officials. The supply is always one team of officials, and the demand is for only one team of officials at each game. The distances in miles for each team of officials to each game location are shown in the following table:

The travel distances to each game for each team of officials

Game Sites

Officials	RALEIGH	ATLANTA	DURHAM	CLEMSON
-----------	---------	---------	--------	---------

A	210	90	180	160
B	100	70	130	200
C	175	105	140	170
D	80	65	105	120

The linear programming formulation of the assignment model is similar to the formulation of the transportation model, except all

the supply values for each source equal one, and all the demand values at each destination equal one. Thus, our example is formulated as follows:

$$\begin{aligned} \text{minimize } Z = & 210x_{AR} + 90x_{AA} + 180x_{AD} + 160x_{AC} + 100x_{BR} + 70x_{BA} + 130x_{BD} \\ & + 200x_{BC} + 175x_{CR} + 105x_{CA} + 140x_{CD} + 170x_{CC} + 80x_{DR} + 65x_{DA} \\ & + 105x_{DD} + 120x_{DC} \end{aligned}$$

subject to

$$x_{AR} + x_{AA} + x_{AD} + x_{AC} = 1$$

$$x_{BR} + x_{BA} + x_{BD} + x_{BC} = 1$$

$$x_{CR} + x_{CA} + x_{CD} + x_{CC} = 1$$

$$x_{DR} + x_{DA} + x_{DD} + x_{DC} = 1$$

$$x_{AR} + x_{BR} + x_{CR} + x_{DR} = 1$$

$$x_{AA} + x_{BA} + x_{CA} + x_{DA} = 1$$

$$x_{AD} + x_{BD} + x_{CD} + x_{DD} = 1$$

$$x_{AC} + x_{BC} + x_{CC} + x_{DC} = 1$$

$$x_{ij} \geq 0$$

This is a *balanced* assignment model. An *unbalanced* model exists when supply exceeds demand or demand exceeds supply.

Computer Solution of an Assignment Problem

The assignment problem can be solved using the assignment modules in QM for Windows and Excel QM, and with Excel spreadsheets. We will solve our example of assigning ACC officials to game sites first by using Excel, followed by Excel QM, and then by using QM for Windows.

Computer Solution with Excel

As is the case with transportation problems, Excel can be used to solve assignment problems, but only as linear programming models. [Exhibit 6.12](#) shows an Excel spreadsheet for our ACC basketball officials example. The objective function in cell C11 was developed by creating a mileage array in cells **C16:F19** and multiplying it by the decision variables in cells

C5:F8. The model constraints for available teams (supply) are contained in the cells in column H, and the constraints, for teams of officials at the game sites (demand) are contained in the cells in row 10.

[Page 236]

Exhibit 6.12.

(This item is displayed on page 235 in the print version)

[\[View full size image\]](#)

Objective function

Decision variables, C5:F8

The screenshot shows an Excel spreadsheet titled "The ACC Basketball Example". The spreadsheet is divided into two main sections: "Team Sides" and "Mileage".

Team Sides Table:

Official Team	Home	Away	Quarter	Season	Available	Assigned
A					1	0
B					1	0
C					1	0
D					1	0

Mileage Table:

Official Team	Home	Away	Quarter	Season
A	210	90	180	180
B	150	70	130	200
C	175	100	140	170
D	80	80	100	120

Formulas and Callouts:

- Objective function:** A callout points to the formula $=C5+D5+E5+F5$ in cell G5.
- Decision variables, C5:F8:** A callout points to the range C5:F8.
- Mileage array:** A callout points to the range D5:D8.
- Formula:** A callout points to the formula $=D5+D6+D7+D8$ in cell G8.

[Exhibit 6.13](#) shows the Solver Parameters screen for our example. Before clicking on "Solve," remember to invoke the Options screen and click on

"Assume Linear Model" to solve as a linear programming model. The optimal solution is shown in [Exhibit 6.14](#). A network diagram of the optimal solution is shown in [Figure 6.5](#).

Exhibit 6.13.

Solver Parameters ✕

Set Target Cell: 

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells: 

Subject to the Constraints:

\$C\$10:\$F\$10 = \$C\$9:\$F\$9

\$H\$5:\$H\$8 = \$G\$5:\$G\$8

^

▼

Exhibit 6.14.

[\[View full size image\]](#)

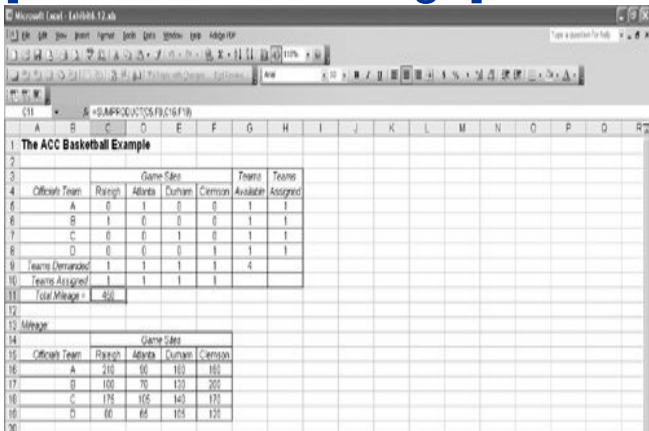
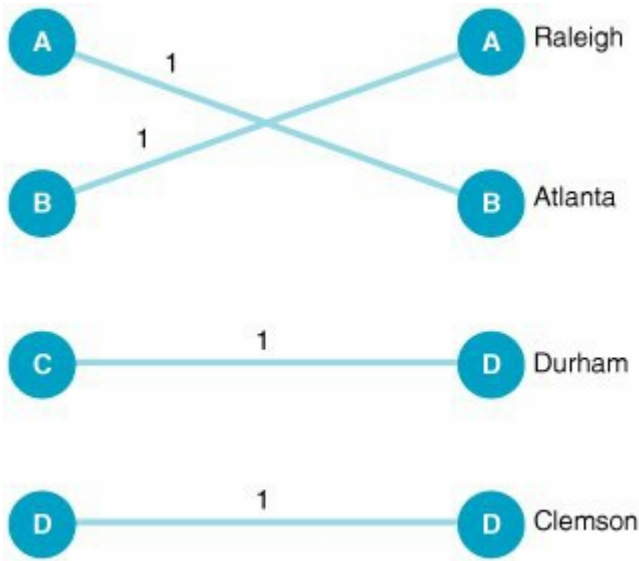


Figure 6.5.
Assignment network

solution for ACC officials example



Computer Solution with Excel QM

Excel QM also includes a spreadsheet macro for assignment problems. It is very similar to the Excel QM macro demonstrated earlier in this chapter for the transportation problem, and the spreadsheet setup is very much like that of the Excel spreadsheet in

[Exhibit 6.14](#). The assignment macro is accessed (from the "QM" menu) and applied similarly in the transportation problem macro. It already includes all the cell and constraint formulas necessary to solve the problem. The solution to our ACC basketball example with Excel QM is shown in [Exhibit 6.15](#).

Exhibit 6.15.

[\[View full size image\]](#)

Microsoft Excel - Exhibit15.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

Formulas: =SUM(CELLS(10:17),E20)

ACC Basketball Example

Assignment

Enter the assignment costs in the shaded area. Then go to TOOLS, SOLVER, SOLVE on the menu bar at the top.
If SOLVER is not a menu option in the Tools menu then go to TOOLS, ADD-INS. If SOLVER is not an add-in option then reinstall Excel.

Data

Mileage	Game 1	Game 2	Game 3	Game 4
Official 1	210	90	100	100
Official 2	100	70	130	200
Official 3	175	105	140	170
Official 4	60	95	105	120

Assignments

Shipments	Game 1	Game 2	Game 3	Game 4	Row Total
Official 1	0	1	0	0	1
Official 2	1	0	0	0	1
Official 3	0	0	1	0	1
Official 4	0	0	0	1	1
Column Total	1	1	1	1	4

Total Cost 450

Computer Solution with QM for Windows

The data input for our example is shown in [Exhibit 6.16](#), followed by the solution in [Exhibit 6.17](#).

Exhibit 6.16.

[\[View full size image\]](#)

The screenshot shows a Microsoft Excel spreadsheet. The title bar reads 'File Edit View Window Format Tools' and 'Excel - [Book1]'. The menu bar includes File, Edit, View, Window, Format, and Tools. The toolbar contains various icons for file operations, editing, and formatting. The status bar at the bottom indicates 'A12: B5: Active Cell'.

The spreadsheet contains a table with the following data:

	Range	Alerts	Duties	Costs
A	210	90	180	180
B	100	70	130	200
C	175	100	140	170
D	80	80	100	120

Exhibit 6.17.

[\[View full size image\]](#)

ACC Basketball Solution				
Optimal solution value = \$450	Raleigh	Atlanta	Durham	Clemson
A	210	Assign 90	180	160
B	Assign 100	70	130	200
C	175	105	Assign 140	170
D	80	65	105	Assign 120

[Page 238]

Management Science

Application: Assigning Managers to Construction Projects

The Nashville office of Heery International contracts with the state of Tennessee, various municipalities, and private firms for construction projects, such as hospitals, office buildings, state park facilities, university libraries, dorms and classroom buildings, and prisons. Projects average about \$2 million, although they can vary in size from \$50,000 up to \$50 million. When assigning managers to projects, Heery's objective is to minimize the total cost while maintaining a balanced workload among its

managers. At any one time, Heery will have approximately 7 managers (who live in different cities) that it must assign to up to 114 projects. Heery developed a modified version of the assignment model that is solved by using an Excel spreadsheet. The solution approach is robust enough to handle new projects, the termination of old projects, the resignation of managers, and the hiring of new managers over the passage of time.

The objective function in Heery's model is to minimize project intensity (instead of cost). Project intensity is a logarithmic function of project cost and manager driving time to a project. This intensity function quantifies a manager's effort by assigning relative (dimensionless) values to the projects, based on a project's dollar value and the time

required for the manager to drive to a project. As projects get larger (in dollar value), there is less difference between the intensities necessary to manage them. The model includes constraints that balance the monthly workload given to each manager. One set of constraints limits each manager's total intensity to a maximum amount each month for the next 12 months. Another set of constraints ensures fairness by specifying that each manager's total monthly intensity must be at least a specified amount. A final set of constraints ensures that each project has exactly one manager assigned to it. In some cases experienced managers are assigned to more important projects *a priori* manually in the spreadsheet. Also, continuity of previous assignments to ongoing projects is necessary. Further, some

projects in an outlying area might require the same manager simply because of their proximity to each other. For example, three projects at Middle Tennessee State University stadium, bookstore, and library needed to be managed by one person.



Because of the natural progression of construction projects over time, Heery updates the spreadsheet model each month at its Nashville

office. The assignment model has saved Heery money by enabling it to reduce its number of managers, reduce travel costs for managers, and reduce the time required to prepare project assignments each month. It has also improved morale because workloads are perceived to be fairer.

Source: L. J. LeBlanc, D. Randels, Jr., and T. K. Swann, "Heery International's Spreadsheet Optimization Model for Assigning Managers to Construction Projects," *Interfaces* 30, no. 6 (November/December 2000): 95-106.

Summary

In this chapter three special types of linear programming problems were presented: the transportation problem, the transshipment problem, and the assignment problem. As mentioned at the beginning of the chapter, they are part of a larger class of linear programming problems, known as network flow problems. In the next chapter we examine several additional examples of

network flow problems, including the shortest route problem, the minimal spanning tree problem, and the maximal flow problem. Although they have different objectives, they share the same general characteristics as the transportation and assignment problems that is, the flow of items from sources to destinations.

References

Ackoff, R. L., and Sasieni, M. W. *Fundamentals of Operations Research*. New York: John Wiley & Sons, 1968.

Charnes, A., and Cooper, W. W. *Management Models and Industrial Applications of Linear Programming*. New York: John Wiley & Sons, 1961.

Churchman, C. W., Ackoff, R. L., and Arnoff, E. L. *Introduction to Operations Research*. New York: John Wiley & Sons, 1957.

Hitchcock, F. L. "The Distribution of a Product from

Several Sources to Numerous Localities." *Journal of Mathematics and Physics* 20 (1941): 224, 230.

Koopmans, T. C., ed. *Activity Analysis of Production and Allocation*. Cowles Commission Monograph No. 13. New York: John Wiley & Sons, 1951.

Orchard-Hays, W. *Advanced Linear Programming Computing Techniques*. New York: McGraw-Hill, 1968.

Taha, H. A. *Operations Research*, 4th ed. New York:

Macmillan, 1987.

Example Problem Solution

This example will demonstrate the procedure for solving a transportation problem.

Problem Statement

A concrete company transports concrete from three plants to three construction sites. The supply capacities of the three plants, the demand

requirements at the three sites, and the transportation costs per ton are as follows:

Plant	Construction Site			Supply (tons)
	A	B	C	
1	\$ 8	\$ 5	\$ 6	120
2	15	10	12	80
3	3	9	10	80
Demand	150	70	100	280

(tons)

Determine the linear programming model formulation for this problem and solve it by using Excel.

Solution

Step 1. The Linear Programming Model Formulation

minimize $Z = \$8x_{1A} + 5x_{1B} + 6x_{1C} + 15x_{2A} + 10x_{2B} + 12x_{2C} + 3x_{3A} + 9x_{3B} + 10x_{3C}$
subject to

$$x_{1A} + x_{1B} + x_{1C} = 120$$

$$x_{2A} + x_{2B} + x_{2C} = 80$$

$$x_{3A} + x_{3B} + x_{3C} = 80$$

$$x_{1A} + x_{2A} + x_{3A} \leq 150$$

$$x_{1B} + x_{2B} + x_{3B} \leq 70$$

$$x_{1C} + x_{2C} + x_{3C} \leq 100$$

$$x_{ij} \geq 0$$

[Page 240]

Step 2. Excel Solution

[\[View full size image\]](#)

Problems

Green Valley Mills produces carpet at pl
The carpet is then shipped to two outle
Atlanta. The cost per ton of shipping ca
to the two warehouses is as follows:

		To (cost)	
From		Chicago	Atlar
<u>1.</u>	St. Louis	\$40	\$6!
	Richmond	70	30

The plant at St. Louis can supply 250 tons per week, the plant at Richmond can supply 400 tons per week. The warehouse has a demand of 300 tons per week, and the store at Kansas City demands 350 tons per week. The company wants to determine the number of tons of carpet to ship from each plant to each warehouse to minimize the total shipping cost. Solve

A transportation problem involves the following data:

		To (cost)			
From		1	2	3	4
1		\$500	\$750	\$300	\$400

2.	2	650	800	400	6
	3	400	700	500	5
	Demand	10	10	10	

Solve this problem by using the compu

[Page 24

Given a transportation problem with th
demand, find the optimal solution by us

To (cost]

3.

From	1	2
A	\$6	\$7
B	5	3
C	8	5
Demand	135	175

Consider the following transportation p

To (cost)

	From	1	2
	A	\$ 6	\$9
4.	B	12	3
	C	4	8
	Demand	80	110

Formulate this problem as a linear prog
using the computer.

Solve the following linear programming

minimize $Z = 3x_{11} + 12x_{12} + 8x_{13} + 10x_{21} + 5x_{22} + 6x_{23} + 6x_{31} + 7x_{32} + 10x_{33}$
 subject to

$$x_{11} + x_{12} + x_{13} = 90$$

$$x_{21} + x_{22} + x_{23} = 30$$

$$x_{31} + x_{32} + x_{33} = 100$$

$$x_{11} + x_{21} + x_{31} \leq 70$$

$$x_{12} + x_{22} + x_{32} \leq 110$$

$$x_{13} + x_{23} + x_{33} \leq 80$$

$$x_{ij} \geq 0$$

5.

Consider the following transportation p

		To (cost)	
From		1	2
	A	\$ 6	\$9

6.	B	12	3
	C	4	8
	Demand	80	110

Solve it by using the computer.

[Page 24

Steel mills in three cities produce the fo

Location

*Weekly Prod
(tons)*

A. Bethlehem	150
B. Birmingham	210
C. Gary	320
	680

These mills supply steel to four cities, with the following demand:

<i>Location</i>	<i>Weekly Demand</i>
-----------------	----------------------

1. Detroit	130
2. St. Louis	70
<u>7.</u>	
3. Chicago	180
4. Norfolk	240
	620

Shipping costs per ton of steel are as follows

To

From	1	2
A	\$14	\$ 9
B	11	8
C	16	12

Because of a truckers' strike, shipment Birmingham to Chicago. Formulate this programming model and solve it by using

8. In [Problem 7](#), what would be the effect reduction in production capacity at the tons per week?

Coal is mined and processed at the following locations in West Virginia, and Virginia:

<i>Location</i>	<i>Capacity (t)</i>
A. Cabin Creek	90
B. Surry	50
C. Old Fort	80
D. McCoy	60
	280

These mines supply the following amount in three cities:

<i>Plant</i>	<i>Demand (t)</i>
--------------	-------------------

1. Richmond	120
-------------	-----

2. Winston-Salem	100
------------------	-----

3. Durham	110
-----------	-----

9.

	330
--	-----

The railroad shipping costs (in thousands) are shown in the following table. Because direct shipments are prohibited from Cabin Creek, the cost of shipping from Cabin Creek to Cabin Creek is \$0.

	To (cost, in \$1,000s)	
From	1	2
A	\$ 7	\$10
B	12	9
C	7	3
D	9	5

Formulate this problem as a linear program using the computer.

Oranges are grown, picked, and then shipped from Miami, and Fresno. These warehouses are located in New York, Philadelphia, Chicago, and Boston. The shipping costs per truckload (in hundreds of dollars) are given below. Because of an agreement between the growers, shipping is prohibited from Miami to Chicago:

		To (cost, in \$1000)			
From		New York	Philadelphia	Chicago	Boston
Miami	to New York	\$ 9	\$14	\$11	\$18
	to Philadelphia	\$12	\$10	\$13	\$15
Fresno	to New York	\$10	\$11	\$12	\$13
	to Philadelphia	\$11	\$12	\$13	\$14
Tampa		\$ 9	\$14	\$11	\$18

10.

Miami	11	10	
Fresno	12	8	
Demand	130	170	1

Formulate this problem as a linear program using the computer.

A manufacturing firm produces diesel engines in Seattle, St. Louis, and Detroit. The company has the following numbers of engines per month

Plant

Production

1. Phoenix

5

2. Seattle

25

3. St. Louis

20

4. Detroit

25

Three trucking firms purchase the following plants in three cities:

Firm

Demand

A. Greensboro	10
---------------	----

B. Charlotte	20
--------------	----

11.

C. Louisville	15
---------------	----

[Page 24

The transportation costs per engine (in sources to destinations are shown in the
Charlotte firm will not accept engines from
firm will not accept engines from Detroit
prohibited:

To (cost, in \$1

From	A	B
1	\$ 7	\$ 8
2	6	10
3	10	4
4	3	9

Formulate this problem as a linear prog
using the computer.

The Interstate Truck Rental firm has ac

of its truck leasing outlets, as shown in

Leasing Outlet	Extra Trucks
-----------------------	---------------------

1. Atlanta	70
2. St. Louis	115
3. Greensboro	60
Total	245

The firm also has four outlets with short-term leases, which are shown in the table that follows:

Leasing Outlet	Truck Shortage
----------------	----------------

A. New Orleans	80
----------------	----

B. Cincinnati	50
---------------	----

12. C. Louisville	90
-------------------	----

D. Pittsburgh	25
---------------	----

Total	245
-------	-----

The firm wants to transfer trucks from those with shortages at the minimum cost of transporting these trucks from city to city.

			To
From	A	B	
1	\$70	\$80	
2	120	40	
3	110	60	

Solve this problem by using the compu

The Shotz Beer Company has breweries that can supply the following numbers of barrels to its company's distributors each month:

Brewery	Monthly Supply (bbl)
A. Tampa	3,500
B. St. Louis	5,000
Total	8,500

The distributors, which are spread throughout the country, have the following total monthly demand:

Distributor	Monthly Demar (bbl)
1. Tennessee	1,600
2. Georgia	1,800
<u>13.</u> 3. North Carolina	1,500
4. South Carolina	950
5. Kentucky	1,250
6. Virginia	1,400
Total	8,500

The company must pay the following s

		To (		
From		1	2	3
A		\$0.50	\$0.35	\$0.60
B		0.25	0.65	0.40

Solve this problem by using the compu

In [Problem 13](#), the Shotz Beer Compar

14. a new shipping contract with a trucking brewery and its distributor in Kentucky. shipping cost per barrel from \$0.80 per will this cost change affect the optimal

Computers Unlimited sells microcomputers on the East Coast and ships them from The firm is able to supply the following the universities by the beginning of the

Distribution Warehouse	Supply (microcomputers)
1. Richmond	420
2. Atlanta	610
3. Washington, DC	340

Total	1,370
-------	-------

[Page 24

Four universities have ordered microcomputers and installed by the beginning of the academic year.

University	Demand (microcomputers)
------------	----------------------------

A. Tech	520
---------	-----

15. B. A & M	250
---------------------	-----

C. State	400
----------	-----

D. Central

380

Total

1,550

The shipping and installation costs per r
distributor to each university are as foll

To

From

A

B

1

\$22

\$17

2

15

35

Solve this problem by using the compu

- 16.** In [Problem 15](#), Computers Unlimited w the four universities it supplies. It is cor expand its warehouse at Richmond to ; equivalent to an additional \$6 in handlin
- purchase a new warehouse in Charlotte shipping costs of \$19 to Tech, \$26 to A Central. Which alternative should mana transportation costs (i.e., no capital co

Computers Unlimited in [Problem 15](#) has unable to meet the demand for microc supplies, the universities tend to purcha

the future. Thus, the firm has estimated microcomputer demanded but not supplied future sales and goodwill. These costs follow:

University	Cost/Microcomputer
17. A. Tech	\$40
B. A & M	65
C. State	25
D. Central	50

Solve [Problem 15](#) with these shortage

total transportation cost and the total :

A severe winter ice storm has swept across the Southeast, followed by over a foot of snow and frigid temperatures. These weather conditions have resulted in downed power lines and power outages, causing danger to the general population. Local utility companies have requested assistance from unaffected utility companies in the Southeast. The following table shows the number of crews available from five different companies in North Carolina, and Florida; the demand for crews in areas that local companies cannot get to; and the cost (in dollars) of a crew going to a specific area (in addition to a company's normal charges, the distance traveled, and living expenses in an area):

[Page 24]

Area (cost, in \$

Crews	NC-E	NC-SW	NC-P	NC-W
-------	------	-------	------	------

18.

GA-1	15.2	14.3	13.9	13.5
------	------	------	------	------

GA-2	12.8	11.3	10.6	12.0
------	------	------	------	------

SC-1	12.4	10.8	9.4	11.3
------	------	------	-----	------

FL-1	18.2	19.4	18.2	17.9
------	------	------	------	------

FL-2	19.3	20.2	19.5	20.2
------	------	------	------	------

Crews Needed	9	7	6	8
-----------------	---	---	---	---

Determine the number of crews that should be assigned to each affected area to minimize total cost.

A large manufacturing company is closing three plants and intends to transfer some of its mobile equipment to plants that will remain open. The number of employees to transfer from each closing plant is as follows:

Closing Plant	Transferable Employees
1	60
2	105
3	70

Total

235

The following number of employees can
plants remaining open:

Open Plants

**Employees
Demanded**

A

45

19.

B

90

C

35

Total

170

Each transferred employee will increase plant, as shown in the following table:

From	To (employee)	
	A	B
1	5	8
2	10	9
3	7	6

The company wants to transfer employment to the other plant to increase in product output. Solve this problem.

[Page 24]

The Sav-Us Rental Car Agency has six lots in the city. Each lot has a certain number of cars available for rental each day for local rental. The agency would like to solve at the end of each day that would require moving cars among the six lots in the minimum number of moves. The distances to travel between the six lots are as follows:

		To (number of lots)		
From	1	2	3	
1		12	17	

	2	14		10
	3	14	10	
	4	8	16	14
20.	5	11	21	16
	6	24	12	9

The agency would like the following number of rental cars at the end of the day. Also shown is the number of rental cars at the end of a particular day. Determine the number of rental cars by using any initial solution and a method:

Cars

1

2

Available

37

20

:

Desired

30

25

2

Bayville has built a new elementary school to four schools Addison, Beeks, Canfield of 400 students. The school board was so that their travel time by bus is as shown board has partitioned the town into five population density north, south, east, west bus travel time from each district to each

		Travel Time	
21.	District	Addison	Beeks
	North	12	23
	South	26	15
	East	18	20
	West	29	24
	Central	15	10

Determine the number of children that district to each school to minimize total

In [Problem 21](#), the school board has decided that any of the schools to be overly crowded. It would like to assign students to schools so that enrollments are evenly balanced across all schools. However, the school board is concerned that this will significantly increase travel time. Determine the number of students to be assigned from each district to each school so that enrollments are evenly balanced. Does this significantly increase travel time per student?

22. Determine the number of students to be assigned from each district to each school so that enrollments are evenly balanced. Does this significantly increase travel time per student?

The Easy Time Grocery chain operates stores along the East Coast. The stores have a "no-frills" policy, low overhead and high volume. They generate high sales at low prices. However, in some cases the stores are not in the other areas and ship it in. They can do this in some cities they operate in compared with cities where they do not operate.

example is baby food. Easy Time purchases baby food from a supplier in Albany, Binghamton, Claremont, Dover, and six stores in and around New York City. The supplier knows what Easy Time is up to, so they can tell Easy Time what baby food Easy Time can purchase. The supplier also knows how much Easy Time makes per case of baby food. Easy Time purchases it and at which store it is sold. The supplier also knows how much per week at purchase locations and the supplier's profit per week at Easy Time store per week:

		Easy Time Store		
23.	Purchase Location	1	2	3
	Albany	\$ 9	\$ 8	\$10
	Binghamton	10	10	8

Claremont	8	6	6
Dover	4	6	9
Edison	12	10	8
Demand	25	15	30

Determine where Easy Time should purchase food should be distributed to maximize

24. Suppose that in [Problem 23](#) Easy Time needs from a New York City distributor profit of \$9 per case at stores 1, 3, and 6; and \$7 per case at store 5. Should I

some of its baby food from the distributor. Can it buy from other stores and truck it in?

25. In [Problem 23](#), if Easy Time could arrange to purchase from one of the outlying locations, which location would provide the most additional cases could be purchased, and what would be the profit?

The Roadnet Transport Company has been purchasing 90 trailer trucks from a contractor. The contractor's company is subsequently located 30 of the trucks at its shipping warehouses in Charlotte, Miami, and St. Louis. The company makes shipments from each terminal in St. Louis, Atlanta, and New York City, making one shipment per week. The trucks have a capacity of extra shipments. The contractor can accommodate 40 additional trucks per week, the contractor can accommodate 60 additional trucks per week, and the contractor can accommodate 50 additional trucks per week. The following profit per truckload shipment is given for each terminal. The profits differ as a result of varying shipping costs, and transport rates:

26.

Terminal (pro**Warehouse**

St. Louis

Atlanta

Charlotte

\$1,800

\$2,100

Memphis

1,000

700

Louisville

1,400

800

Determine how many trucks to assign to terminal) in order to maximize profit

During the Gulf War in Iraq, large amounts of supplies had to be shipped daily from supply depots to bases in the Middle East. The critical factor for the supplies was speed. The following table shows the plane loads of supplies available each day from the depots and the number of daily loads delivered to the bases. (Each plane load is approximately equal to the transport hours per plane, including loading time, and unloading and refueling:

		Military Bases		
Supply Depot	A	B	C	
1	36	40	32	

27.

2

28

27

29

3

34

35

41

4

41

42

35

5

25

28

40

6

31

30

43

Demand

9

6

12

Determine the optimal daily flight schedule to minimize total transport time.

PM Computer Services produces personal computers. The parts it buys on the open market. The maximum of 300 personal computers determine its production schedule for the year. The cost to produce a personal computer is \$1,200. However, PM knows the cost of each month so that the overall cost to produce each month. The cost of holding a computer unit per month. Following is the demand each month:

Month	Demand
January	180
February	260
March	340

28.

Determine a production schedule for PM

[Page 25

In [Problem 28](#), suppose that the demand increases each month, as follows:

Month	Demand
January	410
February	320
March	500

29.

In addition to the regular production capacity, PM Computer Services can also produce 100 units per month by using overtime. Overtime cost of a personal computer.

Determine a production schedule for PM

National Foods Company has five plant locations that produce and package fruits and vegetables. It has sales offices in Texas, Alabama, and Florida. The company has its own trucking system in the past for transporting goods from its suppliers to its plants. However, it is now outsourcing all its shipping to outside trucking companies and its own trucks. It currently spends \$245,000 per month to maintain its own trucking system. It has determined that the cost (in thousands of dollars per ton) of using common carriers from suppliers to each of its plants, as shown in the following table, is

		Processing Plant		
	Suppliers	Location		
		Denver	St. Paul	Los Angeles
30.	Sacramento	\$3.7	\$4.6	
	Bakersfield	3.4	5.1	
	San Antonio	3.3	4.1	
	Montgomery	1.9	4.2	
	Jacksonville	6.1	5.1	
	Ocala	6.6	4.8	

Demand (tons)

20

15

Should National Foods continue to operate its trucks and outsource its shipping?

In [Problem 30](#), National Foods would like to know what would be on the optimal solution and the effect of increasing its shipping if it negotiates with its suppliers, Ocala and Ocala to increase their capacity to 20. What would be the effect of negotiating instead with Montgomery to increase their capacity to 20?

31.

Orient Express is a global distribution company that ships its clients' products to customers in Hong Kong. The products Orient Express ships are sold in two distribution centers: one in Los Angeles, one in Savannah.

the coming month the company has 400 containers available at the Los Angeles port, 350 containers available at Savannah, and 350 containers available at Singapore. The company has orders for 600 containers from Los Angeles, 400 containers from Singapore, and 500 containers from Savannah. The shipping costs per container from each of the overseas ports are shown in the following table.

[Page 25]

		Overseas Ports	
32.	U.S. Distribution Center	Hong Kong	Singapore
	Los Angeles	\$300	
	Savannah	490	

Orient Express, as the overseas broker responsible for unfulfilled orders, and it charges overseas customers if it does not meet their demand. Singapore customers charge \$8 per container, and Taipei customers charge \$10 per container, and solve a transportation model to determine the shipping costs from each U.S. distribution center to each overseas customer. Indicate what portion of the total penalties.

Binford Tools manufactures garden tools and subcontracting to absorb demand. The regular and overtime production capacities are provided in the following table for the

basic line of steel garden tools:

Quarter	Demand	Regular Capacity
1	9,000	9,000
2	12,000	10,000
3	16,000	12,000
4	19,000	12,000

33.

The regular production cost per unit is \$25, the cost to subcontract a unit is \$30, and the cost to hold inventory for one quarter is \$2 per unit. The company has 3

beginning of the year.

Determine the optimal production scheme to minimize total costs.

Al, Barbara, Carol, and Dave have joined to purchase season tickets to the Giants' home football games. For each of the eight home games, each person will get one ticket. Each person has ranked the games they prefer from most preferred and 8 least preferred, as shown below.

P

Game

Al

Barbara

1. Cowboys

1

2

2. Redskins

3

4

3. Cardinals	7	8
--------------	---	---

34.

4. Eagles	2	7
-----------	---	---

5. Bengals	5	6
------------	---	---

6. Packers	6	3
------------	---	---

7. Saints	8	5
-----------	---	---

8. Jets	4	1
---------	---	---

Determine the two games each person result in the groups' greatest degree of participants would think your allocation

World Foods, Inc., imports food products pastries to the United States from warehouses in Marseilles, and Liverpool. Ships from the Norfolk, New York, and Savannah, where the warehouses before being shipped to distribution centers in St. Louis, and Chicago. The products are then sold through catalogs. The shipping costs from the European ports to the U.S. cities are (in \$/lb.) at the European ports are provided

	U.S. City	
European Port	4. Norfolk	5. New York

1. Hamburg	\$420	\$390
------------	-------	-------

2. Marseilles	510	590
---------------	-----	-----

3. Liverpool	450	360
--------------	-----	-----

35.

The transportation costs (\$/1,000 lb.)
distribution centers and the demands (1
centers are as follows:

Distributi

Warehouse

7. Dallas

8. St

4. Norfolk

\$75

\$

5. New York	125	1
6. Savannah	68	8
	60	4

Determine the optimal shipments between warehouses and the distribution center to minimize transportation costs.

A sports apparel company has received an order from a basketball team's national championship to purchase the T-shirts from textile factories in Haiti. The shirts are shipped from the factories in the United States that silk-screen the shirts.

distribution centers. Following are the per-unit production costs (\$/shirt) from the T-shirt factories to the distribution centers, plus the shipping costs and demand for the shirts at the distribution centers.

Silk-screen Costs

T-shirt Factory	4. Miami	5. Atlanta
1. Mexico	\$4	\$6
2. Puerto Rico	3	5
3. Haiti	2	4

		District
Silk-screen Company	7. New York	8.
4. Miami	\$5	
5. Atlanta	7	
6. Houston	8	
Demand (1,000s)	20	

Determine the optimal shipments to minimize transportation costs for the apparel company.

Walsh's Fruit Company contracts with California and New York to purchase grapes. The grapes are harvested at the farms and stored in refrigerated trucks. The grapes are then transported to two plants, where it is processed into juice and concentrate. The juice and concentrate are then transported to food warehouses/distribution centers. The transportation costs from the farms to the plants and from the plants to the food warehouses and the supply at the farms and demand at the food warehouses are summarized in the following tables:

Plant

Farm

4. Indiana

5. Georgia

1. Ohio	\$16	21
2. Pennsylvania	18	16
3. New York	22	25

37.

		Dis
Plant	6. Virginia	
4. Indiana	\$23	

5. Georgia

20

**Demand (1,000
tons)**

90

1. Determine the optimal shipments to distribution centers to minimize total cost.
2. What would be the effect on the total cost if the demand at the plant were 140,000 tons?

A national catalog and Internet retailer has three major distribution centers located around the country. Goods are shipped directly from the warehouses to the customers; however, each of the distribution centers acts as an intermediate transshipment point. The distances between warehouses and distribution centers are given in the following table.

warehouses (100 units), and the demand (100 units) for a specific week are shown below.

[Page 25]

Distribution C

Warehouse

A

B

1

\$12

\$11

2

8

6

3

9

10

38.

Demand

60

100

The transportation costs (\$/unit) between

		Distribu
Distribution Center		A
A		\$
B		1
C		7

Determine the optimal shipments between centers to minimize total transportation costs.

Horizon Computers manufactures laptops in Germany, Italy. Because of high tariffs between Italy and the U.S., it is sometimes cheaper to ship partially completed laptops to Puerto Rico, Mexico, and Panama and then make the final shipment to U.S. distributors in Texas and Virginia. The cost (\$/unit) of the completed laptops plus the shipping cost from the European plants directly to the U.S. demand are as follows:

European Plant	U.S. Distributor		
	7. Texas	8. Virginia	9. Other
1. Germany	\$2,600	\$1,900	\$2,300

2. Belgium	2,200	2,100	2,6
3. Italy	1,800	2,200	2,5
Demand (1,000s)	2.1	3.7	:

Alternatively, the unit costs of shipping plants for finishing before sending them follows:

39.

	Factory	
European Plant	4. Puerto Rico	5. Mexico

1. Germany	\$1,400	\$1,200
2. Belgium	1,600	1,100
3. Italy	1,500	1,400

U.S. Distributors

Factory	7. Texas	8. Virginia	9.
4. Puerto Rico	\$800	\$700	

5. Mexico	600	800	:
-----------	-----	-----	---

6. Panama	900	700	:
-----------	-----	-----	---

Determine the optimal shipments of lap the U.S. distributors at the minimum to

The Midlands Field Produce Company c Colorado, Minnesota, North Dakota, ar shipments. Midlands picks up the potat mostly by truck (and sometimes by rai centers in Ohio, Missouri, and Iowa. At cleaned, rejects are discarded, and the to size and quality. They are then shipp distribution centers in Virginia, Pennsylv the company produces a variety of pot

bags of potatoes to stores. Exceptions which will only accept potatoes from farms in Idaho, Washington, and Wisconsin, and the Texas plant, which does not accept potatoes from Ohio because of disagreements over distribution issues. Following are summaries of the flow of potatoes from the farms to the distribution centers and the processing plants, and from the distribution centers to the plants, as well as the supply at each farm, the processing capacity at each plant, and the final demand at the plants (in billions of pounds).

Distribution Centers

Farm

5. Ohio

1. Colorado

\$

2. Minnesota

0.89

3. North Dakota	0.78
-----------------	------

40.

4. Wisconsin	1.19
--------------	------

Processing Capacity (bushels)	1,800
--	-------

F

Distribution Center	8. Virginia	Penr
----------------------------	-------------	------

5. Ohio	\$4.56	\$
---------	--------	----

6. Missouri	3.43	!
7. Iowa	5.39	(
Demand (bushels)	1,200	

Formulate and solve a linear program optimal monthly shipments from the factories and from the distribution centers to the customers, minimizing the total shipping and processing costs.

KanTech Corporation is a global distributor of computer components. Its customers are electronics manufacturers in the United States, including computer manufacturers. The company contracts for computer components from manufacturers in Russia, Eastern Europe, the Mediterranean, and it has them delivered to its distribution centers in the United States.

European ports, Gdansk, Hamburg, and components and parts are loaded into from U.S. customers. Each port has a l containers available each month. The c overseas by container ships to the port Orleans, and Galveston. From these se typically coupled with trucks and haulec (Virginia), Kansas City, and Dallas. Then haulers available at each port each mo sometimes called "freight villages," or i containers are collected and transferred another (i.e., from truck to rail or vice , the containers are transported to KanT Tucson, Pittsburgh, Denver, Nashville, a handling and shipping costs (\$/contain embarkation and destination points alo and the available containers at each po

**European
Port**4.
Norfolk5.
Jacksonville

1. Gdansk

\$1,725

\$1,800

2. Hamburg

1,825

1,750

3. Lisbon

2,060

2,175

Inland Port**41.****U.S. Port**8.
Dallas9. Kansas
City

4. Norfolk	\$825	\$545
------------	-------	-------

5. Jacksonville	750	675
-----------------	-----	-----

6. New Orleans	325	605
----------------	-----	-----

7. Galveston	270	510
--------------	-----	-----

Intermodal Capacity (containers)	170	240
---	-----	-----

Inland Port	11. Tucson	12. Denver
8. Dallas	\$450	\$830
9. Kansas City	880	520
10. Front Royal	1,350	390
Demand	85	60

Formulate and solve a linear program to determine optimal shipments from each point of origin to each point of destination.

along this supply chain that will result in cost.

In [Problem 41](#), KanTech Corporation is distributors receive shipments in the m are about minimizing their shipping cost distributor receives one major container Following are summaries of the shipping of the embarkation and destination poi supply chain. These times not only enco processing, loading, and unloading time

[Page 25

U.S.

**European
Port**

4. Norfolk

5.
Jacksonville

1. Gdansk	22	24
2. Hamburg	17	20
3. Lisbon	25	21

--	--	--

--	--	--

In

U.S. Port	8. Dallas	9. I
------------------	-----------	------

4. Norfolk	10
------------	----

42.

5. Jacksonville	12
-----------------	----

6. New Orleans	8
----------------	---

7. Galveston	12
--------------	----

Distribu

Inland Port	11. Tucson	12. Denver	F
--------------------	------------	------------	---

8. Dallas	5	6
-----------	---	---

9. Kansas City	6	4
----------------	---	---

10. Front
Royal

10

5

- 1.** Formulate and solve a linear prog the optimal shipping route for each supply chain that will result in the Determine the shipping route and
- 2.** Suppose the European ports can shipments each. How will this affect

Solve the following linear programming

$$\begin{aligned} \text{minimize } Z &= 18x_{11} + 30x_{12} + 20x_{13} + \\ & 22x_{23} \end{aligned}$$

$$+ 16x_{24} + 30x_{31} + 26x_{32} \\ 36x_{42}$$

$$+ 27x_{43} + 29x_{44} + 30x_{51}$$

subject
to

43.

$x_{11} +)$

$x_{12} +)$

$x_{13} +)$

$x_{14} +)$

[Page 25

A plant has four operators to be assign
(minutes) required by each worker to p
machine is shown in the following table

		Mach	
Operator		A	B
44.	1	10	12
	2	5	10
	3	12	14
	4	8	15

Determine the optimal assignment and

A shop has four machinists to be assigned to four machines. The cost of having each machine operated

		Machinist	
		A	B
Machinist	1	\$12	\$11
	2	10	9
	3	14	8
	4	6	8

45.

However, because he does not have er cannot operate machine B.

1. Determine the optimal assignment cost.
2. Formulate this problem as a gene

The Omega pharmaceutical firm has five sales regions. It wants to assign five salespersons to these regions. Given the contacts, the salespersons are able to cover the regions in different amounts of time. The amount of time (in hours) for each salesperson to cover each city is shown below.

Region

Salesperson

A

B

	1	17	10
46.	2	12	9
	3	11	16
	4	14	10
	5	13	12

Which salesperson should be assigned to each region? Identify the optimal assignments and the total time.

The Bunker Manufacturing firm has five and wants to assign the employees to A cost table showing the cost incurred machine follows:

M

Employee

A

B

1

\$12

\$7

2

10

14

3

5

3

4

9

11

47.

48.

2	9	5
3	12	7
4	3	1

An electronics firm produces electronic various electrical manufacturers. Quality different employees produce different average number of defects produced by components is given in the following table

Cor

49.

Employee

A

B

1

30

24

2

22

28

3

18

16

4

14

22

5

25

18

6

32

14

Determine the optimal assignment that number of defects produced by the firm

[Page 26

A dispatcher for Citywide Taxi Company locations and five customers who have from each taxi's present location to ea following table:

		Cu	
Cab		1	2
	A	7	2

50.

B	5	1
C	8	7
D	2	5
E	3	3
F	6	2

Determine the optimal assignment(s) to minimize the total mileage traveled.

The Southeastern Conference has nine teams assigned to three conference games, the

conference office wants to assign the cities so that the total distance they travel will be minimized. The distances from each city to each game travel to each game is given in the following table.

Official	Athens	Columbus
1	165	100
2	75	100
3	180	100
4	220	100

5	410
6	150
7	170
8	105
9	240

Determine the optimal assignment(s) to be traveled by the officials.

In [Problem 51](#), officials 2 and 8 have had one of the coaches in the game in Athe

the coach after several technical fouls.

- 52.** decided that it would not be a good idea to work the Athens game so soon after the first game. He decided that officials 2 and 8 will not be available. How will this affect the optimal solution?

State University has planned six special events for its homecoming football game. The events include a parents' brunch, a booster club lunch, a season ticket holders, a lettermen's dinner, and two major contributors. The university wants to hire as well as the university catering service to cater the events. It asked the caterers to bid on each event (in thousands of dollars) based on menu guidelines for the homecoming. The university are shown in the following table.

Rank	Caterer	Average Price Per Person			
		Alumni Brunch	Parents' Brunch	Booster Club Lunch	Postgame Reception
53.	Al's	\$12.6	\$10.3	\$14.0	\$10.0
	Bon Apétit	14.5	13.0	16.5	11.0
	Custom	13.0	14.0	17.6	20.0
	Divine	11.5	12.6	13.0	11.0
	Epicurean	10.8	11.9	12.9	11.0
	Fouché's	13.5	13.5	15.5	20.0

University	12.5	14.3	16.0	2
------------	------	------	------	---

The Bon Apetit, Custom, and University events, whereas each of the other four. The university is confident that all the c so it wants to select the caterers for th lowest total cost.

Determine the optimal selection of cate

A university department head has five i different courses. All the instructors hav and have been evaluated by the studer for each course is given in the following

Instructor		A	B
54.	1	80	75
	2	95	90
	3	85	95
	4	93	91
	5	91	92

The department head wants to know the best instructor to assign to each of the five courses to maximize the average student rating. Which instructor who is not assigned to teach

grade exams.

The coach of the women's swim team for the conference swim meet and must she will assign to the 800-meter medley consists of four strokes backstroke, breaststroke, freestyle. The coach has computed the time for each of her top six swimmers has achieved for 200 meters in previous swim meets:

[Page 26]

Stroke

Swimmer

Backstroke

Breaststroke

Annie

2.56

3.07

Beth	2.63	3.01
Carla	2.71	2.95
Debbie	2.60	2.87
Erin	2.68	2.97
Fay	2.75	3.10

Determine for the coach the medley relay time.

Biggio's Department Store has six employee departments in the storehome furnishing

jewelry. Most of the six employees have worked in several departments on several occasions in the past that they perform better in some departments than others. The average daily sales for each of the six employees in the three departments are shown in the following table.

	Department		
	Employee	Home Furnishings	China
56.	1	\$340	\$160
	2	560	370
	3	270	

4	360	220
5	450	190
6	280	320

Employee 3 has not worked in the chin manager does not want to assign this c

Determine which employee to assign to the total expected daily sales.

The Vanguard Publishing Company hires salespeople to sell encyclopedias during desires to allocate them to three sales three salespeople, and territories 2 and

It is estimated that each salesperson w
amounts of dollar sales per day in each
in the following table:

1

Salesperson

1

A

\$110

B

90

C

205

D

125

57.

E	140
F	100
G	180
H	110

Help the company allocate the salespeople to the regions so that sales will be maximized.

[Page 26]

Carolina Airlines, a small commuter airline, has flight attendants that it wants to assign in a way that will minimize the number

their homes. The numbers of nights each
home with each schedule are given in the

		Schedule		
Attendant		A	B	C
58.	1	7	4	6
	2	4	5	5
	3	9	9	11
	4	11	6	8
	5	5	8	6

6

10

12

11

Identify the optimal assignments that w
nights the attendants will be away from

The football coaching staff at Tech focu
states, including Georgia, Florida, Virgin
New Jersey. The staff includes seven as
are responsible for Florida, a high schoo
coach is assigned to each of the other
together for a long time and at one tim
have recruited in all the states. The hea
data on the past success rate (i.e., per
signed) for each coach in each state, a

Coach

GA

FL

VA

Allen

62

56

65

Bush

65

70

63

59.

Crumb

46

53

62

Doyle

58

66

70

Evans

77

73

69

Fouch

68

73

72

Determine the optimal assignment of c will maximize the overall success rate a percentage success rate for the staff w

Kathleen Taylor is a freshman at Roano develop her schedule for the spring sen class periods either on Monday and We Thursday for 1 hour and 15 minutes du class periods. For example, a course de Monday and Wednesday from 8:00 A.M. Monday and Wednesday (9M) meets fr class (11M) is from 11:00 A.M. to 12:1 wants to take the following six freshman sections shown in order of her preferer who's teaching the course and the time

Course**Section**

Math

11T, 12T, 9T, 11M,

History

11T, 11M, 14T, 14I

English

9T, 11T, 14T, 11M,

60. Biology

14T, 11M, 12M, 14

Spanish

9T, 11M, 12M, 8T

Psychology

14T, 11T, 12T, 9T, 1

For example, there are eight sections of the 11T section, her first preference is the 11T section, her second preference is the 11T section, and so forth.

- 1.** Determine a class schedule for Kathleen based on her preferences.
- 2.** Determine a class schedule for Kathleen from 11:00 A.M. to noon open for lunch.
- 3.** Suppose Kathleen wants all her classes on Monday and Wednesday or Tuesday and Thursday. Determine schedules for each and indicate whether they meet her preferences.

CareMed, an HMO health care provider, has a clinic in Draper, near the Tech campus. The clinic has staff with doctors and nurses who see patients on a daily appointment schedule. However, Tech students who visit the clinic each week for appointments because their families are in the area. The clinic has 12 nurses who work according to a schedule.

Five nurses are needed from 8:00 A.M. from 4:00 P.M. to midnight, and 3 nurse to 8:00 A.M. The clinic administrator was according to their preferences and senior nurses who prefer a shift exceed the shift assigned according to seniority). While day shift, some prefer other shifts because of schedules of their spouses and families preferences (where 1 is most preferred clinic:

Nurse	8 A.M. to 4 P.M.	4 P.M. to Midnight
Adams	1	2
Baxter	1	3

<u>61.</u> Collins	1	2
Davis	3	1
Evans	1	3
Forrest	1	2
Gomez	2	1
Huang	3	2
Inchio	1	3

Jones	2	1
King	1	3
Lopez	2	3

Formulate and solve a linear program to shifts according to their preferences

Case Problem

THE DEPARTMENT OF MANAGEMENT
SCIENCE AND INFORMATION
TECHNOLOGY AT TECH

The management science and information technology department at Tech offers between 36 and forty 3-hour course sections each semester. Some of the courses are taught by graduate student instructors, whereas 20 of the course sections are taught by

the 10 regular, tenured faculty in the department. Before the beginning of each year, the department head sends the faculty a questionnaire, asking them to rate their preference for each course using a scale from 1 to 5, where 1 is "strongly preferred," 2 is "preferred but not as strongly as 1," 3 is "neutral," 4 is "prefer not to teach but not strongly," and 5 is "strongly prefer not to teach this course." The faculty have returned their preferences, as follows:

Faculty Member	Course				
	3424	3434	3444	3454	4434
Clayton	2	4	1	3	2
Houck	3	3	4	1	2
Huang	2	3	2	1	3
Major	1	4	2	5	1
Moore	1	1	4	4	2
Ragsdale	1	3	1	5	4

Rakes	3	1	2	5	3
Rees	3	4	3	5	5
Russell	4	1	3	2	2
Sumichrast	4	3	1	5	2

For the fall semester the department will offer two sections each of 3424 and 4464; three sections of 3434, 3444, 4434, 4444, and 4454; and one section of 3454.

The normal semester teaching load for a regular faculty member is two sections. (Once the department head determines the courses, he will assign the faculty he schedules the course times so they will not conflict.) Help the department head determine a teaching schedule that will satisfy faculty teaching preferences to the greatest degree possible.

Case Problem

STATELINE SHIPPING AND TRANSPORT COMPANY

Rachel Sundusky is the manager of the South-Atlantic office of the Stateline Shipping and Transport Company. She is in the process of negotiating a new shipping contract with Polychem, a company that manufactures chemicals for industrial use. Polychem wants Stateline to pick up and

transport waste products from its six plants to three waste disposal sites. Rachel is very concerned about this proposed arrangement. The chemical wastes that will be hauled can be hazardous to humans and the environment if they leak. In addition, a number of towns and communities in the region where the plants are located prohibit hazardous materials from being shipped through their municipal limits. Thus, not only will the shipments have to be handled carefully and transported at reduced speeds,

they will also have to traverse circuitous routes in many cases.

Rachel has estimated the cost of shipping a barrel of waste from each of the six plants to each of the three waste disposal sites as shown in the following table:

Waste Disposal Site			
Plant	Whitewater	Los Canos	Duras
Kingsport	\$12	\$15	\$17

Danville	14	9	10
Macon	13	20	11
Selma	17	16	19
Columbus	7	14	12
Allentown	22	16	18

The plants generate the

following amounts of waste products each week:

Plant	Waste per Week (bbl)
Kingsport	35
Danville	26
Macon	42
Selma	53
Columbus	29

The three waste disposal sites at Whitewater, Los Canos, and Duras can accommodate a maximum of 65, 80, and 105 barrels per week, respectively.

In addition to shipping directly from each of the six plants to one of the three waste disposal sites, Rachel is also considering using each of the plants and waste disposal sites as intermediate shipping points.

Trucks would be able to drop a load at a plant or disposal site to be picked up and carried on to the final destination by another truck, and vice versa. Stateline would not incur any handling costs because Polychem has agreed to take care of all local handling of the waste materials at the plants and the waste disposal sites. In other words, the only cost Stateline incurs is the actual transportation cost. So Rachel wants to be able to consider the possibility that it may be cheaper to drop and pick up

loads at intermediate points rather than ship them directly.

Rachel estimates the shipping costs per barrel between each of the six plants to be as follows:

	Plant			
Plant	Kingsport	Danville	Macon	Selma
Kingsport	\$	\$6	\$4	\$9
Danville	6		11	10

Macon	5	11		3
Selma	9	10	3	
Columbus	7	12	7	3
Allentown	8	7	15	16

The estimated shipping cost per barrel between each of the three waste disposal sites is as follows.

	Waste Disposal Site		
Waste Disposal Site	Whitewater	Los Canos	Duras
Whitewater	\$	\$12	\$10
Los Canos	12		15
Duras	10	15	

Rachel wants to determine the shipping routes that will

minimize Stateline's total cost in order to develop a contract proposal to submit to Polychem for waste disposal. She particularly wants to know if it would be cheaper to ship directly from the plants to the waste sites or if she should drop and pick up some loads at the various plants and waste sites. Develop a model to assist Rachel and solve the model to determine the optimal routes.

Case Problem

BURLINGHAM TEXTILE COMPANY

Brenda Last is the personnel director at the Burlingham Textile Company. The company's plant is expanding, and Brenda must fill five new supervisory positions in carding, spinning, weaving, inspection, and shipping. Applicants for the positions are required to take a written psychological and aptitude test.

The test has different modules that indicate an applicant's aptitude and suitability for a specific area and position. For example, one module tests the psychological traits and intellectual skills that are best suited for the inspection department, which are different from the traits and skills required in shipping. Brenda has had 10 applicants for the five positions and has compiled the results from the test. The test scores for each position module for each applicant (where the higher the score,

the better) are as follows:

[Page 268]

Test Module Score

Applicant	<i>Carding</i>	<i>Spinning</i>	<i>Weaving</i>	<i>Inspection</i>
------------------	----------------	-----------------	----------------	-------------------

Roger Acuff	68	75	72	86
-------------	----	----	----	----

Melissa Ball	73	82	66	78
--------------	----	----	----	----

Angela Coe	92	101	90	79
------------	----	-----	----	----

Name	Test Scores			
	Math	Science	Reading	Writing
Maureen Davis	87	98	75	90
Fred Evans	58	62	93	81
Bob Frank	93	79	94	92
Ellen Gantry	77	92	90	81
David Harper	79	66	90	85
Mary Inchavelia	91	102	95	90

Marilu Jones	72	75	67	93
--------------	----	----	----	----

Brenda wants to offer the vacant positions to the five most qualified candidates. Determine an optimal assignment for Brenda.

There is a possibility that one or more of the successful applicants will turn down a position offer, and Brenda wants to be able to hire the

next-best person into a position if someone rejects a job offer. If the applicant selected for the carding job turns it down, whom should Brenda offer this job to next? If the applicants for both the carding and spinning jobs turn them down, which of the remaining applicants should Brenda offer each job to? How would a third applicant be selected if three of the job offers were declined?

Brenda believes this is a particularly good group of applicants. She would like to retain a few of the people for

several more supervisory positions that she believes will open up soon. She has two vacant clerical positions that she can offer to the two best applicants not selected for the five original supervisory positions. Then when the supervisory positions open, she can move these people into them. How should Brenda identify these two people?

Case Problem

THE GRAPHIC PALETTE

The Graphic Palette is a firm in Charleston, South Carolina, that does graphic artwork and produces color and black- and-white posters, lithographs, and banners. The firm's owners, Kathleen and Lindsey Taylor, have been approached by a client to produce a spectacularly colored poster for an upcoming arts festival. The

poster is more complex than anything Kathleen and Lindsey have previously worked on. It requires color screening in three stages, and the processing must proceed rapidly to produce the desired color effect.

By suspending all their other jobs, they can devote three machines to the first stage, four to the second stage, and two to the last stage of the process. Posters that come off the machines at each stage can be processed on any of the machines at the next stage.

However, all the machines are different models and of varying ages, so they cannot process the same number of posters in the specified time frame necessary to complete the job. The different machine capacities at each stage are as follows:

Stage 1	Stage 2	Stage 3
Machine 1 = 750	Machine 4 = 530	Machine 8 = 620
Machine 2 = 900	Machine 5 = 320	Machine 9 = 750

Machine 3 = Machine 6 =
670 450

Machine 7 =
250

Because the machines are of different ages and types, the costs of producing posters on them differ. For example, a poster that starts on machine 1 and then proceeds to machine 4 costs \$18. If this poster at machine 4 is then processed on

machine 8, it costs an additional \$36. The processing costs for each combination of machines for stages 1, 2, and 3 are as follows:

Machine (cost)				
Machine	4	5	6	7
1	\$18	\$23	\$25	\$21
2	20	26	24	19
3	24	24	22	23

Machine (cost)		
Machine	8	9
4	\$36	\$41
5	40	52
6	42	46
7	33	49

Kathleen and Lindsey are unsure how to route the posters from one stage to the next to make as many posters as they possibly can at the lowest cost. Determine how to route the posters through the various stages for Graphic Palette to minimize costs.

Case Problem

SCHEDULING AT HAWK SYSTEMS, INC.

Jim Huang and Roderick Wheeler were sales representatives in a computer store at a shopping mall in Arlington, Virginia, when they got the idea of going into business in the burgeoning and highly competitive microcomputer market. Jim went to Taiwan over the summer to visit relatives and

made a contact with a new firm producing display monitors for microcomputers, which was looking for an East Coast distributor in America. Jim made a tentative deal with the firm to supply a maximum of 500 monitors per month and called Rod to see if he could find a building they could operate out of and some potential customers.

Rod went to work. The first thing he did was send bids to several universities in Maryland, Virginia, and Pennsylvania for contracts as

an authorized vendor for monitors at the schools. Next, he started looking for a facility to operate from. Jim and his operation would provide minor physical modifications to the monitors, including some labeling, testing, packaging, and then storage in preparation for shipping. He knew he needed a building with good security, air-conditioning, and a loading dock. However, his search proved to be more difficult than he anticipated. Building space of the type and size he needed was very

limited in the area and very expensive. Rod began to worry that he would not be able to find a suitable facility at all. He decided to look for space in the Virginia and Maryland suburbs and countryside; and although he found some good locations, the shipping costs out to those locations were extremely high.

Disheartened by his lack of success, Rod sought help from his sister-in-law Miriam, a local real estate agent. Rod poured out the details of his plight to Miriam over dinner at Rod's mother's house, and she was

sympathetic. She told Rod that she owned a building in Arlington that might be just what he was looking for, and she would show it to him the next day. As promised, she showed him the ground floor of the building, and it was perfect. It had plenty of space, good security, and a nice office; furthermore, it was in an upscale shopping area with lots of good restaurants. Rod was elated; it was just the type of environment he had envisioned for them to set up their business in. However, his joy

soured when he asked Miriam what the rent was. She said she had not worked out the details, but the rent would be around \$100,000 per year. Rod was shocked, so Miriam said she would offer him an alternative: a storage fee of \$10 per monitor for every monitor purchased and in stock the first month of operation, with an increase of \$2 per month per unit for the remainder of the year. Miriam explained that based on what he told her about the business, they would not have any sales

until the universities opened around the end of August or the first of September, and that their sales would fall off to nothing in May or June. She said her offer meant that she would share in their success or failure. If they ended up with some university contracts, she would reap a reward along with them; if they did not sell many monitors, she would lose on the deal. But in the summer months after school ended, if they had no monitors in stock, they would pay her nothing.

Rod mulled this over, and it

sounded fair. He loved the building. Also, he liked the idea that they would not be indebted for a flat lease payment and that the rent was essentially on a per-unit basis. If they failed, at least they would not be stuck with a huge lease. So he agreed to Miriam's offer.

When Jim returned from Taiwan, he was skeptical about Rod's lease arrangement with Miriam. He was chagrined that Rod hadn't performed a more thorough analysis of the costs, but Rod explained that it was

pretty hard to do an analysis when he did not know their costs, potential sales, or selling price. Jim said he had a point, and his concern was somewhat offset by the fact that Rod had gotten contracts with five universities as an authorized vendor for monitors at a selling price of \$180 per unit. So the two sat down to begin planning their operation.

First, Jim said he had thought of a name for their enterprise, Hawk Systems, Inc., which he said stood for Huang and Wheeler Computers. When Rod

asked how Jim got a *k* out of *computers*, Jim cited poetic license.

Jim said that he had figured that the total cost of the units for them including the purchase of the units, shipping, and their own material, labor, and administrative costs would be \$100 per unit during the first 4 months but would then drop to \$90 per month for the following 4 months and, finally, to \$85 per month for the remainder of the year. Jim said that the Taiwan firm was anticipating being able to lower the

purchase price because its production costs would go down as it gained experience.

Jim thought their own costs would go down, too. He also explained that they would not be able to return any items, so it was important that they develop a good order plan that would minimize costs. This was now much more important than Jim had originally thought because of their peculiar lease arrangement based on their inventory level. Rod said that he had done some research on past computer sales at the

universities they had contracted with and had come up with the following sales forecast for the next 9 months of the academic year (from September through May):

September	340
October	650
November	420
December	200
January	660

February	550
March	390
April	580
May	120

Rod explained to Jim that computer equipment purchases at universities go up in the fall, then drop until January, and then peak again in April, just

before university budgets are exhausted at the end of the academic year.

[Page 270]

Jim then asked Rod what kind of monthly ordering schedule from Taiwan they should develop to meet demand while minimizing their costs. Rod said that it was a difficult question, but he remembered that when he was in college in a management science course, he had seen a production schedule developed using a

transportation model. Jim suggested he get out his old textbook and get busy, or they would be turning over all their profits to Miriam.

However, before Rod was able to develop a schedule, Jim got a call from the Taiwan firm, saying that it had gotten some more business later in the year and it could no longer supply up to 500 units per month. Instead, it could supply 700 monitors for the first 4 months and 300 for the next 5. Jim and Rod worried about what this would do to their inventory

costs.

- 1.** Formulate and solve a transportation model to determine an optimal monthly ordering and distribution schedule for Hawk Systems that will minimize costs.
- 2.** If Hawk Systems has to borrow approximately \$200,000 to start up the business, will it end up making anything the first year?
- 3.** What will the change in the

supply pattern from the Taiwan firm cost Hawk Systems?

- 4.** How did Miriam fare with her alternative lease arrangement? Would she have been better off with a flat \$100,000 lease payment?

Case Problem

GLOBAL SHIPPING AT ERKEN APPAREL INTERNATIONAL

Erken Apparel International manufactures clothing items around the world. It has currently contracted with a U.S. retail clothing wholesale distributor for men's goatskin and lambskin leather jackets for the next Christmas season. The distributor has distribution centers in Indiana, North

Carolina, and Pennsylvania. The distributor supplies the leather jackets to a discount retail chain, a chain of mall boutique stores, and a department store chain. The jackets arrive at the distribution centers unfinished, and at the centers the distributor adds a unique lining and label specific to each of its customer. The distributor has contracted with Erken to deliver the following number of leather jackets to its distribution centers in late fall:

Distribution Center	Goatskin Jackets	Lambskin Jackets
Indiana	1,000	780
North Carolina	1,400	950
Pennsylvania	1,600	1,150

Erken has tanning factories and clothing manufacturing plants to produce leather jackets in Spain, France, Italy, Venezuela, and Brazil. Its tanning facilities are in Mende in France, Foggia

in Italy, Saragosa in Spain, Feira in Brazil, and El Tigre in Venezuela. Its manufacturing plants are in Limoges, Naples, and Madrid in Europe and in Sao Paulo and Caracas in South America. Following are the supplies of available leather from each tanning facility and the processing capacity at each plant (in pounds) for this particular order of leather jackets:

Tanning Factory	Goatskin Supply (lb.)	Lambskin Supply (lb.)
----------------------------	----------------------------------	----------------------------------

Mende	4,000	4,400
Foggia	3,700	5,300
Saragosa	6,500	4,650
Feira	5,100	6,850
El Tigre	3,600	5,700

Plant	Production Capacity (lb.)	
Madrid	7,800	

Naples	5,700
Limoges	8,200
Sao Paulo	7,600
Caracas	6,800

In the production of jackets at the plants, 37.5% of the goatskin leather and 50% of the lambskin leather is waste (i.e., it is discarded during the

production process and sold for other byproducts). After production, a goatskin jacket weighs approximately 3 pounds, and a lambskin jacket weighs approximately 2.5 pounds (neither with linings, which are added in the United States).

Following are the costs per pound, in U.S. dollars, for tanning the uncut leather, shipping it, and producing the leather jackets at each plant:

Plant (\$/lb.)

Tanning Factory	Madrid	Naples	Limoges	Sao Paulo	Caracas
Mende	\$24	\$22	\$16	\$21	\$22
Foggia	31	17	22	19	2
Saragosa	18	25	28	23	2
Feira				16	1
El Tigre				14	1

Note that the cost of jacket production is the same for goatskin and lambskin. Also, leather can be tanned in France, Spain, and Italy and shipped directly to the South American plants for jacket production, but the opposite is not possible due to high tariff restrictions (i.e., tanned leather is not shipped to Europe for production).

Once the leather jackets are produced at the plants in Europe and South America,

Erken transports them to ports in Lisbon, Marseilles, and Caracas and then from these ports to U.S ports in New Orleans, Jacksonville, and Savannah. The available shipping capacity at each port and the transportation costs from the plants to the ports are as follows:

Plant	Port (\$/lb.)		
	Lisbon	Marseilles	Caracas
Madrid	0.75	1.05	

Naples	3.45	1.35	
Limoges	2.25	0.60	
Sao Paulo			1.15
Caracas			0.20
Capacity (lb.)	8,000	5,500	9,000

The shipping costs (\$/lb.) from each port in Europe and South

America to the U.S. ports and the available truck and rail capacity for transport at the U.S. ports are as follows:

	U.S. Port (\$/lb.)		
Port	New Orleans	Jacksonville	Savannah
Lisbon	\$2.35	\$1.90	\$1.80
Marseilles	3.10	2.40	2.00
Caracas	1.95	2.15	2.40

Capacity (lb.)	8,000	5,200	7,500
-------------------	-------	-------	-------

The transportation costs (\$/lb.) from the U.S. ports to the three distribution centers are as follows:

	Distribution Center (\$/lb.)		
U.S. Port	Indiana	North Carolina	Pennsylvania
New			

Orleans	\$0.65	\$0.52	\$0.87
Jacksonville	0.43	0.41	0.65
Savannah	0.38	0.34	0.50

Erken wants to determine the least costly shipments of material and jackets that will meet the demand at the U.S. distribution centers. Develop a transshipment model for Erken that will result in a minimum cost solution.

Chapter 7. Network Flow Models

[Page 273]

A **network** is an arrangement of paths connected at various points, through which one or more items move from one point to another. Everyone is familiar with such networks as highway systems, telephone networks, railroad systems, and television networks. For example, a railroad network consists of a number of fixed

rail routes (paths) connected by terminals at various junctions of the rail routes.

*A **network** is an arrangement of paths connected at various points, through which items move.*

In recent years using network models has become a very popular management science technique for a couple of very

important reasons. First, a network is drawn as a diagram, which literally provides a *picture* of the system under analysis. This enables a manager to visually interpret the system and thus enhances the manager's understanding. Second, a large number of real-life systems can be modeled as networks, which are relatively easy to conceive and construct.

Networks are popular because they provide a picture of a system and because a large

*number of systems
can be easily
modeled as networks.*

In this and the next chapter we will look at several different types of network models. In this chapter we will present a class of network models directed at the *flow of items* through a system. As such, these models are referred to as **network flow models**. We will discuss the use of network flow

models to analyze three types of problems: the shortest route problem, the minimal spanning tree problem, and the maximal flow problem. In [Chapter 8](#) we will present the network techniques PERT and CPM, which are used extensively for project analysis.

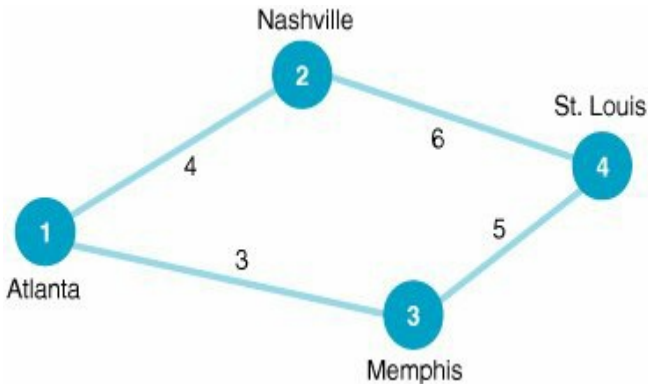
Network Components

Networks are illustrated as diagrams consisting of two main components: nodes and branches. **Nodes** represent junction points for example, an intersection of several streets. **Branches** connect the nodes and reflect the flow from one point in the network to another. Nodes are denoted in the network diagram by *circles*, and branches are represented by *lines* connecting the nodes.

Nodes typically represent localities, such as cities, intersections, or air or railroad terminals; branches are the paths connecting the nodes, such as roads connecting cities and intersections or railroad tracks or air routes connecting terminals. For example, the different railroad routes between Atlanta, Georgia, and St. Louis, Missouri, and the intermediate terminals are shown in [Figure 7.1](#).

Figure 7.1. Network

of railroad routes



Nodes, denoted by circles, represent junction points

connecting branches.

Branches,
*represented as lines,
connect nodes and
show flow from one
point to another.*

The network shown in [Figure 7.1](#) has four nodes and four branches. The node

representing Atlanta is referred to as the *origin*, and any of the three remaining nodes could be the *destination*, depending on what we are trying to determine from the network. Notice that a number has been assigned to each node. Numbers provide a more convenient means of identifying the nodes and branches than do names. For example, we can refer to the origin (Atlanta) as node 1 and the branch from Atlanta to Nashville as branch 12.

Typically, a value that

represents a distance, length of time, or cost is assigned to each branch. Thus, the purpose of the network is to determine the shortest distance, shortest length of time, or lowest cost between points in the network. In [Figure 7.1](#), the values 4, 6, 3, and 5, corresponding to the four branches, represent the lengths of time, in hours, between the attached nodes. Thus, a traveler can see that the route to St. Louis through Nashville requires 10 hours and the route to St. Louis through Memphis requires 8 hours.

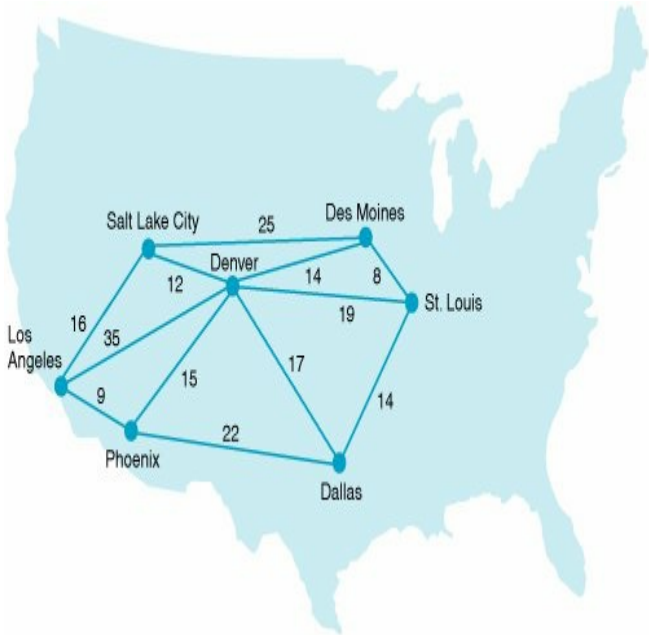
The values assigned to branches typically represent distance, time, or cost.

The Shortest Route Problem

The **shortest route problem** is to determine the shortest distance between an originating point and several destination points. For example, the Stagecoach Shipping Company transports oranges by six trucks from Los Angeles to six cities in the West and Midwest. The different routes between Los Angeles and the destination cities and the length of time, in

hours, required by a truck to travel each route are shown in [Figure 7.2](#).

Figure 7.2. Shipping routes from Los Angeles



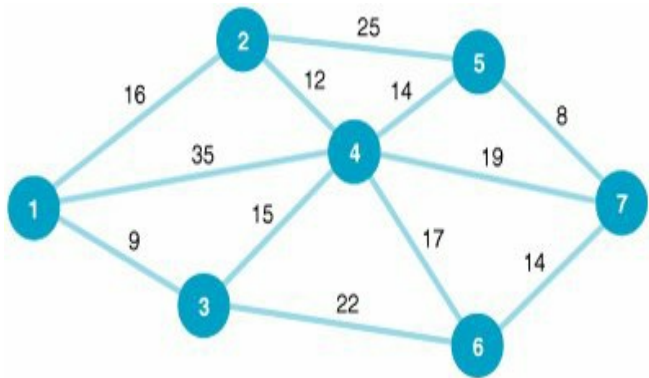
*The **shortest route***

problem is to find
the shortest distance
between an origin
and various
destination points.

The shipping company manager wants to determine the best routes (in terms of the minimum travel time) for the trucks to take to reach their destinations. This problem can be solved by using the shortest route solution technique. In

applying this technique, it is convenient to represent the system of truck routes as a network, as shown in [Figure 7.3](#).

Figure 7.3. Network of shipping routes



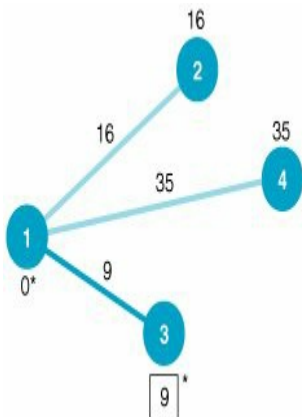
To repeat our objective as it relates to [Figure 7.3](#), we want to determine the shortest routes from the origin (node 1) to the six destinations (nodes 2 through 7).

The Shortest Route Solution Approach

We begin the shortest route solution technique by starting at node 1 (the origin) and determining the shortest time required to get to a directly connected (i.e., adjacent) node. The three nodes directly connected to node 1 are 2, 3, and 4, as shown in [Figure 7.4](#). Of these three nodes, the shortest time is 9 hours to

node 3. Thus, we have determined our first shortest route from nodes 1 to 3 (i.e., from Los Angeles to Phoenix). We will now refer to nodes 1 and 3 as the permanent set to indicate that we have found the shortest route to these nodes. (Because node 1 has no route to it, it is automatically in the permanent set.)

Figure 7.4. Network with node 1 in the permanent set



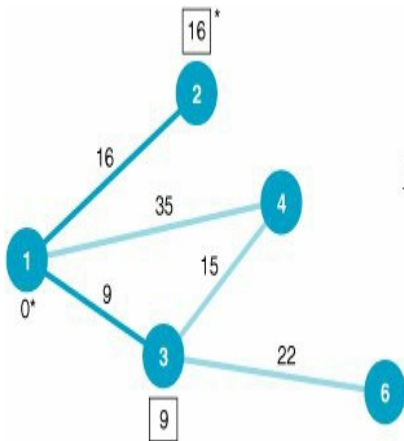
Permanent set	Branch	Time
$\{1\}$	1-2	16
	1-4	35
	1-3	9 [*]

Determine the initial shortest route from the origin (node 1) to the closest node (3).

Notice in [Figure 7.4](#) that the shortest route to node 3 is drawn with a heavy line, and the shortest time to node 3 (9 hours) is enclosed by a box. The table accompanying [Figure 7.4](#) describes the process of selecting the shortest route. The permanent set is shown to contain only node 1. The three branches from node 1 are 12, 14, and 13, and this last branch has the minimum time of 9 hours.

Next, we will repeat the foregoing steps used to determine the shortest route to node 3. First, we must *determine all the nodes directly connected to the nodes in the permanent set* (nodes 1 and 3). Nodes 2, 4, and 6 are all directly connected to nodes 1 and 3, as shown in [Figure 7.5](#).

Figure 7.5. Network with nodes 1 and 3 in the permanent set



Permanent set	Branch	Time
{1, 3}	1-2	16^*
	1-4	35
	3-4	24
	3-6	31

Determine all nodes directly connected to the permanent set.

The next step is to determine the shortest route to the three nodes (2, 4, and 6) directly connected to the permanent set nodes. There are two branches starting from node 1 (12 and 14) and two branches from node 3 (34 and 36). The branch with the shortest time is to node 2, with a time of 16 hours. Thus, node 2 becomes part of the permanent set. Notice in our computations accompanying [Figure 7.5](#) that the time to node 6 (branch 36)

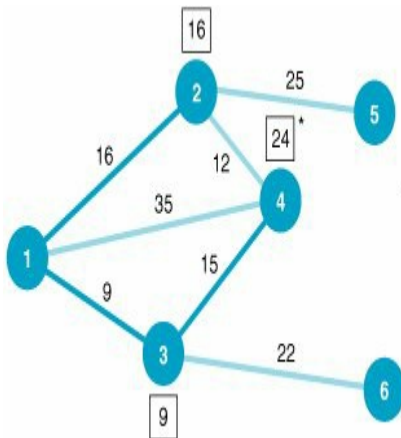
is 31 hours, which is determined by adding the branch 36 time of 22 hours to the shortest route time of 9 hours at node 3.

[Page 276]

As we move to the next step, the permanent set consists of nodes 1, 2, and 3. This indicates that we have found the shortest route to nodes 1, 2, and 3. We must now determine which nodes are directly connected to the permanent set nodes. Node 5 is

the only *adjacent* node not presently connected to the permanent set, so it is connected directly to node 2. In addition, node 4 is now connected directly to node 2 (because node 2 has joined the permanent set). These additions are shown in [Figure 7.6](#).

Figure 7.6. Network with nodes 1, 2, and 3 in the permanent set



Permanent set	Branch	Time
{1, 2, 3}	1-4	35
	2-4	28
	2-5	41
	3-4	24 *
	3-6	31

Redefine the permanent set.

Five branches lead from the permanent set nodes (1, 2, and 3) to their directly connected nodes, as shown in the table accompanying [Figure 7.6](#). The branch representing the route with the shortest time is 34, with a time of 24 hours. Thus, we have determined the shortest route to node 4, and it joins the permanent set. Notice that the shortest time to node 4 (24 hours) is the route from node 1 through node 3. The other routes to node 4 from node 1 through node 2 are longer; therefore, we will not

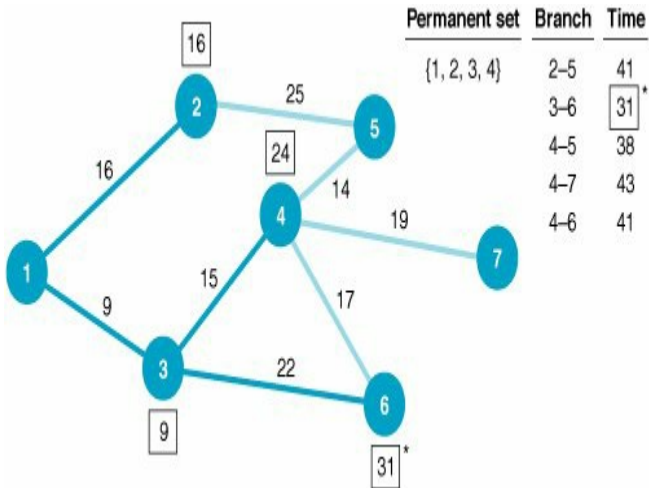
consider them any further as possible routes to node 4.

To summarize, the shortest routes to nodes 1, 2, 3, and 4 have all been determined, and these nodes now form the permanent set.

Next, we repeat the process of determining the nodes directly connected to the permanent set nodes. These directly connected nodes are 5, 6, and 7, as shown in [Figure 7.7](#). Notice in [Figure 7.7](#) that we have eliminated the branches from nodes 1 and 2 to node 4

because we determined that the route with the shortest time to node 4 does not include these branches.

Figure 7.7. Network with nodes 1, 2, 3, and 4 in the permanent set



[Page 277]

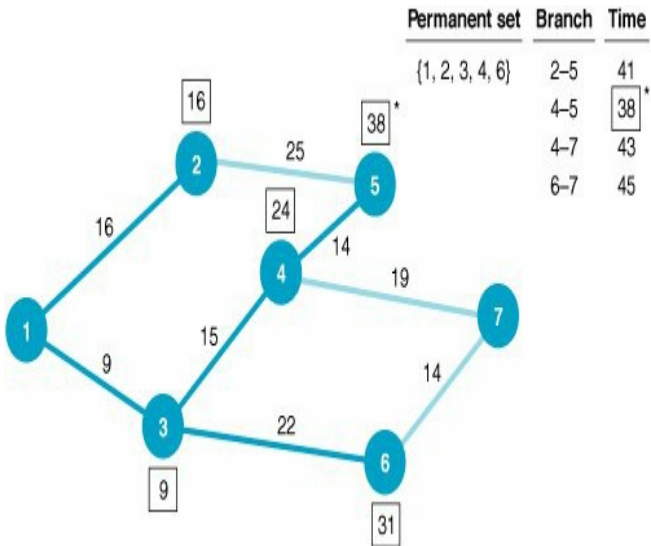
From the table accompanying [Figure 7.7](#) we can see that of

the branches leading to nodes 5, 6, and 7, branch 36 has the shortest *cumulative* time, of 31 hours. Thus, node 6 is added to our permanent set. This means that we have now found the shortest routes to nodes 1, 2, 3, 4, and 6.

Repeating the process, we observe that the nodes directly connected (adjacent) to our permanent set are nodes 5 and 7, as shown in [Figure 7.8](#). (Notice that branch 46 has been eliminated because the best route to node 6 goes through node 3 instead of node

4.)

**Figure 7.8. Network
with nodes 1, 2, 3, 4,
and 6 in the
permanent set**



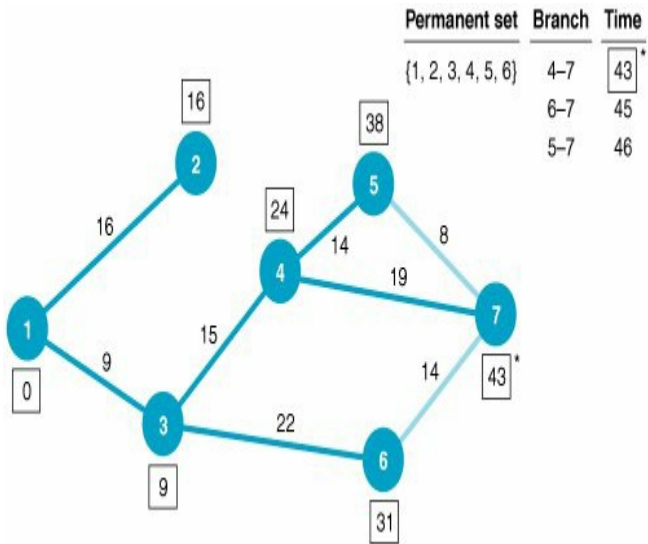
Of the branches leading from the permanent set nodes to nodes 5 and 7, branch 45 has

the shortest cumulative time, of 38 hours. Thus, node 5 joins the permanent set. We have now determined the routes with the shortest times to nodes 1, 2, 3, 4, 5, and 6 (as denoted by the heavy branches in [Figure 7.8](#)).

The only remaining node directly connected to the permanent set is node 7, as shown in [Figure 7.9](#). Of the three branches connecting node 7 to the permanent set, branch 47 has the shortest time, of 43 hours. Therefore, node 7 joins the permanent

set.

Figure 7.9. Network with nodes 1, 2, 3, 4, 5, and 6 in the permanent set



The routes with the shortest times from the origin (node 1) to each of the other six nodes

and their corresponding travel times are summarized in [Figure 7.10](#) and [Table 7.1](#).

[Page 278]

Figure 7.10. Network with optimal routes from Los Angeles to all destinations

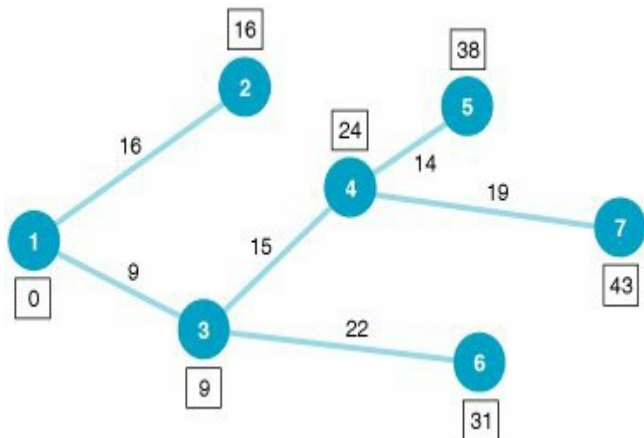


Table 7.1. Shortest travel time from origin to each destination

From Los Angeles to:	Route	Total Hours
----------------------	-------	-------------

Salt Lake City (node 2)	12	16
Phoenix (node 3)	13	9
Denver (node 4)	134	24
Des Moines (node 5)	1345	38
Dallas (node 6)	136	31
St. Louis (node 7)	1347	43

In summary, the steps of the shortest route solution method are as follows:

- 1.** Select the node with the shortest direct route from the origin.
- 2.** Establish a *permanent set* with the origin node and the node that was selected in step 1.
- 3.** Determine all nodes directly connected to the permanent set nodes.

4. Select the node with the shortest route (branch) from the group of nodes directly connected to the permanent set nodes.
5. Repeat steps 3 and 4 until all nodes have joined the permanent set.

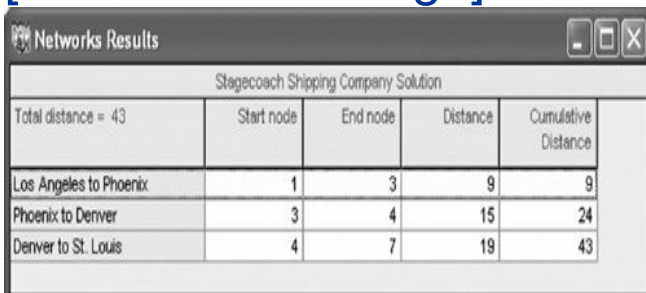
Steps of the shortest route solution method.

Computer Solution of the Shortest Route Problem with QM for Windows

QM for Windows includes modules for each of the three network flow models that will be presented in this chaptershortest route, minimal spanning tree, and maximal flow. The QM for Windows solution for the shortest route from node 1 to node 7 for the Stagecoach Shipping Company example is shown in [Exhibit 7.1](#).

Exhibit 7.1.

[\[View full size image\]](#)



The screenshot shows a window titled "Networks Results" with standard Windows window controls (minimize, maximize, close) in the top right corner. The window content is titled "Stagecoach Shipping Company Solution". It displays a table with shipping routes and distances. The first row of the table indicates the total distance is 43. The subsequent rows list the route segments: Los Angeles to Phoenix, Phoenix to Denver, and Denver to St. Louis, along with their respective start and end nodes, individual distances, and cumulative distances.

Stagecoach Shipping Company Solution					
Total distance = 43	Start node	End node	Distance	Cumulative Distance	
Los Angeles to Phoenix	1	3	9	9	
Phoenix to Denver	3	4	15	24	
Denver to St. Louis	4	7	19	43	

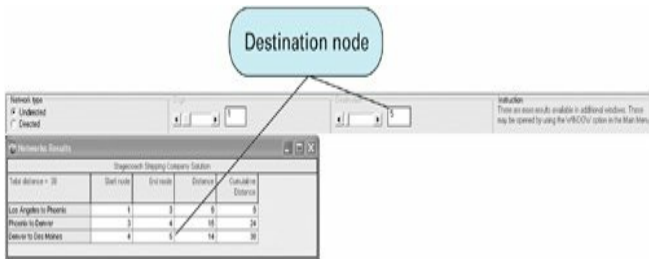
[Page 279]

The solution screen in [Exhibit 7.1](#) provides the shortest route

from the start node, 1 (Los Angeles), to the end node, 7 (St. Louis). The shortest route from any node in the network to any other node can be determined by indicating the desired origin and destination nodes at the top of the solution screen. For example, the shortest route from node 1 to node 5 is shown in [Exhibit 7.2](#).

Exhibit 7.2.

[\[View full size image\]](#)



Computer Solution of the Shortest Route Problem with Excel

We can also solve the shortest route problem with Excel spreadsheets by formulating

and solving the shortest route network as a 01 integer linear programming problem. To formulate the linear programming model, we first define a decision variable for each branch in the network, as follows:

$x_{ij} = 0$ if branch ij is not selected as part of the shortest route and 1 if branch ij is selected

To reduce the complexity and size of our model formulation, we will also assume that items can flow only from a lower

node number to a higher node number (e.g., 3 to 4 but not 4 to 3). This greatly reduces the number of variables and branches to consider.

The objective function is formulated by minimizing the sum of the products of each branch value times the travel time for each branch:

$$\begin{aligned} \text{minimize } Z = & 16x_{12} + 9x_{13} + \\ & 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} \\ & + 22x_{36} + 14x_{45} + 17x_{46} + \\ & 19x_{47} + 8x_{57} + 14x_{67} \end{aligned}$$

There is one constraint for each node, indicating that whatever comes into a node must also go out. This is referred to as *conservation of flow*. This means that one "truck" leaves node 1 (Los Angeles) either through branch 12, branch 13, or branch 14. This constraint for node 1 is formulated as

$$x_{12} + x_{13} + x_{14} = 1$$

At node 2, a truck would come in through branch 12 and depart through 24 or 25, as follows:

$$X_{12} = X_{24} + X_{25}$$

This constraint can be rewritten as

$$X_{12} - X_{24} - X_{25} = 0$$

[Page 280]

Constraints at nodes 3, 4, 5, 6, and 7 are constructed similarly, resulting in the complete linear programming model, summarized as follows:

$$\text{minimize } Z = 16x_{12} + 9x_{13} + 35x_{14} + 12x_{24} + 25x_{25} + 15x_{34} + 22x_{36} + 14x_{45} + 17x_{46} \\ + 19x_{47} + 8x_{57} + 14x_{67}$$

subject to

$$x_{12} + x_{13} + x_{14} = 1$$

$$x_{12} - x_{24} - x_{25} = 0$$

$$x_{13} - x_{34} - x_{36} = 0$$

$$x_{14} + x_{24} + x_{34} - x_{45} - x_{46} - x_{47} = 0$$

$$x_{25} + x_{45} - x_{57} = 0$$

$$x_{36} + x_{46} - x_{67} = 0$$

$$x_{47} + x_{57} + x_{67} = 1$$

$$x_{ij} = 0 \text{ or } 1$$

This model is solved in Excel much the same way as any other linear programming problem, using the "Solver" option from the "Tools" menu at the top of the spreadsheet.

[Exhibit 7.3](#) shows an Excel spreadsheet set up to solve the

Stagecoach Shipping Company example. The decision variables, x_{ij} , are represented by cells **A6:A17**. Thus, a value of 1 in one of these cells means that branch has been selected as part of the shortest route. Cells **F6:F17** contain the travel times (in hours) for each branch, and the objective function formula is contained in cell F18, shown on the formula bar at the top of the screen. The model constraints reflecting the flow through each node are included in the box on the right side of the

spreadsheet. For example, cell I6 contains the constraint formula for node 1, **=A6+A7+A8**, and cell I7 contains the constraint formula for node 2, **=A6-A9-A10**.

Exhibit 7.3.

[\[View full size image\]](#)

Click on "Tools"
to invoke "Solver."

Total hours

First constraint;
 $=A6+A7+A8$

Constraint for node 2;
= A6-A9-A10

Decision variables

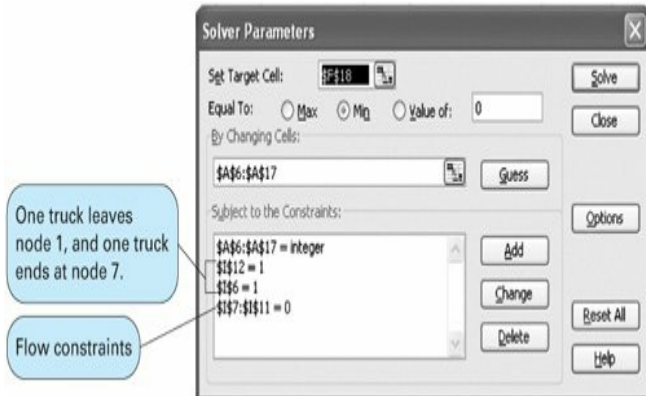
Exhibit 7.4 shows the Solver screen, which is accessed from the "Tools" menu at the top of

the screen. Notice that the constraint for node 1 embedded in cell I6 equals 1, indicating that one truck is leaving node 1, and the constraint for node 7, embedded in cell I12, also equals 1, indicating that one truck will end up at node 7.

[Page 281]

Exhibit 7.4.

[\[View full size image\]](#)



The Excel solution is shown in [Exhibit 7.5](#). Notice that cells A7, A11, and A15 have values of 1 in them, indicating that these branches, 13, 34, and 47, are on the shortest route.

The total distance in travel time is 43 hours, as shown in cell F18.

Exhibit 7.5.

[\[View full size image\]](#)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Stagescoach Shipping Company: Shortest Route Problem																
2																	
3																	
4	Select																
5	Branch	Node	City	Node	City	Distance (hours)											
6	0	1	Los Angeles	2	Salt Lake City	16											
7	1	1	Los Angeles	3	Phoenix	8											
8	0	1	Los Angeles	4	Denver	28											
9	0	2	Salt Lake City	4	Denver	12											
10	0	2	Salt Lake City	6	Des Moines	28											
11	1	3	Phoenix	4	Denver	15											
12	0	3	Phoenix	6	Dallas	22											
13	0	4	Denver	6	Des Moines	14											
14	0	4	Denver	6	Dallas	17											
15	1	4	Denver	7	St. Louis	19											
16	0	5	Des Moines	7	St. Louis	8											
17	0	6	Dallas	7	St. Louis	14											
18					Total	43											
19																	
20																	

Node	Network Flow
1	1
2	0
3	0
4	0
5	0
6	0
7	1

One truck flows out of node 1; one truck flows into node 7.

The Minimal Spanning Tree Problem

In the shortest route problem presented in the previous section, the objective was to determine the shortest routes between the origin and the destination nodes in the network. In our example, we determined the best route from Los Angeles to each of the six destination cities. The **minimal spanning tree problem** is similar to the shortest route

problem, except that the objective is to connect all the nodes in the network so that the total branch lengths are minimized. The resulting network *spans* (connects) all the points in the network at a minimum total distance (or length).

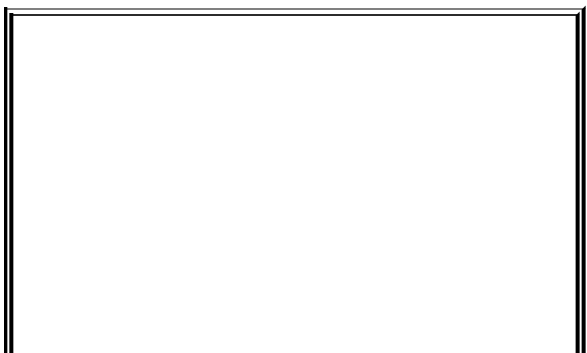
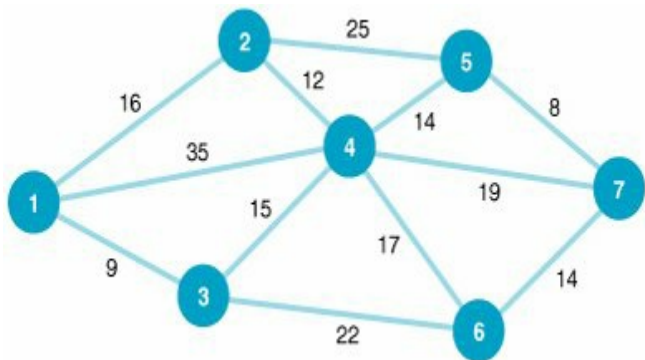
*The **minimal spanning tree problem** is to connect all nodes in a network so that the total branch lengths are minimized.*

To demonstrate the minimal spanning tree problem, we will consider the following example. The Metro Cable Television Company is to install a television cable system in a community consisting of seven suburbs. Each of the suburbs must be connected to the main cable system. The cable television company wants to lay out the main cable network in a way that will minimize the total length of cable that must be installed. The possible paths

available to the cable television company (by consent of the town council) and the feet of cable (in thousands of feet) required for each path are shown in [Figure 7.11](#).

[Page 282]

Figure 7.11. Network of possible cable TV paths



Management Science

Application: Reducing Travel Costs at the Defense Contract Management Agency

The Defense Contract Management Agency (DCMA), headquartered at Fort Belvoir, Virginia, has 15,000 people stationed in offices around the United States. These federal employees have recurring professional development training requirements, and the associated costs for travel to training sites is a substantial portion of the DCMA's annual budget. In 1998 the DCMA faced a 25% budget cut, without any reduction in training requirements. At that time training sites were selected

arbitrarily at the discretion of training planners. Because travel costs vary with the location, the selection of training sites can have a dramatic impact on the total temporary duty (i.e., travel) costs for employees. With the help of consultants, DCMA developed a decision support system (DSS) called OffSite that would select the minimum-cost training sites from among many cities. The DSS employs a shortest route model to determine the lowest-cost location for program participants arriving from geographically dispersed locations.

A shortest route network was created that connected 261 U.S. cities with government-contracted airfares. The minimum-cost training location is based on the federal per-diem travel rate (including meals and lodging) for a specific city, the duration of the training event, the

number of participants at the city, and the cost of travel to the city from other cities. The shortest route method determines the minimum-cost route from each of the 261 cities to a specific training site, enabling training planners to assess the overall costs of bringing attendees to various alternate locations. OffSite was tested using data from a 2-day training event in Monterey, California, attended by representatives from every DCMA office. The total cost of this meeting was \$37,582, which was over \$10,000 (i.e., 39%) more than the least-cost alternative identified by OffSite, which was Tulsa, Oklahoma. Since OffSite's development in 1998, it has been used throughout the federal government. At the Defense Logistics Agency alone, savings have been over \$1 million.



Source: J. L. Huisingsh, H. M. Yamauchi, and R. Zimmerman, "Saving Federal Travel Dollars," *Interfaces* 31, no. 5 (September/October 2001): 1323.

In [Figure 7.11](#) the branch from node 1 to node 2 represents the available cable path between suburbs 1 and 2. The branch requires 16,000 feet of cable. Notice that the network shown in [Figure 7.11](#) is identical to the network in [Figure 7.2](#) that we used to demonstrate the shortest route problem. The networks were intentionally made identical to demonstrate the difference between the results of the two types of network models.

The Minimal Spanning

Tree Solution Approach

The solution approach to the minimal spanning tree problem is actually easier than the shortest route solution method. In the minimal spanning tree solution approach, we can start at any node in the network. However, the conventional approach is to start with node 1. Beginning at node 1, we select the closest node (i.e., the shortest branch) to join our spanning tree. The shortest branch from node 1 is to node 3, with a length of 9 (thousand

feet). This branch is indicated with a heavy line in [Figure 7.12](#). Now we have a *spanning tree* consisting of two nodes: 1 and 3. The next step is to select the closest node not now in the spanning tree. The node closest to either node 1 or node 3 (the nodes in our present spanning tree) is node 4, with a branch length of 15,000 feet. The addition of node 4 to our spanning tree is shown in [Figure 7.13](#).

Figure 7.12. Spanning

tree with nodes 1 and 3

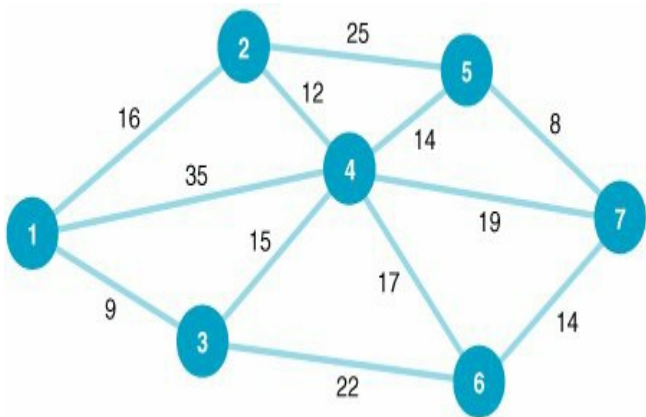
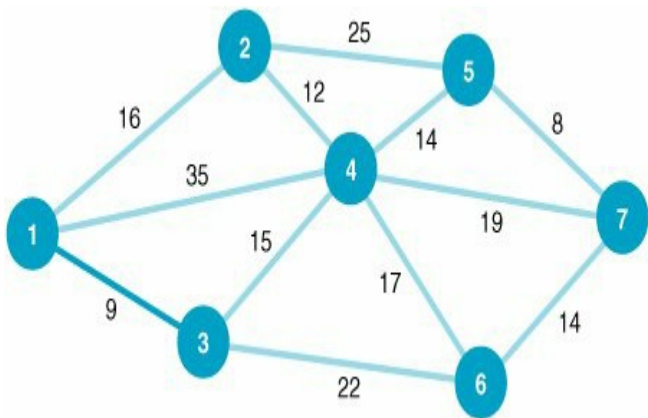


Figure 7.13. Spanning tree with nodes 1, 3, and 4



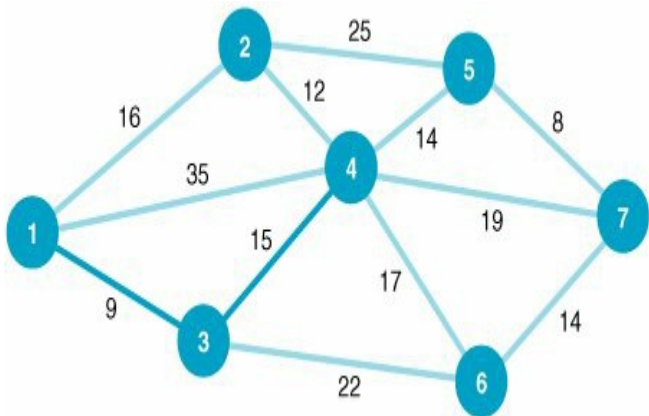
Start with any node in the network and select the closest node to join the spanning tree. Select the closest node to any node in the spanning area.

Next, we repeat the process of selecting the closest node to our present spanning tree (nodes 1, 3, and 4). The closest node not now connected to the

nodes in our spanning tree is node 2. The length of the branch from node 4 to node 2 is 12,000 feet. The addition of node 2 to the spanning tree is shown in [Figure 7.14](#).

[Page 284]

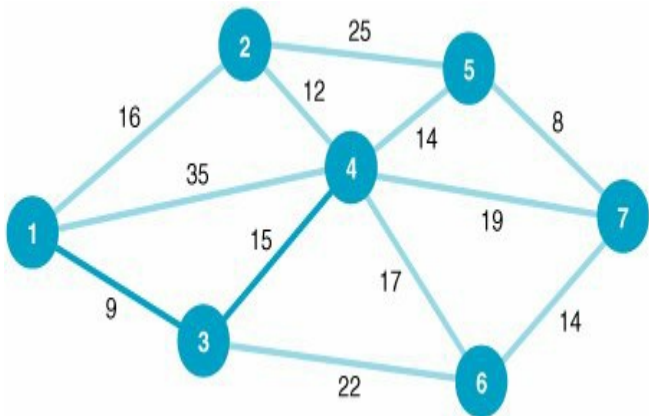
Figure 7.14. Spanning tree with nodes 1, 2, 3, and 4



Our spanning tree now consists of nodes 1, 2, 3, and 4. The node closest to this spanning tree is node 5, with a branch length of 14,000 feet to node 4. Thus, node 5 joins our

spanning tree, as shown in [Figure 7.15](#).

Figure 7.15. Spanning tree with nodes 1, 2, 3, 4, and 5

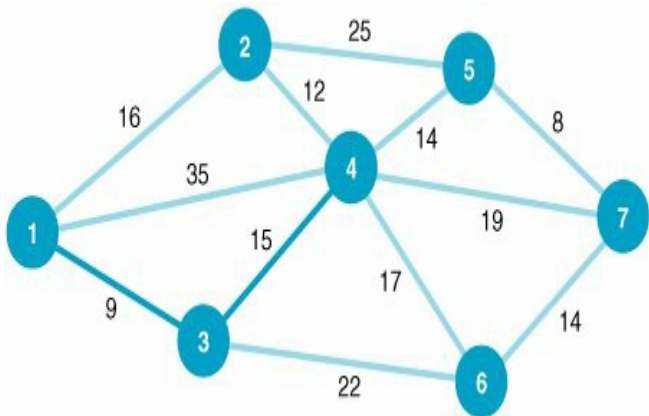


The spanning tree now contains nodes 1, 2, 3, 4, and 5. The closest node not currently connected to the spanning tree is node 7. The branch connecting node 7 to node 5

has a length of 8,000 feet.

[Figure 7.16](#) shows the addition of node 7 to the spanning tree.

Figure 7.16. Spanning tree with nodes 1, 2, 3, 4, 5, and 7

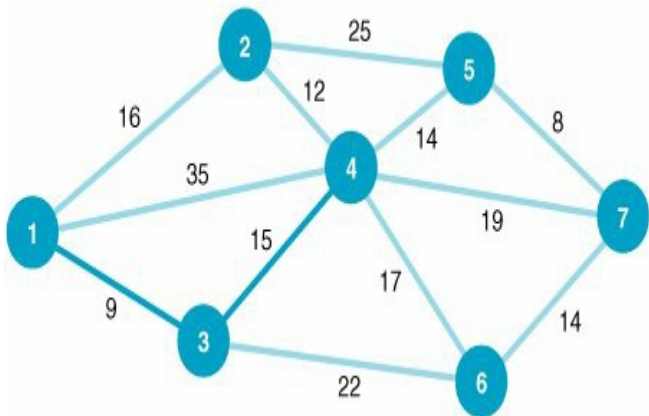


Now our spanning tree includes nodes 1, 2, 3, 4, 5, and 7. The only remaining node not connected to the spanning tree is node 6. The node in the spanning tree closest to node 6

is node 7, with a branch length of 14,000 feet. The complete spanning tree, which now includes all seven nodes, is shown in [Figure 7.17](#).

Figure 7.17. Minimal spanning tree for cable TV network

(This item is displayed on page 285 in the print version)



The spanning tree shown in [Figure 7.17](#) requires the minimum amount of television cable to connect the seven suburbs 72,000 feet. This same minimal spanning tree could

have been obtained by starting at any of the six nodes other than node 1.

[Page 285]

Notice the difference between the minimal spanning tree network shown in [Figure 7.17](#) and the shortest route network shown in [Figure 7.10](#). The shortest route network represents the shortest paths between the origin and each of the destination nodes (i.e., six different routes). In contrast, the minimal spanning tree

network shows how to connect all seven nodes so that the total distance (length) is minimized.

In summary, the steps of the minimal spanning tree solution method are as follows:

- 1.** Select any starting node (conventionally, node 1 is selected).
- 2.** Select the node closest to the starting node to join the spanning tree.
- 3.** Select the closest node not

presently in the spanning tree.

4. Repeat step 3 until all nodes have joined the spanning tree.

Steps of the minimal spanning tree solution method.

Computer Solution of the Minimal Spanning Tree

Problem with QM for Windows

The minimal spanning tree solution for our Metro Cable Television Company example using QM for Windows is shown in [Exhibit 7.6](#).

Exhibit 7.6.

[\[View full size image\]](#)

Networks Results



Metro Cable Television Company Solution

Branch name	Start node	End node	Cost	Include	Cost
1	0	2	16		
2	1	3	9	Y	9
3	1	4	35		
4	2	4	12	Y	12
5	2	5	25		
6	3	4	15	Y	15
7	3	6	22		
8	4	5	14	Y	14
9	4	6	17		
10	4	7	19		
11	5	7	8	Y	8
12	6	7	14	Y	14
Total					72

The Maximal Flow Problem

In the shortest route problem we determined the shortest truck route from the origin (Los Angeles) to six destinations. In the minimal spanning tree problem we found the shortest connected network for television cable. In neither of these problems was the capacity of a branch limited to a specific number of items. However, there are network

problems in which the branches of the network have limited flow capacities. The objective of these networks is to maximize the total amount of flow from an origin to a destination. These problems are referred to as **maximal flow problems**.

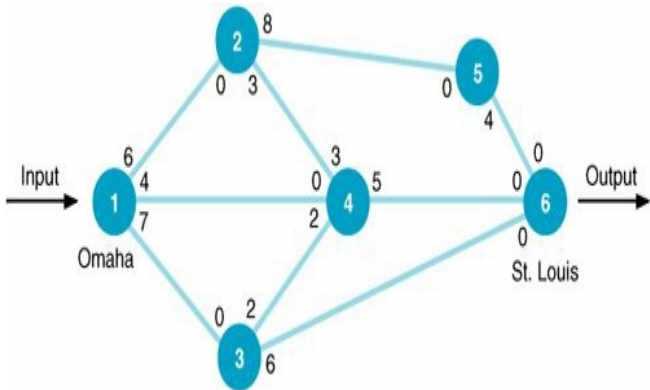
*The **maximal flow problem** is to maximize the amount of flow of items from an origin to a destination.*

Maximal flow problems can involve the flow of water, gas, or oil through a network of pipelines; the flow of forms through a paper processing system (such as a government agency); the flow of traffic through a road network; or the flow of products through a production line system. In each of these examples, the branches of the network have limited and often different flow capacities. Given these conditions, the decision maker

wants to determine the maximum flow that can be obtained through the system.

An example of a maximal flow problem is illustrated by the network of a railway system between Omaha and St. Louis shown in [Figure 7.18](#). The Scott Tractor Company ships tractor parts from Omaha to St. Louis by railroad. However, a contract limits the number of railroad cars the company can secure on each branch during a week.

Figure 7.18. Network of railway system



Given these limiting conditions, the company wants to know the

maximum number of railroad cars containing tractor parts that can be shipped from Omaha to St. Louis during a week. The number of railroad cars available to the tractor company on each rail branch is indicated by the number on the branch to the *immediate right of each node* (which represents a rail junction). For example, six cars are available from node 1 (Omaha) to node 2, eight cars are available from node 2 to node 5, five cars are available from node 4 to node 6 (St. Louis), and so forth. The

number on each branch to the *immediate* left of each node is the number of cars available for shipping in the opposite direction. For example, no cars are available from node 2 to node 1. The branch from node 1 to node 2 is referred to as a *directed* branch because flow is possible in only one direction (from node 1 to node 2, but not from 2 to 1). Notice that flow is possible in both directions on the branches between nodes 2 and 4 and nodes 3 and 4. These are referred to as *undirected branches*.

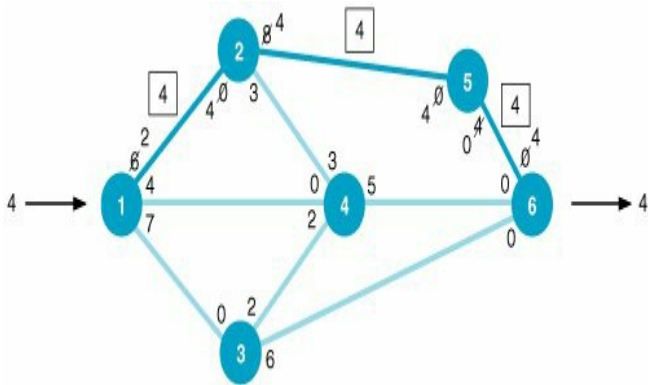
For a directed branch, flow is possible in only one direction.

The Maximal Flow Solution Approach

The first step in determining the maximum possible flow of railroad cars through the rail system is to *choose any path*

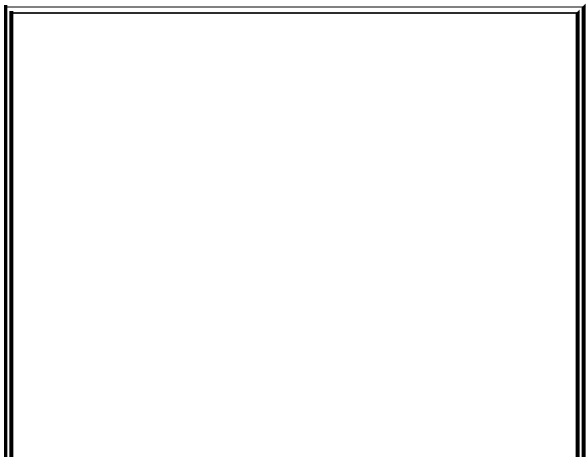
arbitrarily from origin to destination and ship as much as possible on that path. In [Figure 7.19](#) we will arbitrarily select the path 1256. The maximum number of railroad cars that can be sent through this route is four. We are limited to four cars because that is the maximum amount available on the branch between nodes 5 and 6. This path is shown in [Figure 7.19](#).

Figure 7.19. Maximal flow for path 1256



*Arbitrarily choose
any path through the*

*network from origin
to destination and
ship as much as
possible.*



Time Out: For E. W. Dijkstra, L. R. Ford, Jr., and D. R. Fulkerson

In 1959 E. W. Dijkstra of the Netherlands proposed the solution procedures (shown in this chapter) for both the minimal spanning tree problem and the shortest route problem. Earlier, in 1955, L. R. Ford, Jr., and D. R. Fulkerson (colleagues of George Dantzig's) of the RAND Corporation introduced the procedure for solving the maximal flow network problem, which evolved from the study of transportation problems. The original problem analyzed was of a rail network connecting cities in which each rail link had capacity limitations.

Notice that the remaining capacities of the branches from node 1 to node 2 and from node 2 to node 5 are two and four cars, respectively, and that no cars are available from node 5 to node 6. These values were computed by subtracting the flow of four cars from the number available. The actual flow of four cars along each branch is shown enclosed in a box. Notice that the present input of four cars into node 1 and output of four cars from

node 6 are also designated.

*Recompute branch
flow in both
directions.*

The final adjustment on this path is to add the designated flow of four cars to the values at the immediate left of each node on our path, 1256. These are the flows in the opposite direction. Thus, the value 4 is added to the zeros at nodes 2,

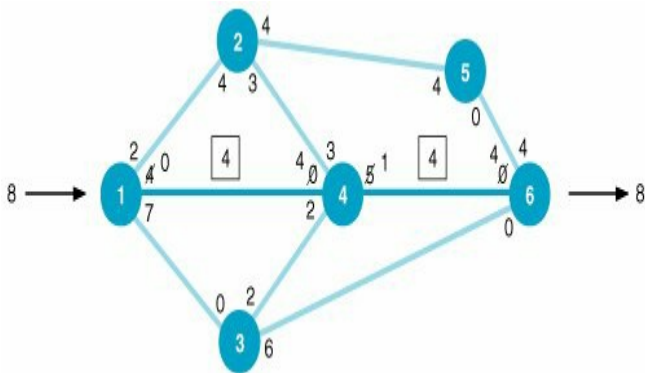
5, and 6. It may seem incongruous to designate flow in a direction that is not possible; however, it is the means used in this solution approach to compute the *net flow* along a branch. (If, for example, a later iteration showed a flow of one car from node 5 to node 2, then the net flow in the correct direction would be computed by subtracting this flow of one in the wrong direction from the previous flow of four in the correct direction. The result would be a net flow of three in

the correct direction.)

We have now completed one iteration of the solution process and must repeat the preceding steps. Again, we arbitrarily select a path. This time we will select path 146, as shown in [Figure 7.20](#). The maximum flow along this path is four cars, which is subtracted at each of the nodes. This increases the total flow through the network to eight cars (because the flow of four along path 146 is added to the flow previously determined in [Figure 7.19](#)).

Figure 7.20. Maximal flow for path 146

(This item is displayed on page 288 in the print version)



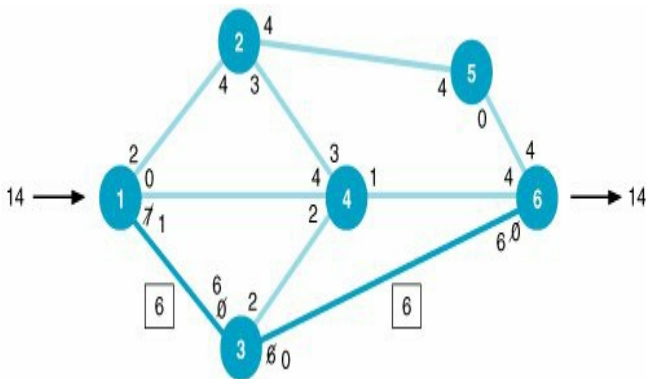
Select other feasible paths arbitrarily and determine maximum flow along the paths until flow is no longer possible.

As a final step, the flow of four cars is added to the flow along the path in the opposite direction at nodes 4 and 6.

Now we arbitrarily select another path. This time we will choose path 136, with a maximum possible flow of six cars. This flow of six is subtracted from the branches along path 136 and added to the branches in the opposite direction, as shown in [Figure 7.21](#). The flow of 6 for this path is added to the previous flow of 8, which results in a total flow of 14 railroad cars.

Figure 7.21. Maximal flow for path 136

(This item is displayed on page 288 in the print version)



Management Science

Application: Improving

Service for Yellow Freight

System's Terminal Network

(This item is displayed on page 288 in
the print version)

Yellow Freight System is one of the largest motor freight carriers in the United States, handling more than 15 million shipments over a network of 380 terminals annually. Revenues in 1997 exceeded \$2.5 billion. Yellow Freight primarily concentrates on the less-than-truckload (LTL) market. LTL shipments are typically less than 10,000 pounds, and because tractor-trailer trucks can normally accommodate about 45,000 pounds,

LTL shipments are consolidated to form economical, full-truck shipments. This required an extensive network of terminals where the individual LTL shipments are consolidated and loaded (i.e., break bulking) for individual delivery in the area. In order to manage its increasingly complex terminal network system, Yellow Freight developed a network modeling approach called SYSNET to improve customer service and reliability, and, secondarily, to improve productivity and lower costs. SYSNET is used for monthly load planning that governs how shipments are consolidated and handled throughout the network and for long-range planning, including the location and size requirements of the company's facilities.



Source: J. W. Braklow, et al.,
"Interactive Optimization Improves
Service and Performance for Yellow
Freight System," *Interfaces* 22, no.
1 (January/February 1992): 14772.

Notice that at this point the
number of paths we can take is
restricted. For example, we

cannot take the branch from node 3 to node 6 because zero flow capacity is available.

Likewise, no path that includes the branch from node 1 to node 4 is possible.

The available flow capacity along the path 1346 is one car, as shown in [Figure 7.22](#). This increases the total flow from 14 cars to 15 cars. The resulting network is shown in [Figure 7.23](#). Close observation of the network in [Figure 7.23](#) shows that there are no more paths with available flow capacity. All paths out of nodes 3, 4, and 5

show zero available capacity,
which prohibits any further
paths through the network.

Figure 7.22. Maximal flow for path 1346

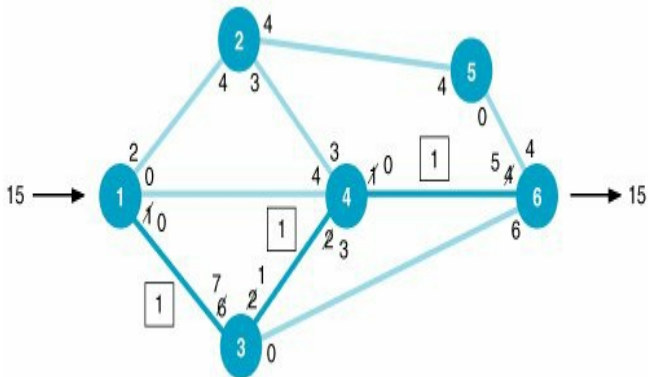
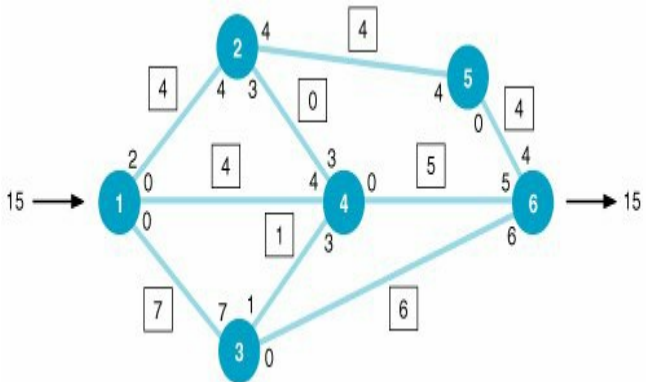


Figure 7.23. Maximal flow for railway network



This completes the maximal flow solution for our example problem. The maximum flow is 15 railroad cars. The flows that will occur along each branch appear in boxes in [Figure 7.23](#).

In summary, the steps of the maximal flow solution method are as follows:

- 1.** Arbitrarily select any path in the network from origin to destination.
- 2.** Adjust the capacities at

each node by subtracting the maximal flow for the path selected in step 1.

- 3.** Add the maximal flow along the path in the opposite direction at each node.
- 4.** Repeat steps 1, 2, and 3 until there are no more paths with available flow capacity.

Steps of the maximal flow solution method.

Computer Solution of the Maximal Flow Problem with QM for Windows

The program input and maximal flow solution for the Scott Tractor Company example using QM for Windows is shown in [Exhibit 7.7](#). (Note that this problem has multiple optimal solutions, and the branch flows are slightly different for this solution.)

Exhibit 7.7.

[\[View full size image\]](#)

Networks Results					
Scott Tractor Company Solution					
Branch name	Start node	End node	Capacity	Reverse capacity	Flow
Maximal Network Flow	15				
1	1	2	6	0	5
2	1	3	7	0	6
3	1	4	4	0	4
4	2	4	3	3	1
5	2	5	8	0	4
6	3	4	2	2	0
7	3	6	6	0	6
8	4	6	5	0	5
9	5	6	4	0	4

Computer Solution of the Maximal Flow Problem with Excel

The maximal flow problem can also be solved with Excel, much the same way as we solved the shortest route problem, by formulating it as an integer linear programming model and solving it by using the "Solver" option from the "Tools" menu.

The decision variables

represent the flow along each branch, as follows:

x_{ij} = flow along branch ij and integer

To reduce the size and complexity of the model formulation, we will also eliminate flow along a branch in the opposite direction (e.g., flow from 4 to 2 is zero).

Before proceeding with the model formulation and developing the objective function, we must alter the network slightly to be able to

solve it as a linear programming problem. We will create a new branch from node 6 back to node 1, x_{61} . The flow along this branch from the end of the network back to the start corresponds to the maximum amount that can be shipped from node 6 to node 1 and then back through the network to node 6. In effect, we are creating a continual flow through the network so that the most that goes through and comes out at node 6 is the most that can go through the network beginning at node 1.

As a result, the objective is to maximize the amount that flows from node 6 back to node 1:

$$\text{maximize } Z = x_{61}$$

The constraints follow the same premise as the shortest route problem; that is, whatever flows into a node must flow out. Thus, at node 1 we have the following constraint, showing that the amount flowing from 61 must flow out through branches 12, 13, and 14:

$$x_{61} = x_{12} + x_{13} + x_{14}$$

This constraint can be rewritten as

$$x_{61} - x_{12} - x_{13} - x_{14} = 0$$

Similarly, the constraint at node 2 is written as

$$x_{12} - x_{24} - x_{25} = 0$$

We must also develop a set of constraints reflecting the capacities along each branch, as follows:

$$x_{12} \leq 6$$

$$x_{34} \leq 2$$

$$x_{13} \leq 7$$

$$x_{36} \leq 6$$

$$x_{14} \leq 4$$

$$x_{46} \leq 5$$

$$x_{24} \leq 3$$

$$x_{56} \leq 4$$

$$x_{25} \leq 8$$

$$x_{61} \leq 17$$

[Page 291]

The capacity for x_{61} can be any

relatively large number (compared to the other branch capacities); we set it at the sum of the capacities on the branches leaving node 1.

The complete linear programming model for our Scott Tractor Company example is summarized as follows:

maximize $Z = x_{61}$

subject to

$$x_{61} - x_{12} - x_{13} - x_{14} = 0$$

$$x_{12} - x_{24} - x_{25} = 0$$

$$x_{13} - x_{34} - x_{36} = 0$$

$$x_{14} + x_{24} + x_{34} - x_{46} = 0$$

$$x_{25} - x_{56} = 0$$

$$x_{36} + x_{46} + x_{56} - x_{61} = 0$$

$$x_{12} \leq 6$$

$$x_{13} \leq 7$$

$$x_{14} \leq 4$$

$$x_{24} \leq 3$$

$$x_{25} \leq 8$$

$$x_{34} \leq 2$$

$$x_{36} \leq 6$$

$$x_{46} \leq 5$$

$$x_{56} \leq 4$$

$$x_{61} \leq 17$$

$$x_{ij} \geq 0 \text{ and integer}$$

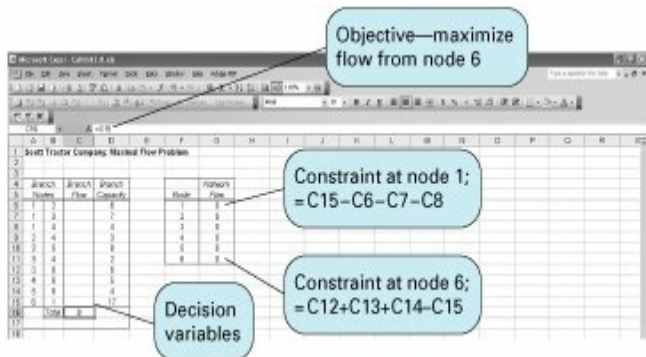
The Excel spreadsheet set up to

solve this integer linear programming model for the Scott Tractor Company example is shown in [Exhibit 7.8](#). The decision variables, x_{ij} , are represented by cells **C6:C15**. Cells **D6:D15** include the branch capacities. The objective function, which is to maximize the flow along branch 61, is included in cell C16. The model constraints reflecting the flow through each node are included in the box on the right side of the spreadsheet. For example, cell G6 contains the constraint formula at node 1,

=C15C6C7C8, and cell G7 contains the constraint formula for node 2, **=C6C9C10**. The constraints for the branch capacities are obtained by adding the formula **C6:C15 ≤ D6:D15** in Solver.

Exhibit 7.8.

[\[View full size image\]](#)



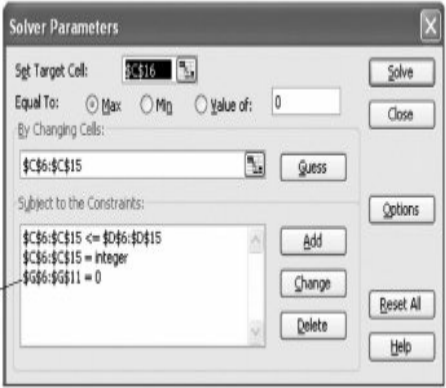
[Page 292]

[Exhibit 7.9](#) shows the Solver Parameters screen for this model.

Exhibit 7.9.

[\[View full size image\]](#)

Flow into and out of nodes must equal each other.



The image shows the 'Solver Parameters' dialog box in Microsoft Excel. The 'Set Target Cell' is '\$C\$16'. The 'Equal To' section has three radio buttons: 'Max' (selected), 'Min', and 'Value of:'. The 'Value of' field is set to '0'. The 'By Changing Cells' field is '\$C\$6:\$C\$15'. The 'Subject to the Constraints' list contains three constraints: '\$C\$6:\$C\$15 <= \$D\$6:\$D\$15', '\$C\$6:\$C\$15 = integer', and '\$G\$6:\$G\$11 = 0'. A light blue callout bubble points to the constraints list with the text 'Flow into and out of nodes must equal each other.' The dialog box has buttons for 'Solve', 'Close', 'Options', 'Add', 'Change', 'Delete', 'Reset All', and 'Help'.

Solver Parameters

Set Target Cell:

Equal To: ☒ Max ☐ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-
-

Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

The Excel solution is shown in [Exhibit 7.10](#). Notice that the

flows along each branch are shown in cells **C6:C15** and the total network flow is 15 in cell C16.

Exhibit 7.10.

[\[View full size image\]](#)

G11 = \$=C12+C13+C14+C15

1 Scott Tractor Company: Maximal Flow Problem

	Branch		Branch		Branch		Network	
	Nodes		Flow		Capacity		Node Flow	
6	1	2	4		8		1	0
7	1	3	7		7		2	0
8	1	4	4		4		3	0
9	2	4	0		3		4	0
10	2	5	4		8		5	0
11	3	4	1		2		6	0
12	3	6	8		8			
13	4	6	5		5			
14	5	6	4		4			
15	6	1	15		17			
16	Total		15					

Summary

In this chapter we examined a class of models referred to as network flow models. These included the shortest route network, the minimal spanning tree network, and the maximal flow network model. These network models are all concerned with the flow of an item (or items) through an arrangement of paths (or routes).

We demonstrated solution approaches for each of the three types of network models presented in this chapter. At times it may have seemed tiresome to go through the various steps of these solution methods, when the solutions could have more easily been found by simply looking closely at the networks. However, as the sizes of networks increase, intuitive solution by observation becomes more difficult, thus creating the need for a solution procedure. Of course, as with the other

techniques in this text, when a network gets extremely large and complex, computerized solution becomes the best approach.

[Page 293]

In the next chapter we will continue our discussion of networks by presenting the network analysis techniques known as CPM and PERT. These network techniques are used primarily for project analysis and are not only the most popular types of network

analysis but also two of the most widely applied management science techniques.

References

Ford, L. R., Jr., and Fulkerson, D. R. *Flows in Networks*. Princeton, NJ: Princeton University Press, 1962.

Hillier, F. S., and Lieberman, G. J. *Operations Research*, 4th ed. San Francisco: Holden-Day, 1986.

Hu, T. C. *Integer Programming and Network Flows*. Reading, MA: Addison-Wesley, 1969.

Trueman, R. E. *Quantitative Methods for Decision Making in Business*. Hinsdale, IL: The Dryden Press, 1981.

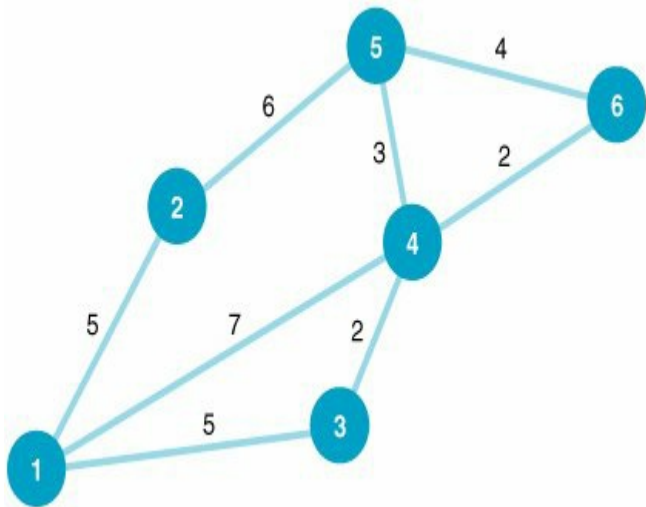
Example Problem Solution

The following example illustrates the solution methods for the shortest route and minimal spanning tree network flow problems.

Problem Statement

A salesman for Healthproof Pharmaceutical Company travels each week from his

office in Atlanta to one of five cities in the Southeast where he has clients. The travel time (in hours) between cities along interstate highways is shown along each branch in the following network:



- 1.** Determine the shortest route from Atlanta to each of the other five cities in the network.
- 2.** Assume that the network

now represents six different communities in a city and that the local transportation authority wants to design a rail system that will connect all six communities with the minimum amount of track. The miles between each community are shown on each branch. Develop a minimal spanning tree for this problem.

Solution

(Part A): Determine the Shortest Route Solution

[Page 294]

1.
Permanent Branch Time
Set

{1}

12

5

13

5

14 7

2.

Permanent Branch Time
Set

$\{1, 2\}$ 13 5

14 7

25 11

3. Branch Time
Permanent
Set

$\{1, 2, 3\}$ 14 7

25 11

34 7

4. Branch Time
Permanent
Step 1. Set

{1, 2, 3, 45 10
4}

46 9

5.
Permanent Branch Time
Set

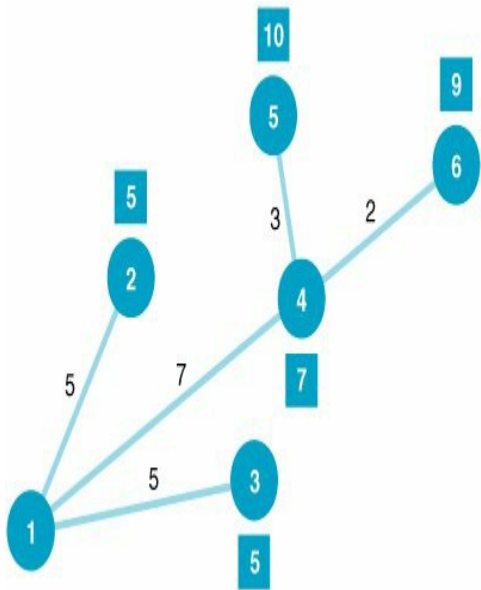
{1, 2, 3, 45 10
4, 6}

65 13

6. Permanent Set

$\{1, 2, 3,$
 $4, 5, 6\}$

The shortest route
network follows:



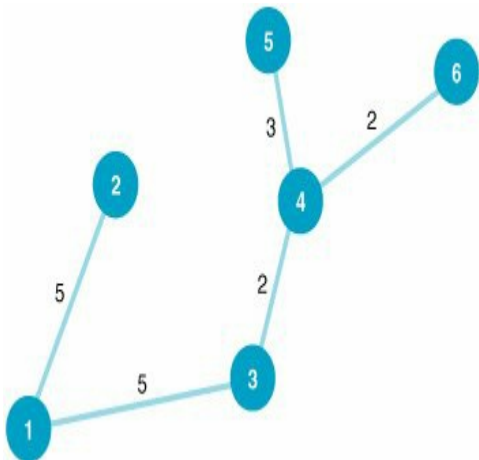
**(Part B): Determine
the Minimal
Spanning Tree**

- 1.** The closest unconnected node to node 1 is node 2.
- 2.** The closest unconnected node to 1 and 2 is node 3.
- 3.** The closest unconnected node to 1, 2, and 3 is node 4.
- 4.** The closest unconnected node

Step 2. to 1, 2, 3, and 4 is node 6.

5. The closest unconnected node to 1, 2, 3, 4, and 6 is node 5.

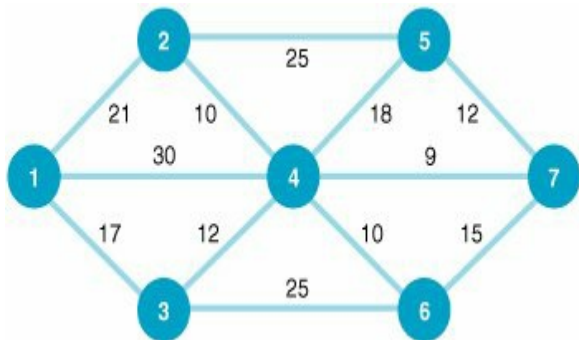
The minimal spanning tree follows; the shortest total distance is 17 miles:



Problems

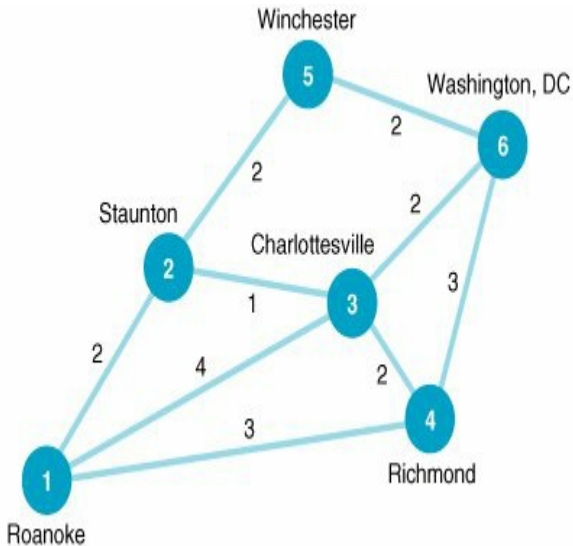
Given the following network with the indicated distances between nodes (in miles), determine the shortest route from node 1 to each of the other six nodes (2, 3, 4, 5, 6, and 7):

1.



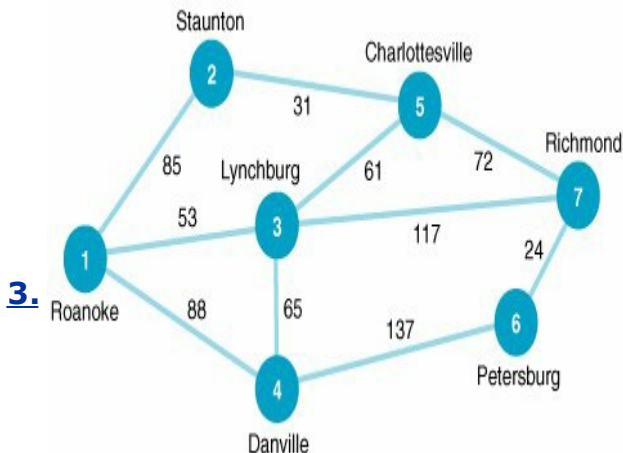
Frieda Millstone and her family live in Roanoke, Virginia, and they are planning an auto vacation across Virginia, their ultimate destination being Washington, DC. The family has developed the following network of possible routes and cities to visit on their trip:

2.



The time, in hours, between cities (which is affected by the type of road and the number of intermediate towns) is shown along each branch. Determine the shortest route that the Millstone family can travel from Roanoke to Washington, DC.

The Roanoke, Virginia, distributor of Rainwater Beer delivers beer by truck to stores in six other Virginia cities, as shown in the following network:



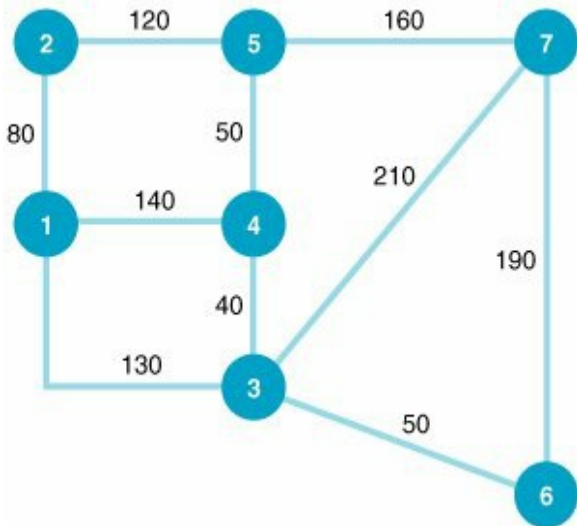
[Page 296]

The mileage between cities is shown

along each branch. Determine the shortest truck route from Roanoke to each of the other six cities in the network.

The plant engineer for the Bitco manufacturing plant is designing an overhead conveyor system that will connect the distribution/inventory center to all areas of the plant. The network of possible conveyor routes through the plant, with the length (in feet) along each branch, follows:

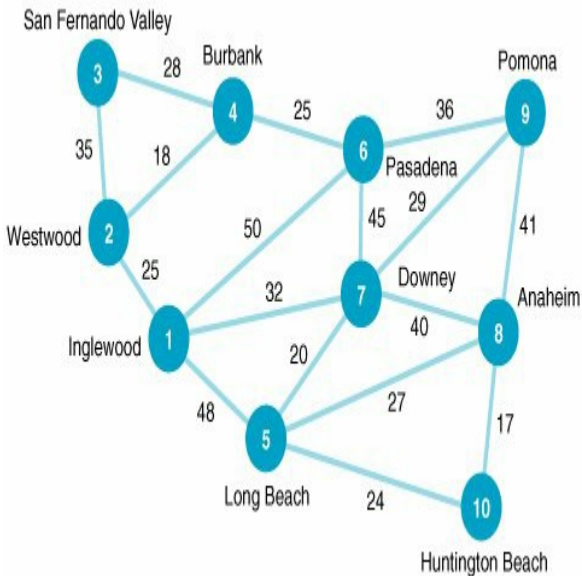
4.



Determine the shortest conveyor route from the distribution/inventory center at node 1 to each of the other six areas of the plant.

The Burger Doodle restaurant franchises in Los Angeles are supplied from a central warehouse in Inglewood. The location of the warehouse and its proximity, in minutes of travel time, to the franchises are shown in the following network:

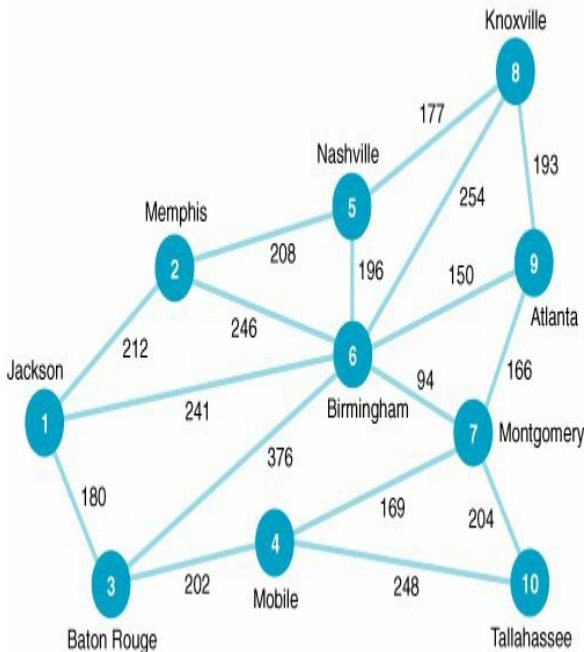
5.



Trucks supply each franchise on a daily basis. Determine the shortest route from the warehouse at Inglewood to each of the nine franchises.

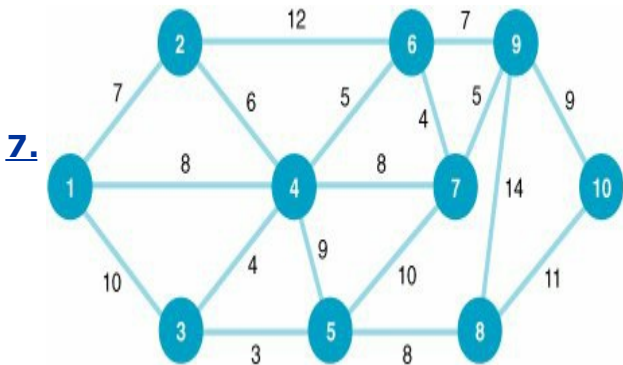
The Petroco gasoline distributor in Jackson, Mississippi, supplies service stations in nine other southeastern cities, as shown in the following network:

6.



The distance, in miles, is shown on each branch. Determine the shortest route from Jackson to the nine other cities in the network.

The Hylton Hotel has a limousine van that transports guests to various business and tourist locations around the city. The following network indicates the different routes the limousine could follow from the hotel at node 1 to the nine locations (nodes 2 through 10):



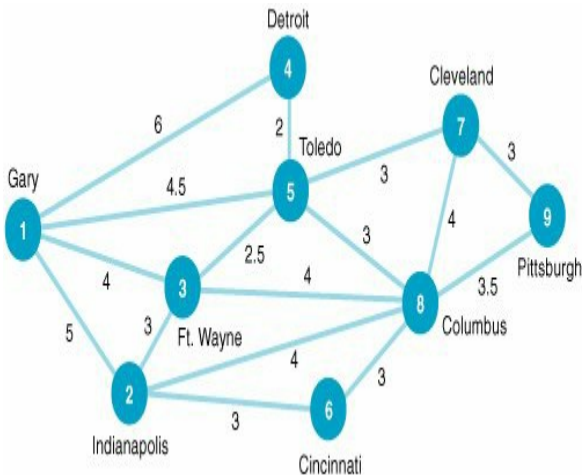
The values on each branch in the network are the distances, in miles, between the locations. Determine the

shortest route from the hotel to each of the nine destinations and indicate the distance for each route.

[Page 298]

A steel mill in Gary supplies steel to manufacturers in eight other midwestern cities via truck, as shown in the following network:

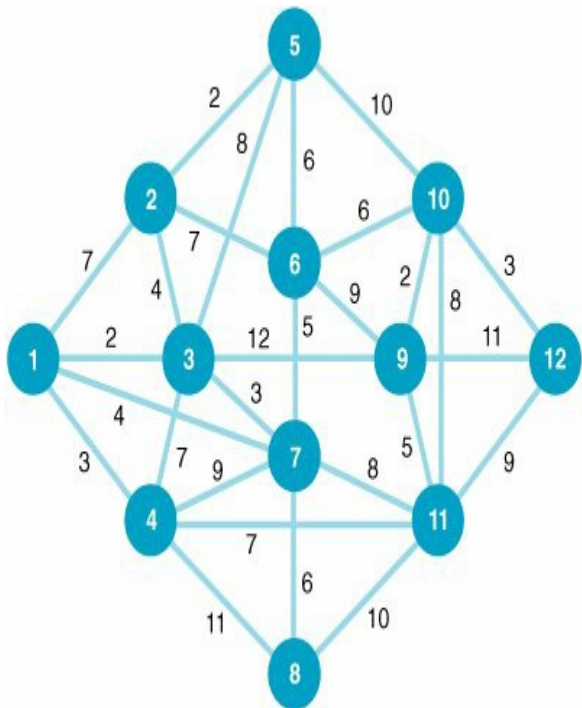
8.



The travel time between cities, in hours, is shown along each branch. Determine the shortest route from Gary to each of the other eight cities in the network.

Determine the shortest route from node 1 (origin) to node 12

(destination) for the following network. Distances are given along the network branches:



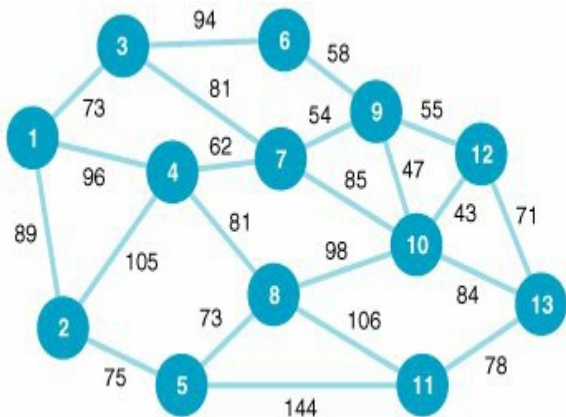
9.

The members of the Vistas Club are planning a month-long hiking and camping trip through the Blue Ridge Mountains, from north Georgia to Virginia. There are a number of trails

through the mountains that connect at various campgrounds, crossings, and cabins. The following network shows the different connecting trails and the distance of each connecting branch, in miles:

[Page 299]

10.

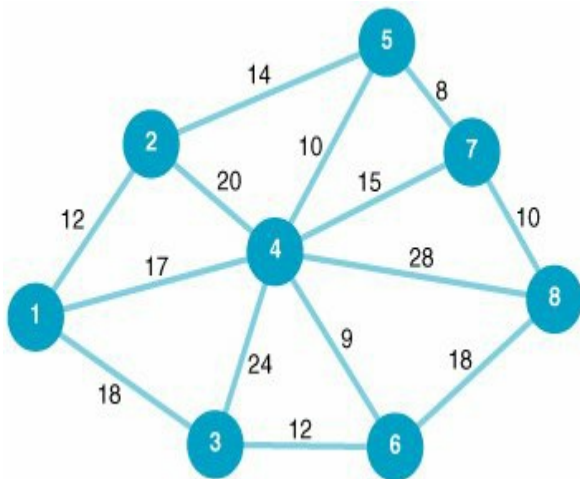


Determine the shortest path from

node 1 in north Georgia to node 13 in Virginia and indicate the total mileage of the trail.

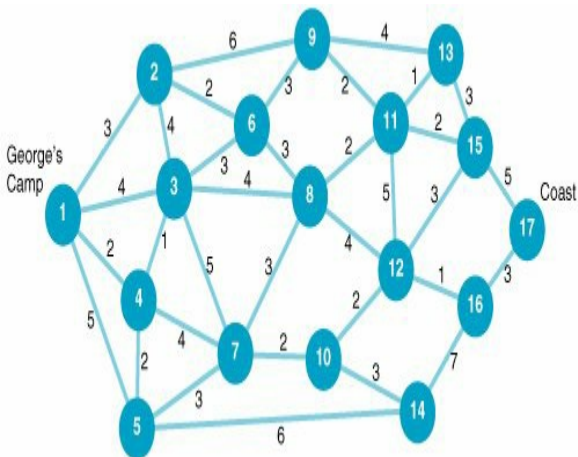
Hard Rock Concrete Supply makes concrete at its plant in Centerville, Virginia, and delivers it to construction sites throughout the metropolitan Washington, DC, area. The following network shows the possible routes and distances (in miles) from the concrete plant to seven construction sites:

11.



Determine the shortest route a concrete truck would take from the plant at node 1 to node 8 and the total distance for this route, using the shortest route method.

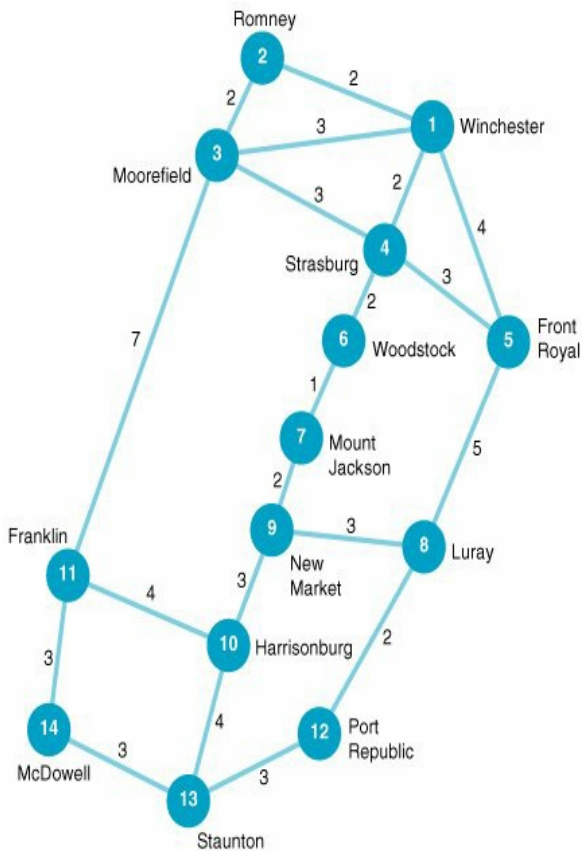
George is camped deep in the jungle, and he wants to make his way back to the coast and civilization. Each of the paths he can take through the jungle has obstacles that can delay him, including hostile natives, wild animals, dense forests and vegetation, swamps, rivers and streams, snakes, insects, and mountains. Following is the network of paths that George can take, with the time (in days) to travel each branch:



Determine the shortest (time) path for George to take and indicate the total time.

In 1862, during the second year of the Civil War, General Thomas J. "Stonewall" Jackson fought a brilliant military campaign in the Shenandoah Valley in Virginia. One of his victories

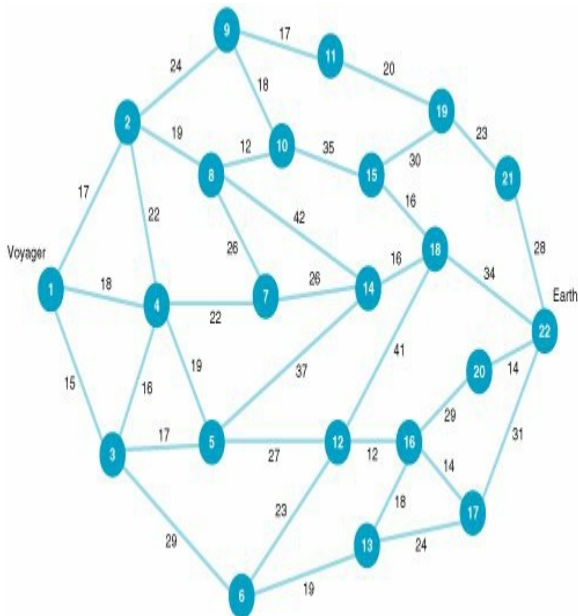
was at the Battle of McDowell. Using the following figure and your imagination, determine the shortest path and how long it will take (in days) for General Jackson to move his army from Winchester to McDowell to fight the battle:



The *Voyager* spacecraft has been transported by an alien being to the Delta Quadrant of space, millions of light-years from Earth. Captain Janeway and her crew are attempting to plot the shortest course home. Following is a network of the possible routes through the Delta Quadrant, where the nodes are different worlds, planetary systems, and space anomalies, and the values on the branches are the actual times, in

14. years, between nodes:

[\[View full size image\]](#)



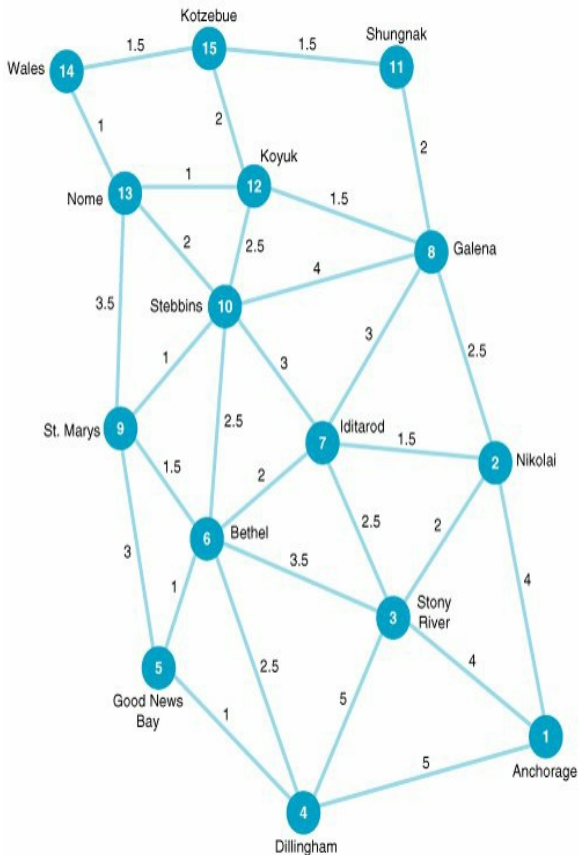
Determine the shortest route from the *Voyager's* present location (at node 1) to Earth.

John Clooney, a bush pilot in Alaska, makes regular charter flights in his floatplane to various towns and cities in western Alaska. His passengers include hunters, fishermen, backpackers and campers, and tradespeople hired for jobs in the different localities. He also carries some cargo for delivery. The following network shows the possible air routes between various towns and cities John might take (with the times, in hours):

[Page 302]

[\[View full size image\]](#)

15.



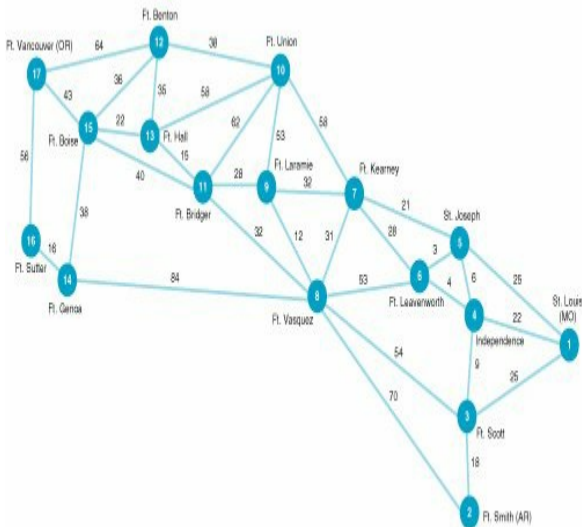
For safety reasons, he flies point-to-point, at least flying over a town along a route, even though he might not land there. In the upcoming week John has scheduled charter flights for Kotzebue, Nome, and Stebbins. Determine the shortest route between John's home base in Anchorage and each of these destinations.

From 1840 to 1850, more than 12,000 pioneers migrated west in wagon trains. It was typically about a 2,000-mile journey, and pioneers averaged about 10 miles per day. One pioneer family, the Smiths, is planning to join a wagon train traveling west to Oregon. The destination is Fort Vancouver, which is near present-day Portland. The family plans to join a

wagon train in St. Louis. However, the trains follow various trails west that are mostly determined by the location of forts and trading posts along the way. The Smith family wants to choose a wagon train that will get them to Oregon in the shortest amount of time. They have checked around with the different wagon train leaders plus immigrants, soldiers, fur traders, and scouts who have previously made the trip west, and from the information they have gathered, they have developed the following network, with estimated times (in days) along each branch:

[Page 303]

16. [\[View full size image\]](#)



[Page 304]

1. Determine the shortest route for the Smiths from St. Louis to Ft. Vancouver.
2. Another family, the Steins, is

leaving from Ft. Smith, Arkansas. Determine the shortest route for the Steins to Ft. Vancouver, Oregon.

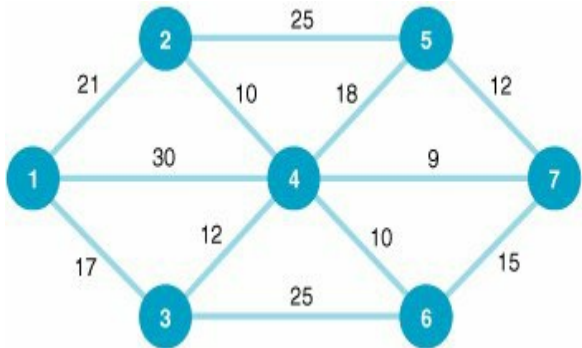
A new police car costs the Bay City Police Department \$26,000. The annual maintenance cost for a car depends on the age of the car at the beginning of the year. (All cars accumulate approximately the same mileage each year.) The maintenance costs increase as a car ages, as shown in the following table:

	Car Age (yr.)	Annual Maintenance Cost	Used Selling Price
	0	\$ 3,000	\$
	1	4,500	15,000
	2	6,000	12,000
	3	8,000	8,000
<u>17.</u>	4	11,000	4,000
	5	14,000	2,000
	6		0

In order to avoid the increasingly high maintenance costs, the police department can sell a car and purchase a new car at the end of any year. The selling price for a car at the end of each year of use is also shown in the table. It is assumed that the price of a new car will increase \$500 each year. The department's objective is to develop a car replacement schedule that will minimize total cost (i.e., the purchase cost of a new car plus maintenance costs minus the money received for selling a used car) during the next 6 years. Develop a car replacement schedule, using the shortest route method.

Given the following network, with the indicated distances between nodes, develop a minimal spanning tree:

18.



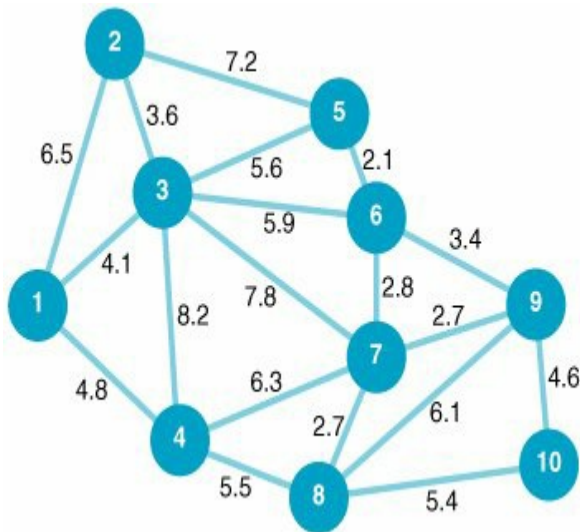
A developer is planning a development that includes subdivisions of houses, cluster houses, town-houses, apartment complexes, shopping areas, a daycare center and playground, a community center, and a school, among other facilities. The

developer wants to connect the facilities and areas in the development with the minimum number of streets possible. The following network shows the possible street routes and distances (in thousands of feet) between 10 areas in the development that must be connected:



[Page 305]

19.

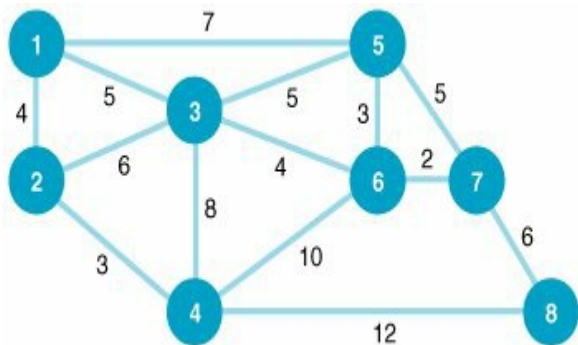


Determine a minimal spanning tree network of streets to connect the 10 areas, and indicate the total streets (in thousands of feet) needed.

One of the opposing forces in a simulated army battle wishes to set up

a communications system that will connect the eight camps in its command. The following network indicates the distances (in hundreds of yards) between the camps and the different paths over which a communications line can be constructed:

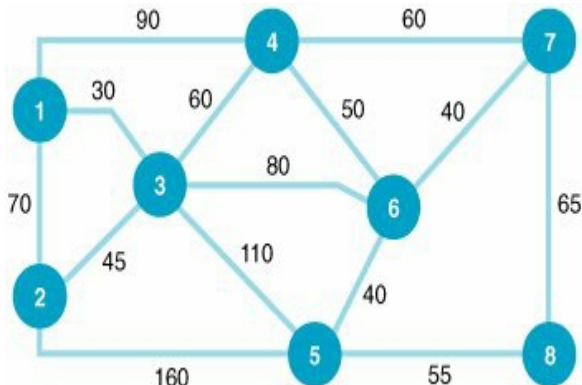
20.



Using the minimal spanning tree approach, determine the minimum distance communication system that will connect all eight camps.

The management of the Dynaco manufacturing plant wants to connect the eight major manufacturing areas of its plant with a forklift route. Because the construction of such a route will take a considerable amount of plant space and disrupt normal activities, management wants to minimize the total length of the route. The following network shows the distance, in yards, between the manufacturing areas (denoted by nodes 1 through 8):

21.



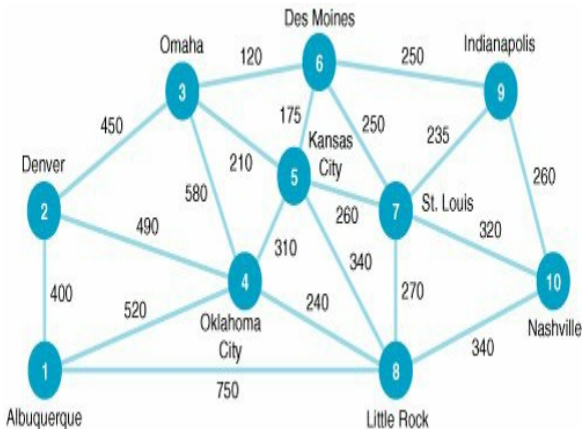
[Page 306]

Determine the minimal spanning tree forklift route for the plant and indicate the total yards the route will require.

Several oil companies are jointly planning to build an oil pipeline to connect several southwestern, southeastern, and midwestern cities, as shown in the following network:

[\[View full size image\]](#)

22.

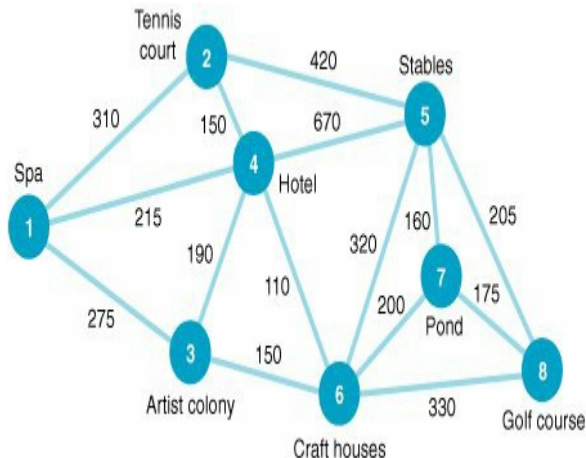


The miles between cities are shown on each branch. Determine a pipeline system that will connect all 10 cities, using the minimum number of miles of pipe, and indicate how many miles of pipe will be used.

A major hotel chain is constructing a new resort hotel complex in

Greenbranch Springs, West Virginia. The resort is in a heavily wooded area, and the developers want to preserve as much of the natural beauty as possible. To do so, the developers want to connect all the various facilities in the complex with a combination walking/riding path that will minimize the amount of pathway that will have to be cut through the woods. The following network shows possible connecting paths and corresponding distances (in yards) between the facilities:

23.



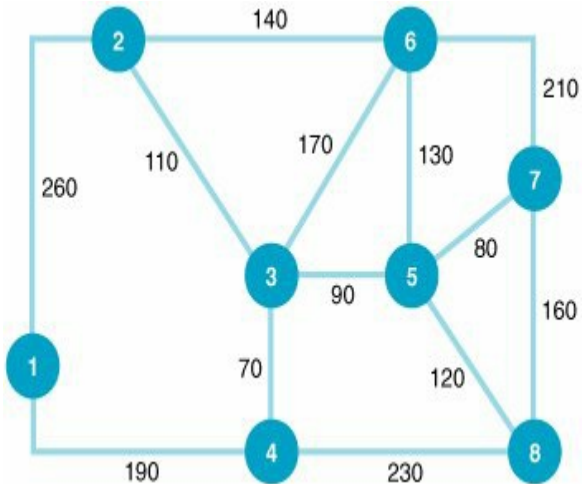
Determine the path that will connect all the facilities with the minimum amount of construction and indicate the total length of the pathway.

[Page 307]

The Barrett Textile Mill is remodeling its plant and installing a new ventilation

system. The possible ducts connecting the different rooms and buildings at the plant, with the length (in feet) along each branch, are shown in the following network:

24.

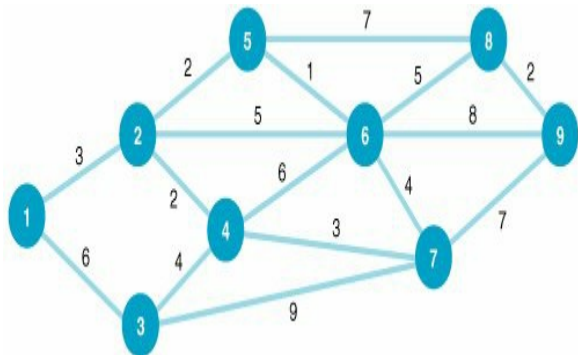


Determine the ventilation system that will connect the various rooms and buildings of the plant, using the

minimum number of feet of ductwork, and indicate how many feet of ductwork will be used.

The town council of Whitesville has decided to construct a bicycle path that will connect the various suburbs of the town with the shopping center, the downtown area, and the local college. The council hopes the local citizenry will use the bike path, thus conserving energy and decreasing traffic congestion. The various paths that can be constructed and their lengths (in miles) are shown in the following network:

25.



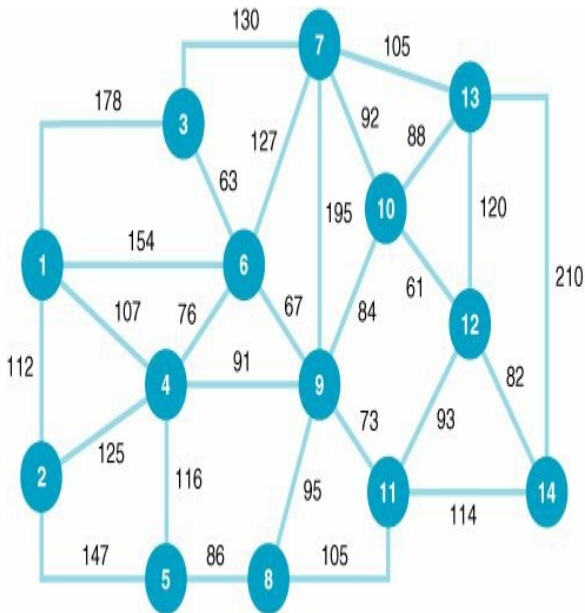
Determine the bicycle pathway that will require the minimum amount of construction to connect all the areas of the town. Indicate the total length of the path.

- 26.** Determine the minimal spanning tree for the network in [Problem 9](#).

State University has decided to

reconstruct the sidewalks throughout the east side of its campus to provide wheelchair access. However, upgrading sidewalks is a very expensive undertaking, so for the first phase of this project, university administrators want to make sure they connect all buildings with wheelchair access with the minimum number of refurbished sidewalks possible. Following is a network of the existing sidewalks on the east side of campus, with the feet between each building shown on the branches:

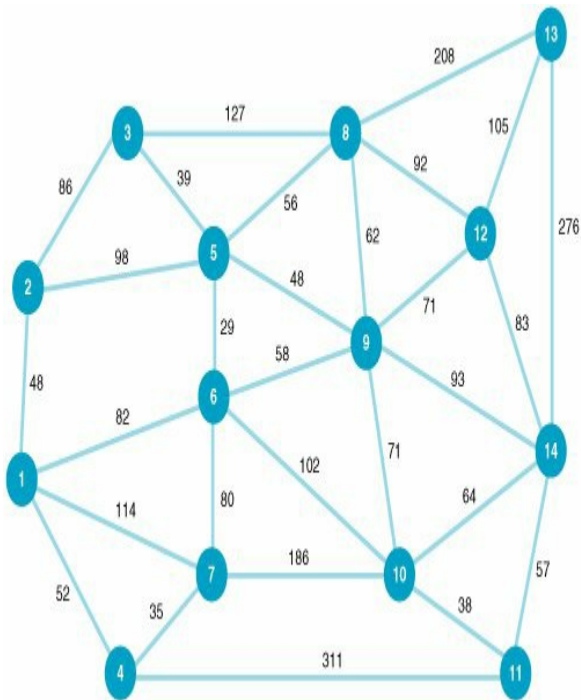
27.



Determine a minimal spanning tree network that will connect all the buildings on campus with wheelchair access sidewalks and indicate the number of feet of sidewalk.

Tech wants to develop an area network that will connect its server at its computer and satellite center with the main campus buildings to improve Internet service. The cable will be laid primarily through existing electrical tunnels, although some cable will have to be buried underground. The following network shows the possible cable connections between the computer center at node 1 and the various buildings, with the distances, in feet, along the branches:

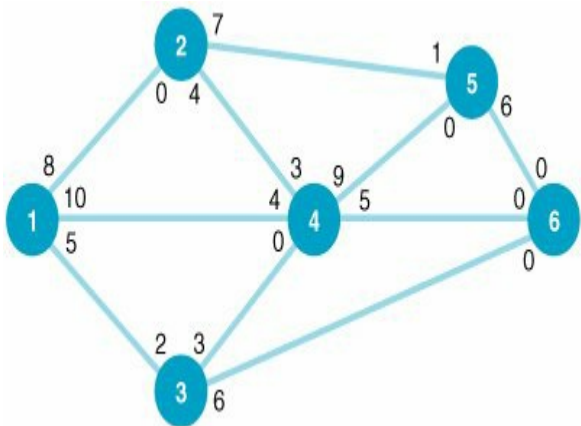
[\[View full size image\]](#)



Determine a minimal spanning tree network that will connect all the buildings and indicate the total amount of cable that will be needed to do so.

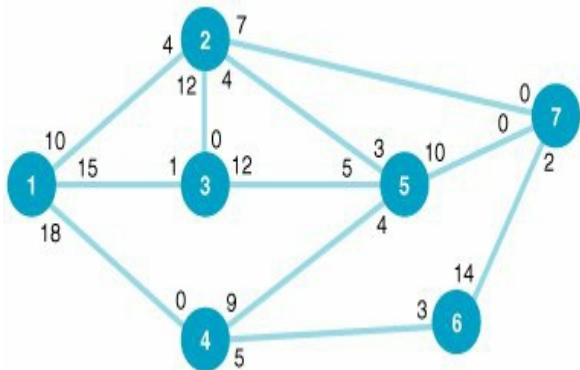
Given the following network, with the indicated flow capacities of each branch, determine the maximum flow from source node 1 to destination node 6 and the flow along each branch:

29.



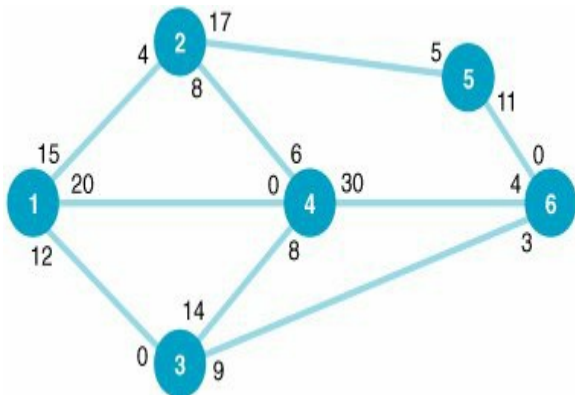
Given the following network, with the indicated flow capacities along each branch, determine the maximum flow from source node 1 to destination node 7 and the flow along each

30. branch:



Given the following network, with the indicated flow capacities along each branch, determine the maximum flow from source node 1 to destination node 6 and the flow along each branch:

31.

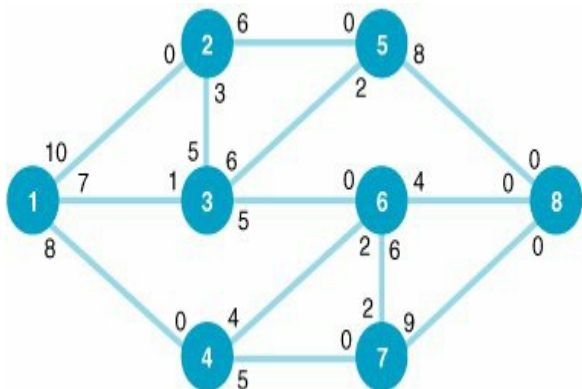


[Page 310]

A new stadium complex is being planned for Denver, and the Denver traffic engineer is attempting to determine whether the city streets between the stadium complex and the interstate highway can accommodate the expected flow of 21,000 cars after each game. The various traffic arteries

between the stadium (node 1) and the interstate (node 8) are shown in the following network:

32.

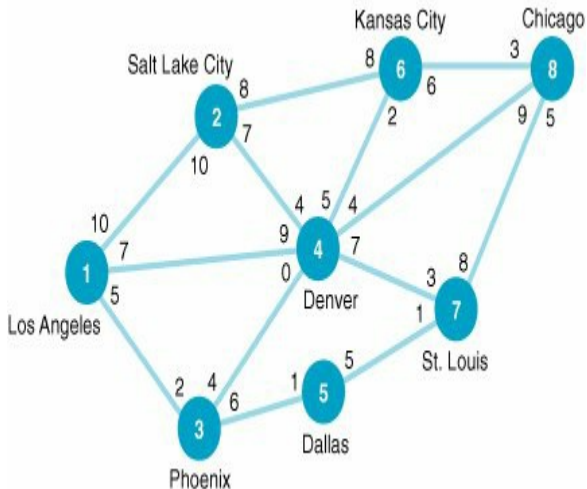


The flow capacities on each street are determined by the number of available lanes, the use of traffic police and lights, and whether any lanes can be opened or closed in either direction. The flow capacities are given in thousands of cars. Determine the maximum traffic flow the streets can

accommodate and the amount of traffic along each street. Will the streets be able to handle the expected flow after a game?

The FAA has granted a license to a new airline, Omniair, and awarded it several routes between Los Angeles and Chicago. The flights per day for each route are shown in the following network:

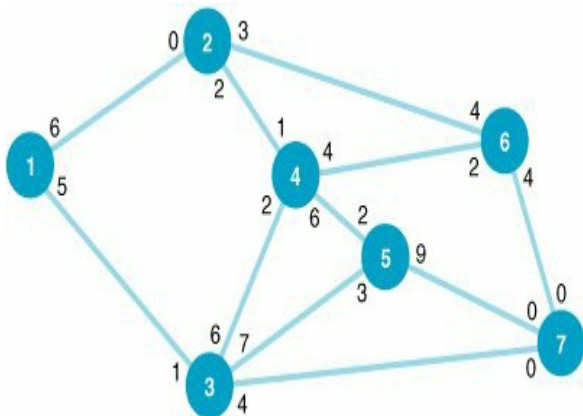
33.



Determine the maximum number of flights the airline can schedule per day from Chicago to Los Angeles and indicate the number of flights along each route.

The National Express Parcel Service has established various truck and air

34. routes around the country over which it ships parcels. The holiday season is approaching, which means a dramatic increase in the number of packages that will be sent. The service wants to know the maximum flow of packages it can accommodate (in tons) from station 1 to station 7. The network of routes and the flow capacities (in tons of packages per day) along each route are shown in the following network:

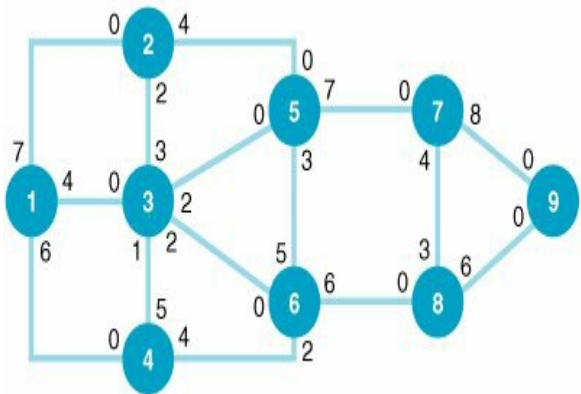


Determine the maximum tonnage of packages that can be transported per day from station 1 to station 7 and indicate the flow along each branch.

The traffic management office in Richmond is attempting to analyze the potential traffic flow from a new office complex under construction to an interstate highway interchange during

the evening rush period. Cars leave the office complex via one of three exits, and then they travel through the city streets until they arrive at the interstate interchange. The following network shows the various street routes (branches) from the office complex (node 1) to the interstate interchange (node 9):

35.



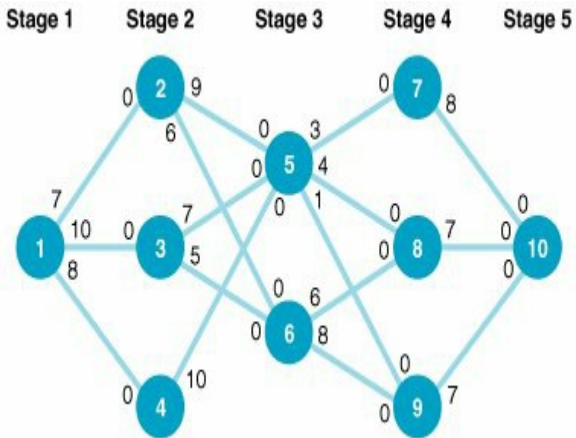
All intermediate nodes represent street intersections, and the values

accompanying the branches emanating from the nodes represent the traffic capacities of each street, expressed in thousands of cars per hour. Determine the maximum flow of cars that the street system can absorb during the evening rush hour.

The Dynaco Company manufactures a product in five stages. Each stage of the manufacturing process is conducted at a different plant. The following network shows the five different stages and the routes over which the partially completed products

are shipped to the various plants at the different stages:

[Page 312]



36.

Stage 5 (at node 10) is the distribution center in which final products are stored. Although each node represents a different plant, plants at the same stage perform the

same operation. (For example, at stage 2 of the manufacturing process, plants 2, 3, and 4 all perform the same manufacturing operation.) The values accompanying the branches emanating from each node indicate the maximum number of units (in thousands) that a particular plant can produce and ship to another plant at the next stage. (For example, plant 3 has the capacity to process and ship 7,000 units of the product to plant 5.) Determine the maximum number of units that can be processed through the five-stage manufacturing process and the number of units processed at each plant.

Suppose in [Problem 36](#) that the processing cost per unit at each plant is different because of different machinery, workers' abilities, overhead, and so on, according to the following table:

Stage 1	Stage 2	Stage 3	Stage 4
---------	---------	---------	---------

1. \$3	2.\$5	5.\$22	7.\$12
--------	-------	--------	--------

37.

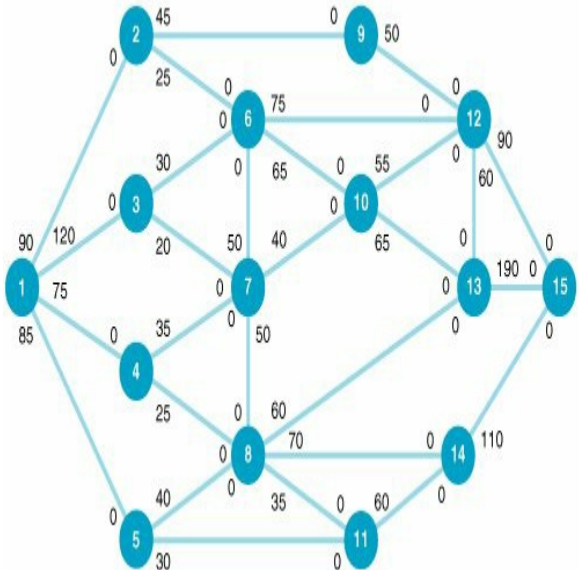
3.7	6.19	8.14
-----	------	------

4.4	9.16
-----	------

There are no differentiated costs at stage 5, the distribution center. Given a budget of \$700,000, determine the maximum number of units that can be processed through the five-stage manufacturing process.

Given the following network, with the indicated flow capacities along each branch, determine the maximum flow from source node 1 to destination node 10 and the flow along each path:

showing the various work centers in the plant, the daily capacities at each work center, and the flow of the partially completed products between work centers:



39.

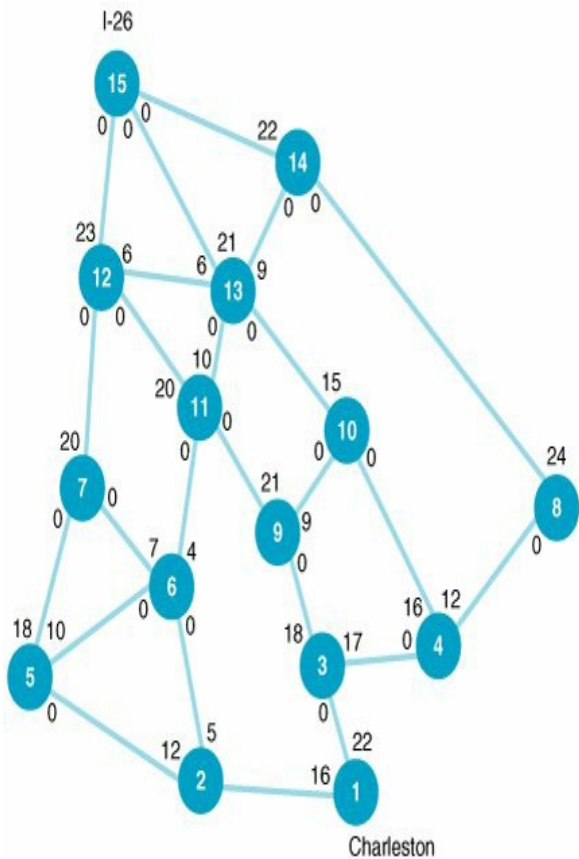
Node 1 represents the point where

raw materials enter the process, and node 15 is the packaging and distribution center. Determine the maximum number of units that can be completed each day and the number of units processed at each work center.

The following network shows the major roads that would be part of the

hurricane evacuation routes for
Charleston, South Carolina:

40.



The destination is the interchange of Interstate 26 and Interstate 526, north of Charleston. The nodes represent road intersections, and the values at the nodes are the traffic capacities, in thousands of cars. Determine the maximum flow of cars that can leave Charleston during a hurricane evacuation and the optimal flow along each road segment (branch 1).

In [Problem 41](#) in [Chapter 6](#), KanTech Corporation ships electronic components in containers from seaports in Europe to U.S. ports, from which the containers are transported by truck or rail to inland ports and then transferred to an alternative

mode of transportation before being shipped to KanTech's U.S. distributors. Using the shipping capacity at each port, develop a maximal flow network to determine the number of containers to ship through each port to meet demand at the distribution centers.

Case Problem

THE PEARLSBURG RESCUE SQUAD

The Pearlsburg (West Virginia) Rescue Squad serves a mountainous, rural area in southern West Virginia. The only access to the homes, farms, and small crossroad communities and villages is a network of dirt, gravel, and poorly paved roads. The rescue squad had just returned from an emergency at Blake's

Crossing, and two of the squad members, Melanie Hart and Ben Cross, were cleaning up and getting the truck back in proper order for the next call.

"You know, Ben," said Melanie, "that was close. We could have lost little Randy if we had been a few minutes later in getting to their farm."

"Yep," said Ben.

After a moment Melanie continued, "I was sort of wondering about the route Dave took to get there. It

seems to me if we had gone around by Cedar Creek, we could have gotten there a few minutes sooner. And as close as things got, a few minutes could have made a big difference, don't you think?"

"Yep," said Ben.

"Well, I was wondering, Ben, why don't we study all these different ways to get to the little communities and farms around here so we will always know what the quickest way to all the different places is?"

Ben thought a moment before answering. "It would take us a while to time all the different ways you could get from here to everywhere we go."

"That's true," Melanie answered, "but I've been studying a way to work out this sort of problem in one of my college courses. All we have to do is get the times it takes to travel each piece of road between all these little communities, and I think I can do the rest. Will you help me?"

"Sure," Ben said.

"Okay, then, here's what we'll do. I'll write down all the routes you should time, and I'll time the rest all the way over to Holbrook."

Here is the combined list of times (in minutes) Melanie and Ben compiled for all the route segments between Pearlsburg and Holbrook:

[Page 315]

Segment

Time

Pearlsburg to Kitchen Corner	10
Pearlsburg to Quarry	15
Pearlsburg to Morgan Creek	12
Kitchen Corner to Cutter's Store	20
Kitchen Corner to Stone House	14
Kitchen Corner to Quarry	8
Quarry to Blake's	

Crossing	18
----------	----

Quarry to Cedar Creek	9
--------------------------	---

Morgan Creek to Quarry	16
---------------------------	----

Morgan Creek to Cedar Creek	7
--------------------------------	---

Morgan Creek to Homer	18
--------------------------	----

Morgan Creek to McKinney Farm	11
----------------------------------	----

Stone House to Cutter's Store	10
Stone House to Blake's Crossing	6
Cedar Creek to Blake's Crossing	10
Cedar Creek to Wellis Farm	17
Cedar Creek to Homer	5
Cutter's Store to Blake's Crossing	12

Cutter's Store to Bottom Town	14
Blake's Crossing to Bottom Town	6
Blake's Crossing to Holbrook	15
Blake's Crossing to Wellis Farm	9
Homer to Wellis Farm	11
Homer to McKinney Farm	8
McKinney Farm to	

Wellis Farm	21
-------------	----

Wellis Farm to Holbrook	10
----------------------------	----

Bottom Town to Holbrook	12
----------------------------	----

Determine the shortest routes from Pearlsburg to all the different communities and farms visited by the rescue squad.

Case Problem

AROUND THE WORLD IN 80 DAYS

In the novel *Around the World in 80 Days* by Jules Verne, Phileas T. Fogg wagered four of his fellow members of the Reform Club in London £5,000 apiece that he could travel around the world in 80 days. This would have been an astounding feat in 1872, the year in which the novel is set. Then, the fastest modes of

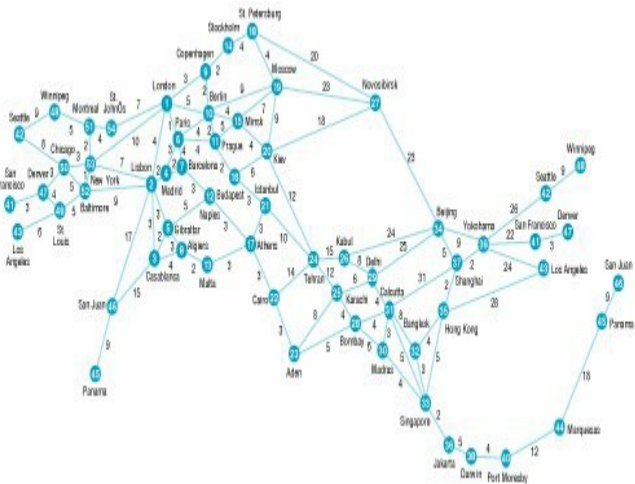
travel were rail and ship; however, much of the world still traveled by coach, wagon, horse, elephant, or donkey, or on foot. Phileas Fogg was not a frivolous man. He would not have undertaken such a large wager if he had not carefully researched the feasibility of such a trip and been confident of his chances. Although he undoubtedly analyzed various routes to circumnavigate the globe, he did not have knowledge of techniques such as the shortest route method, nor did he have a computer to

help him select an optimal route. If he had, he might have chosen a route that would have completed his trip in less than 80 days or possibly would not have been so quick to make his wager. On the next page is a network of the various routes of the day for traveling around the world from London. Fogg traveled eastward. The travel time, in days, is shown on each branch. Travel times are based as much on available modes of transportation at the time as on distance. Using the shortest route method (and the

computer), select the best route for Mr. Fogg. Would the route you determined have won him his wager?

[Page 316]

[\[View full size image\]](#)



Case Problem

THE BATTLE OF THE BULGE

On December 16, 1944, in the last year of World War II, two German panzer armies, supported by a third army of infantry, together totaling more than 250,000 men, staged a massive counteroffensive in northern France, overwhelming the American First Army in the Ardennes. The offensive emanated from the German

defensive line along the Our River, north of the city of Luxembourg, and was directed almost due west toward Namur and Liège in Belgium. The result, after several days of fighting, was a huge "bulge" in the Allied line and, therefore, this last major ground battle of World War II became known as the Battle of the Bulge.

On December 20, General Dwight D. Eisenhower, Supreme Allied Commander, called on General George Patton to attack the German offensive with his Third Army, which was

then situated near Verdun, approximately 100 miles due south of the German left flank. Patton's immediate objective was to relieve the 101st Airborne and elements of Patton's own 9th and 10th Armored Divisions surrounded at Bastogne. Within 48 hours, on December 22, Patton was able to begin his counteroffensive, with three divisions totaling approximately 62,000 men.

The winter weather was cold with snow and fog, and the roads were icy, making the

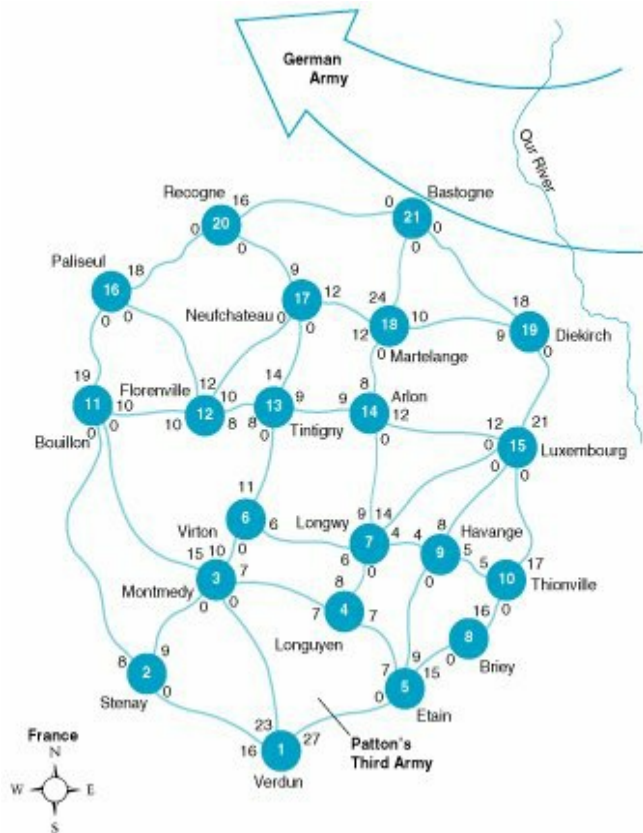
movement of troops, tanks, supplies, and equipment a logistical nightmare.

Nevertheless, on December 26 Bastogne was relieved, and on January 12, 1945, the Battle of the Bulge effectively ended in one of the great Allied victories of the war.

General Patton's staff did not have knowledge of the maximal flow technique nor access to computers to help plan the Third Army's troop movements during the Battle of the Bulge. However, the following figure shows the road network

between Verdun and Bastogne, with (imagined) troop capacities (in thousands) along each road branch between towns. Using the maximal flow technique (and your imagination), determine the number of troops that should be sent along each road in order to get the maximum number of troops to Bastogne. Also indicate the total number of troops that would be able to get to Bastogne.

[\[View full size image\]](#)



Case Problem

NUCLEAR WASTE DISPOSAL AT PAWV POWER AND LIGHT

PAWV Power and Light has contracted with a waste disposal firm to have nuclear waste from its nuclear power plants in Pennsylvania disposed of at a government-operated nuclear waste disposal site in Nevada. The waste must be shipped in reinforced container trucks across the country, and

all travel must be confined to the interstate highway system. The government insists that the waste transport must be completed within 42 hours and that the trucks travel through the least populated areas possible. The following network shows the various interstate segments the trucks might use from Pittsburgh to the Nevada waste site and the travel time (in hours) estimated for each road segment.

The approximate population (in millions) for the metropolitan areas the trucks might travel

through are as follows:

City	Population (1,000,000s)	City
Akron	0.50	Las Vegas
Albuquerque	1.00	Lexington
Amarillo	0.30	Little Rock
Charleston	1.30	Louisville
Cheyenne	0.16	Memphis
Chicago	10.00	Nashville

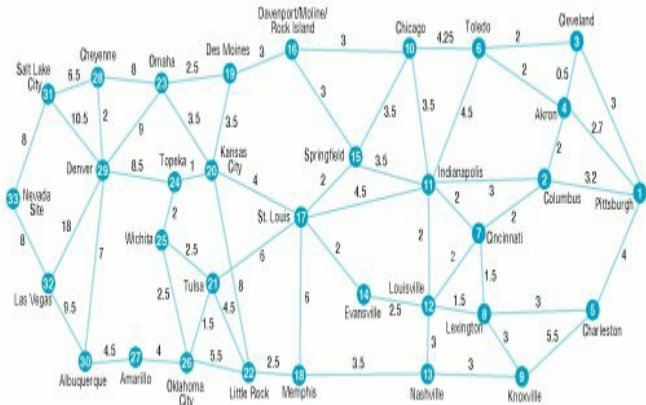
Cincinnati	1.20	Oklahoma City
Cleveland	1.80	Omaha
Columbus	0.75	Salt Lake City
Davenport/Moline/		Springfield
Rock Island	1.00	St. Louis
Denver	2.20	Toledo
Des Moines	0.56	Topeka

Evansville	0.30	Tulsa
Indianapolis	1.60	Wichita
Kansas City	2.10	
Knoxville	0.54	

Determine the optimal route the trucks should take from Pittsburgh to the Nevada site to complete the trip within 42 hours and expose the trucks to the least number of people

possible.

[\[View full size image\]](#)



Chapter 8. Project Management

[Page 321]

One of the most popular uses of networks is for project analysis. Such projects as the construction of a building, the development of a drug, or the installation of a computer system can be represented as networks. These networks illustrate the way in which the parts of the project are organized, and they can be

used to determine the time duration of the projects. The network techniques that are used for project analysis are CPM and PERT. CPM stands for *critical path method*, and PERT is an acronym for *project evaluation and review technique*. These two techniques are very similar.

There were originally two primary differences between CPM and PERT. With CPM, a single, or deterministic, estimate for activity time was used, whereas with PERT probabilistic time estimates

were employed. The other difference was related to the mechanics of drawing the project network. In PERT, activities were represented as arcs, or arrowed lines, between two nodes, or circles, whereas in CPM, activities were represented as the nodes or circles. However, these were minor differences, and over time CPM and PERT have been effectively merged into a single technique, conventionally referred to as simply CPM/PERT.

CPM and PERT were developed at approximately the same time

(although independently) during the late 1950s. The fact that they have already been so frequently and widely applied attests to their value as management science techniques.

The Elements of Project Management

Management is generally perceived to be concerned with the planning, organization, and control of an ongoing process or activity such as the production of a product or delivery of a service. *Project* management is different in that it reflects a commitment of resources and people to a typically important activity for a relatively short time frame,

after which the management effort is dissolved. Projects do not have the continuity of supervision that is typical in the management of a production process. As such, the features and characteristics of project management tend to be somewhat unique.

[Figure 8.1](#) provides an overview of project management, which encompasses three major processes: planning, scheduling, and control. It also includes a number of the more prominent elements of these processes. In

the remainder of this section we will discuss some of the features of the project planning process.

Figure 8.1. The project management process

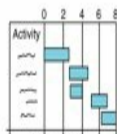
(This item is displayed on page 322 in the print version)

[\[View full size image\]](#)

Planning



Scheduling



Gantt chart



CPM/PERT

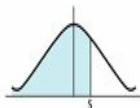


Resources

Control

September						
S	M	T	W	T	F	S
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

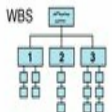
Stay on
schedule



Earned value analysis (EVA)



Cost control



OBS	WBS			
	1.1	1.2	1.3	1.4
Activity 1	✓		✓	
Activity 2		✓		✓
Activity 3			✓	
Activity 4				✓

RAM

Project Planning

Project plans generally include the following basic elements:

- *Objectives* A detailed statement of what is to be accomplished by the project, how it will achieve the company's goals and meets the strategic plan, and an estimate of when it needs to be completed, the cost, and the return.
- *Project scope* A discussion

of how to approach the project, the technological and resource feasibility, the major tasks involved, and a preliminary schedule; it includes a justification of the project and what constitutes project success.

- *Contract requirements* A general structure of managerial, reporting, and performance responsibilities, including a detailed list of staff, suppliers, subcontractors, managerial requirements

and agreements, reporting requirements, and a projected organizational structure.

- *Schedules* A list of all major events, tasks, and subschedules, from which a master schedule is developed.
 - *Resources* The overall project budget for all resource requirements and procedures for budgetary control.
-

- *Personnel* Identification and recruitment of personnel required for the project team, including special skills and training.
- *Control* Procedures for monitoring and evaluating progress and performance, including schedules and cost.
- *Risk and problem analysis*
Anticipation and assessment of

uncertainties, problems, and potential difficulties that might increase the risk of project delays and/or failure and threaten project success.

The Project Team

A project team typically consists of a group of individuals selected from other areas in the organization, or from consultants outside the organization, because of their special skills, expertise, and

experience related to the project activities. Members of the engineering staff, particularly industrial engineering, are often assigned to project work because of their technical skills. The project team may also include various managers and staff personnel from specific areas related to the project. Even workers can be involved on the project team if their jobs are a function of the project activity. For example, a project team for the construction of a new loading dock facility at a plant might

logically include truck drivers; forklift operators; dock workers; and staff personnel and managers from purchasing, shipping, receiving, and packaging; as well as engineers to assess vehicle flow, routes, and space considerations.

Project teams are made up of individuals from various areas and departments within a company.

Assignment to a project team is usually temporary and thus can have both positive and negative repercussions. The temporary loss of workers and staff from their permanent jobs can be disruptive for both the employees and the work area. An employee must sometimes "serve two masters," in a sense, reporting to both the project manager and a regular supervisor. Alternatively,

because projects are usually "exciting," they provide an opportunity to do work that is new and innovative, and the employee may be reluctant to report back to a more mundane, regular job after the project is completed.

The most important member of a project team is the *project manager*. The job of managing a project is subject to a great deal of uncertainty and the distinct possibility of failure. Because each project is unique and usually has not been attempted previously, the

outcome is not as certain as the outcome of an ongoing process would be. A degree of security is attained in the supervision of a continuing process that is not present in project management. The project team members are often from diverse areas of the organization and possess different skills, which must be coordinated into a single, focused effort to successfully complete the project. In addition, the project is invariably subject to time and budgetary constraints that are

not the same as normal work schedules and resource consumption in an ongoing process. Overall, there is usually more perceived and real pressure associated with project management than in a normal management position. However, there are potential rewards, including the ability to demonstrate one's management abilities in a difficult situation, the challenge of working on a unique project, and the excitement of doing something new.

The project manager is often under great pressure.

Scope Statement

A ***scope statement*** is a document that provides a common understanding of a project. It includes a justification for the project that describes what factors have created a need within the

company for the project. It also includes an indication of what the expected results of the project will be and what will constitute project success. Further, the scope statement might include a list of the types of planning reports and documents that are part of the project management process.

A scope statement includes a project justification and the expected results.

A similar planning document is the *statement of work (SOW)*. In a large project, the SOW is often prepared for individual team members, groups, departments, subcontractors, and suppliers. This statement describes the work in sufficient detail so that the team member responsible for it knows what is required and whether he or she has sufficient resources to accomplish the work successfully and on time. For suppliers and subcontractors, it

is often the basis for determining whether they can perform the work and for bidding on it. Some companies require that an SOW be part of an official contract with a supplier or subcontractor.

Work Breakdown Structure

The **work breakdown structure (WBS)** is an organizational chart used for project planning. It organizes the work to be done on a

project by breaking down the project into its major components, referred to as *modules*. These components are then subdivided into more detailed subcomponents, which are further broken down into activities, and, finally, into individual tasks. The end result is an organizational structure of the project made up of different levels, with the overall project at the top level and the individual tasks at the bottom. A WBS helps identify activities and determine individual tasks, project

workloads, and the resources required. It also helps to identify the relationships between modules and activities and avoid unnecessary duplication of activities. A WBS provides the basis for developing and managing the project schedule, resources, and modifications.

A work breakdown structure is an organization chart that break down the project into modules for planning.

There is no specific model for a WBS, although it is most often in the form of a chart or a table. In general, there are two good ways to develop a WBS. One way is to start at the top and work your way down, asking, "What components constitute this level?" until the WBS is developed in sufficient detail. Another way is to brainstorm the entire project, writing down each item on a sticky note and then organizing the sticky notes into a WBS.

The upper levels of the WBS tend to contain the summary activities, major components or functional areas involved in the project that indicate what is to be done. The lower levels tend to describe the detailed work activities of the project within the major components or modules. They typically indicate how things are done.

[Page 325]

[Figure 8.2](#) shows a WBS for a project for installing a new computerized order processing

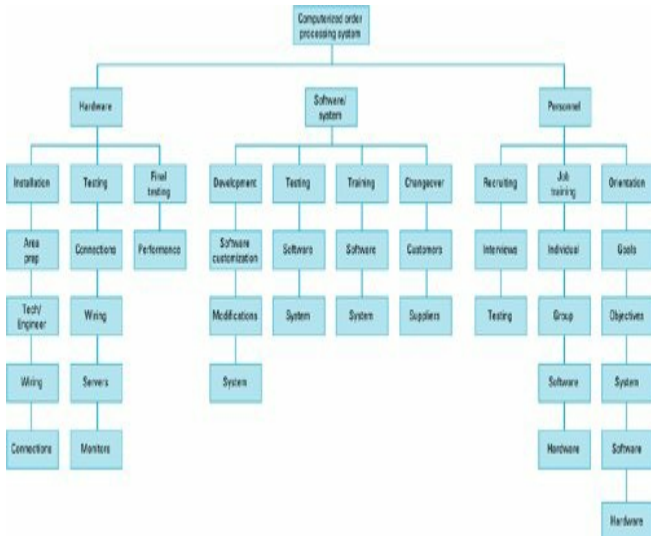
system for a manufacturing company that links customers, the manufacturer, and the manufacturer's suppliers. The WBS is organized according to the three major project categories for the development of the system: hardware, software/system, and personnel. Within each of these categories, the major tasks and activities under those tasks are detailed. For example, under hardware, a major task is installation, and activities required in installation include area preparation,

technical/engineering layouts and configurations, wiring, and electrical connections.

Figure 8.2. Work breakdown structure for computer order-processing system project

(This item is displayed on page 324 in the print version)

[\[View full size image\]](#)



Responsibility Assignment Matrix

After the WBS is developed, to organize the project work into smaller, manageable elements, the project manager assigns the work elements to organizational units—departments, groups, individuals, or subcontractors—by using an *organizational breakdown structure (OBS)*. An OBS is a table or chart that shows which organizational units are responsible for work items. After the OBS is developed, the project manager can then develop a ***responsibility assignment***

matrix (RAM). A RAM shows who in the organization is responsible for doing the work in the project. [Figure 8.3](#) shows a RAM for the "Hardware/Installation" category from the WBS for the computerized order-processing project shown in [Figure 8.2](#). Notice that there are three levels of work assignments in the matrix, reflecting who is responsible for the work, who actually performs the work, and who performs support activities. As with the WBS, there are many different forms

both the OBS and RAM can take, depending on the needs and preferences of the company, project team, and project manager.

Figure 8.3. A responsibility matrix

OBS Units	WBS Activities		
	1.1.1 Area prep	1.1.2 Tech/Engineer	1 W
Hardware engineering	3	1	

Systems engineering		3	
Software engineering		3	
Technical support	1	2	
Electrical staff	2		
Hardware vendor	3	3	
Quality manager			
Customer/supplier liaison			

Level of responsibility: 1 = Overall responsibility Performance responsibility 3 = Support			

A responsibility assignment matrix
is a table or chart that shows who is responsible for project work.

Project Scheduling

A project schedule evolves from the planning documents discussed previously. It is typically the most critical element in the project management process, especially during the implementation phase (i.e., the actual project work), and it is the source of most conflict and problems. One reason is that frequently the single most important criterion for the success of a project is that it be finished on time. If a stadium is

supposed to be finished in time for the first game of the season and it's not, there will be a lot of angry ticket holders; if a school building is not completed by the time the school year starts, there will be a lot of angry parents; if a shopping mall is not completed on time, there will be a lot of angry tenants; if a new product is not completed by the scheduled launch date, millions of dollars can be lost; and if a new military weapon is not completed on time, it could affect national security. Also,

time is a measure of progress that is very visible. It is an absolute with little flexibility; you can spend less money or use fewer people, but you cannot slow down or stop the passage of time.

[Page 326]

Time Out: For Henry Gantt

Both CPM and PERT are based on the network diagram, which is an outgrowth of the bar, or Gantt, chart that was designed to control the time element of a program. Henry Gantt, a pioneer in the field of industrial engineering, first employed the Gantt chart in the artillery ammunition shops at the Frankford Arsenal in 1914, when World War I was declared. The chart graphically depicted actual versus estimated production time.

Developing a schedule encompasses four basic steps.

First, *define the activities* that must be performed to complete the project; second, *sequence the activities* in the order in which they must be completed; third, *estimate the time* required to complete each activity; and fourth, *develop the schedule* based on the sequencing and time estimates of the activities.

Because scheduling involves a quantifiable measure, time, there are several quantitative techniques available that can be used to develop a project schedule, including the Gantt

chart and CPM/PERT networks. There are also various computer software packages that can be used to schedule projects, including the popular Microsoft Project. Later in this chapter we will discuss CPM/PERT and Microsoft Project in greater detail. For now, we will describe one of the oldest and most widely used scheduling techniques, the Gantt chart.

The Gantt Chart

Using a **Gantt chart** is a traditional management technique for scheduling and planning small projects that have relatively few activities and precedence relationships. This scheduling technique (also called a *bar chart*) was developed by Henry Gantt, a pioneer in the field of industrial engineering at the artillery ammunition shops of the Frankford Arsenal in 1914. The Gantt chart has been a popular project scheduling tool since its inception and is still widely used today. It is the direct

precursor of the CPM/PERT technique, which we will discuss later.

*A **Gantt chart** is a graph or bar chart with a bar for each project activity that shows the passage of time.*

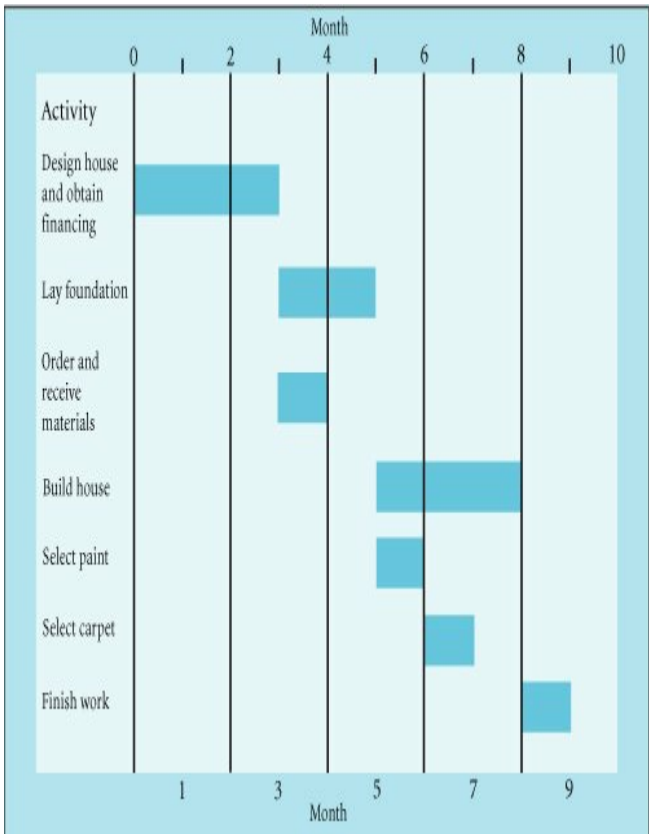
The Gantt chart is a graph with a bar representing time for each activity in the project

being analyzed. [Figure 8.4](#) illustrates a Gantt chart of a simplified project description for building a house. The project contains only seven general activities, such as designing the house, laying the foundation, ordering materials, and so forth. The first activity is "design house and obtain financing," and it requires 3 months to complete, shown by the bar from left to right across the chart. After the first activity is finished, the next two activities, "lay foundation" and "order and receive

materials," can start simultaneously. This set of activities demonstrates how a precedence relationship works; the design of the house and the financing must precede the next two activities.

Figure 8.4. A Gantt chart

(This item is displayed on page 327 in the print version)



The activity "lay foundation" requires 2 months to complete, so it will be finished, at the earliest, at the end of month 5. "Order and receive materials" requires 1 month to complete, and it could be finished after month 4. However, observe that it is possible to delay the start of this activity 1 month, until month 4. This delay would still enable the activity to be completed by the end of month 5, when the next activity, "build house," is scheduled to

start. This extra time for the activity "order and receive materials" is called **slack**. Slack is the amount by which an activity can be delayed without delaying any of the activities that follow it or the project as a whole. The remainder of the Gantt chart is constructed in a similar manner, and the project is scheduled to be completed at the end of month 9.

[Page 327]

Slack is the amount
of time an activity

*can be delayed
without delaying the
project.*

A Gantt chart provides a visual display of a project schedule, indicating when activities are scheduled to start and to finish and where extra time is available and activities can be delayed. A project manager can use a Gantt chart to monitor the progress of activities and see which ones are ahead of

schedule and which ones are behind schedule. A Gantt chart also indicates the precedence relationships between activities; however, these relationships are not always easily discernible. This problem is one of the disadvantages of the Gantt chart method, and it limits the chart's use to smaller projects with relatively few activities. The CPM/PERT network technique, which we will talk about later, does not suffer this disadvantage.

Project Control

Project control is the process of making sure a project progresses toward successful completion. It requires that the project be monitored and progress measured so that any deviations from the project plan, and particularly the project schedule, are minimized. If the project is found to be deviating from the plan (i.e., it is not on schedule, cost overruns are occurring, activity results are not as expected), corrective action must be taken. In the rest of this section we will describe

several key elements of project control, including time and cost management and performance monitoring.

[Page 328]

Time management is the process of making sure a project schedule does not slip and that a project is on time. This requires monitoring of individual activity schedules and frequent updates. If the schedule is being delayed to an extent that jeopardizes the project success, it may be

necessary for the project manager to shift resources to accelerate critical activities. Some activities may have slack time, so resources can be shifted from them to activities that are not on schedule. This is referred to as *timecost trade-off*. However, this can also push the project cost above the budget. In some cases it may be that the work needs to be corrected or made more efficient. In other cases, it may occur that original activity time estimates upon implementation prove to be unrealistic and the

schedule must be changed, and the repercussions of such changes on project success must be evaluated.

Cost management is often closely tied to time management because of the timecost trade-off occurrences mentioned previously. If the schedule is delayed, costs tend to go up in order to get the project back on schedule. Also, as a project progresses, some cost estimates may prove to be unrealistic or erroneous. Therefore, it may be necessary to revise cost estimates and

develop budget updates. If cost overruns are excessive, corrective actions must be taken.

Management Science

Application: Project "Magic" at Disney Imagineering

(This item is displayed on page 328 in the
print version)

The Walt Disney Company theme parks generate \$6 billion in revenue each year. Projects for the development of new attractions and rides at the Disney theme parks are unique and challenging, like all other operations at Disney. Budgets are tight, schedules are aggressive, and Disney standards for design quality and guest experience are high. It takes more than "pixie dust" to successfully manage a project to completion.

Walt Disney Imagineering (WDI) is the group at Disney that manages and delivers

the magic required in its projects. An Imagineering project team is organized to support the special demands of a Disney theme park project. It includes a project manager and show producer, as well as core team members and project control specialists. Each project team is headed by a show producer responsible for the creative vision and a project manager responsible for project scheduling and delivery, plus more than 100 specialized work groups in multiple locations, including writers, artists, model makers, sculptors, set designers, lighting designers, special effects designers, audio/video designers, show/ride mechanical engineers, show/ride electronic software engineers, show production and tooling specialists, show animators and programmers, graphic designers, architects, film production specialists, interior designers, landscape specialists, and themed painters. These specialty groups must all

be coordinated so that they are engaged at the appropriate time and deliver their assigned tasks on time to keep the project on schedule to meet the opening day deadline.

Each new project evolves through a process that establishes the project life cycle and the project schedule. The first phase, *concept*, is the evolutionary phase that is used to ensure that an attraction can be designed to integrate a Disney story line into three dimensions, using feasible technology. The next phase, *design*, includes the complete schematic design for the attraction, including the building and show/ride layouts. In the *implementation* phase, the show/ride design requirements are completed so that bids can be solicited, show/ride production is started, the facilities are constructed, and the shows and rides are installed, tested, and adjusted. In the final phase, *close out*, operations training is

completed. A fully integrated project master schedule is developed by the project team to successfully deliver the project, based on critical path method (CPM) logic.



The key individual in the project management process is the project manager, who must lead the team to success. The project manager has full

accountability for project results. He or she assigns work to divisions and groups, is the project facilitator, must resolve conflicts between team members, and must deal with problems and obstacles from a single, coordinated viewpoint.

Source: F. Addeman, "Managing the Magic," *PM Network* 13, no. 7 (July 1999): 3136.

Performance management is the process of monitoring a project and developing timed (i.e., daily, weekly, monthly) status reports to make sure that goals are being met and

the plan is being followed. It compares planned target dates for events, milestones, and work completion with dates actually achieved to determine whether the project is on schedule or behind schedule. Key measures of performance include deviation from the schedule, resource usage, and cost overruns. The project manager and individuals and organizational units with performance responsibility develop these status reports.

Earned value analysis (EVA) is a specific system for

performance management. Activities "earn value" as they are completed. EVA is a recognized standard procedure for numerically measuring a project's progress, forecasting its completion date and final cost, and providing measures of schedule and budget variation as activities are completed. For example, an EVA metric such as *schedule variance* compares the work performed during a time period with the work that was scheduled to be performed; a negative variance means the project is behind schedule. *Cost*

variance is the budgeted cost of work performed minus the actual cost of the work; a negative variance means the project is over budget. EVA works best when it is used in conjunction with a WBS that compartmentalizes project work into small packages that are easier to measure. The drawbacks of EVA are that it is sometimes difficult to measure work progress, and the time required for data measurement can be considerable.

CPM/PERT

Critical path method (CPM)
and **project evaluation and
review technique (PERT)**

were originally developed as separate techniques. (See the TIME OUT box on page 330.)

Both are derivatives of the Gantt chart and, as a result, are very similar. There were originally two primary differences between CPM and PERT. With CPM, a single estimate for activity time was

used that did not allow for variation; activity times were treated as if they were known with certainty. With PERT, multiple time estimates were used for each activity that reflected variation; activity times were treated as probabilistic. The other difference was in the mechanics of drawing a network. In PERT, activities were represented as arcs, or lines with arrows, between circles called *nodes*, whereas in CPM activities were represented by the nodes, and the arrows between them

showed precedence relationships (i.e., which activity came before another). However, over time CPM and PERT have effectively merged into a single technique conventionally known as CPM/PERT.

The advantage of CPM/PERT over the Gantt chart is in the use of a network (instead of a graph) to show the precedence relationships between activities. A Gantt chart does not clearly show precedence relationships, especially for larger networks. Put simply, a

CPM/PERT network provides a better picture; it is visually easier to use, which makes using CPM/PERT a popular technique for project planners and managers.

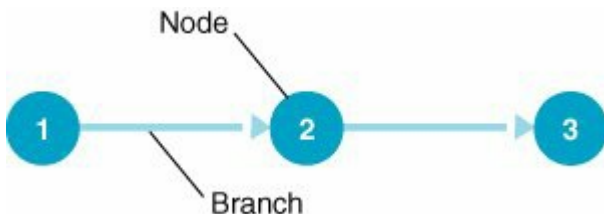
AOA Networks

A CPM/PERT network is drawn with branches and nodes, as shown in [Figure 8.5](#). As previously mentioned, when CPM and PERT were first developed, they employed different conventions for

drawing a network. With CPM, the nodes in [Figure 8.5](#) represent the project activities. The branches with arrows in between the nodes indicate the precedence relationships between activities. For example, in [Figure 8.5](#), activity 1, represented by node 1, precedes activity 2, and 2 must be finished before 3 can start. This approach to constructing a network is called *activity-on-node (AON)*. Alternatively, with PERT, the branches in between the nodes represent activities, and the nodes reflect events or

points in time such as the end of one activity and the start of another. This approach is called *activity-on-arrow (AOA)*, and the activities are identified by the node numbers at the start and end of an activity (for example, activity 1 $\rightarrow$ 2, which precedes activity 2 $\rightarrow$ 3 in [Figure 8.5](#)). In this chapter we will focus on the AON convention, but we will first provide an overview of the AOA network.

Figure 8.5. Nodes and branches

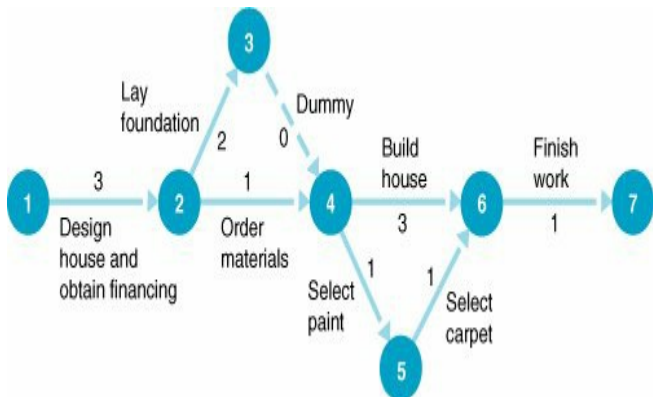


To demonstrate how an AOA network is drawn, we will revisit the example of building a house that we used as a Gantt chart example in [Figure 8.4](#). The comparable CPM/PERT

network for this project is shown in [Figure 8.6](#). The precedence relationships are reflected in this network by the arrangement of the arrowed (or directed) branches. The first activity in the project is to design the house and obtain financing. This activity must be completed before any subsequent activities can begin. Thus, activity 2 $\rightarrow$ 3, laying the foundation, and activity 2 $\rightarrow$ 4, ordering and receiving materials, can start only when node 2 is *realized*, indicating that activity 1 $\rightarrow$ 2 has been

completed. The number 3 above this branch signifies a time estimate of 3 months for the completion of this activity. Activity 2 $\rightarrow$ 3 and activity 2 $\rightarrow$ 4 can occur concurrently; neither depends on the other, and both depend only on the completion of activity 1 $\rightarrow$ 2.

Figure 8.6. Expanded network for building a house, showing concurrent activities



Time Out: For Morgan R. Walker, James E. Kelley, Jr., and D. G. Malcolm

In 1956 a research team at E. I. du Pont de Nemours & Company, Inc., led by a du Pont engineer, Morgan R. Walker, and a Remington-Rand computer specialist, James E. Kelley, Jr., initiated a project to develop a computerized system to improve the planning, scheduling, and reporting of the company's engineering programs (including plant maintenance and construction projects). The resulting network approach is known as the critical path method (CPM). At virtually the same time, the U.S. Navy established a research team composed of members of the Navy Special Projects Office, Lockheed

(the prime contractor), and the consulting firm of Booz, Allen, Hamilton, led by D. G. Malcolm, that developed PERT for the design of a management control system for the Polaris Missile Project (a ballistic missilefiring nuclear submarine). The Polaris project eventually included 23 PERT networks encompassing 2,000 events and 3,000 activities.

[Page 331]

When the activities of laying the foundation ($2 \rightarrow 3$) and ordering and receiving materials ($2 \rightarrow 4$) are

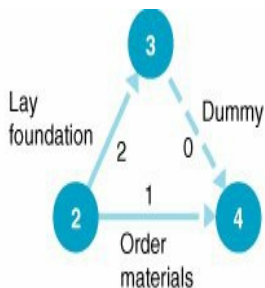
completed, activities $4 \rightarrow 5$ and $4 \rightarrow 6$ can begin simultaneously. However, notice activity $3 \rightarrow 4$, which is referred to as a **dummy activity**. A dummy activity is used in an AOA network to show a precedence relationship, but it does not represent any actual passage of time. These activities have the precedence relationship shown in **Figure 8.7**(a). However, in an AOA network, two or more activities are not allowed to share the same start and ending nodes because that

would give them the same name designation (i.e., $2 \rightarrow 3$). Activity $3 \rightarrow 4$ is inserted to give the two activities separate end modes and thus separate identities, as shown in [Figure 8.7\(b\)](#).

Figure 8.7. A dummy activity



(a) Incorrect precedence relationship



(b) Correct precedence relationship

Returning to the network shown in [Figure 8.6](#), we see that two activities start at node 4. Activity $4 \rightarrow 6$ is the actual building of the house, and activity $4 \rightarrow 5$ is the search for and selection of the paint for

the exterior and interior of the house. Activity 4 $\rightarrow$ 6 and activity 4 $\rightarrow$ 5 can begin simultaneously and take place concurrently. Following the selection of the paint (activity 4 $\rightarrow$ 5) and the realization of node 5, the carpet can be selected (activity 5 $\rightarrow$ 6) because the carpet color is dependent on the paint color. This activity can also occur concurrently with the building of the house (activity 4 $\rightarrow$ 6). When the building is completed and the paint and carpet are selected, the house can be

finished (activity 6 $\rightarrow$ 7).

AON Networks

[Figure 8.8](#) shows the comparable AON network to the AOA network in [Figure 8.6](#) for our house-building project. Notice that the activities and activity times are on the nodes and not on the activities, as with the AOA network. The branches or arrows simply show the precedence relationships between the activities. Also, notice that

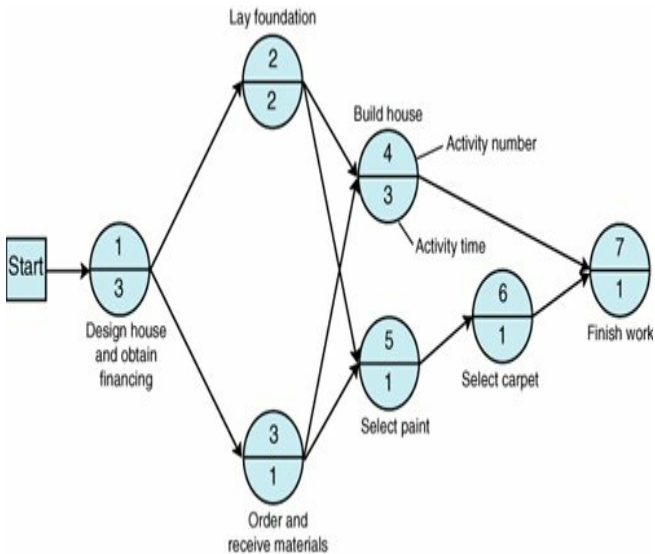
there is no dummy activity; dummy activities are not required in an AON network because no two activities have the same start and ending nodes, so they will never be confused. This is one of the advantages of the AON convention, although both AOA and AON have minor advantages and disadvantages. In general, the two methods both accomplish the same thing, and the one that is used is usually a matter of individual preference. However, for our purposes, the AON network has

one distinct advantage: It is the convention used in the popular Microsoft Project software package. Because we will demonstrate how to use that software later in this chapter, we will use AON.

Figure 8.8. AON network for house-building project

(This item is displayed on page 332 in the print version)

[\[View full size image\]](#)



The Critical Path

A network path is a sequence of connected activities that runs from the start to the end of the network. The network shown in [Figure 8.8](#) has several paths. In fact, close observation of this network shows four paths, identified in [Table 8.1](#) as A, B, C, and D.

Table 8.1. Paths through the house-building network

(This item is displayed on page 332 in the print version)

Path	Events
------	--------

A	1 → 2 → 4 → 7
---	---------------

B 1 → 2 → 5 → 6 → 7

C 1 → 3 → 4 → 7

D 1 → 3 → 5 → 6 → 7

The project cannot be completed (i.e., the house cannot be built) until the longest path in the network is completed. This is the minimum time in which the project can be completed. The longest path is referred to as the **critical**

path. To better understand the relationship between the minimum project time and the critical path, we will determine the length of each of the four paths shown in [Figure 8.8](#).

[Page 332]

*The **critical path** is the longest path through the network; it is the minimum time in which the network can be completed.*

By summing the activity times (shown in [Figure 8.8](#)) along each of the four paths, we can compute the length of each path, as follows:

path A: $1 \rightarrow 2 \rightarrow 4 \rightarrow 7$

$$3 + 2 + 3 + 1 = 9 \text{ months}$$

path B: $1 \rightarrow 2 \rightarrow 5 \rightarrow 6 \rightarrow 7$

$$3 + 2 + 1 + 1 + 1 = 8 \text{ months}$$

path C: $1 \rightarrow 3 \rightarrow 4 \rightarrow 7$

$$3 + 1 + 3 + 1 = 8 \text{ months}$$

path D: $1 \rightarrow 3 \rightarrow 5 \rightarrow 6 \rightarrow 7$

$$3 + 1 + 1 + 1 + 1 = 7 \text{ months}$$

Because path A is the longest, it is the critical path; thus, the

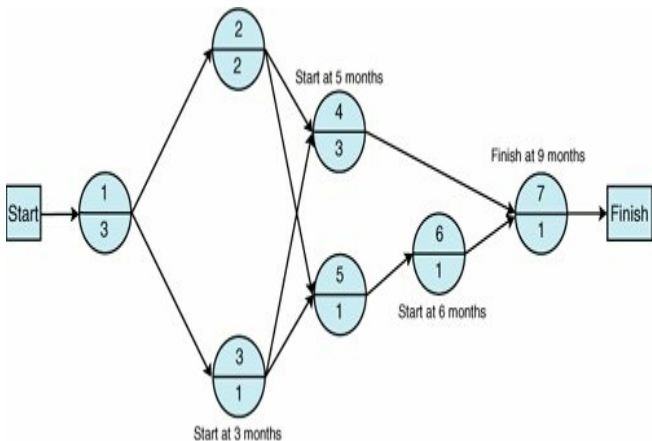
minimum completion time for the project is 9 months. Now let us analyze the critical path more closely. From [Figure 8.9](#) we can see that activity 3 cannot start until 3 months have passed. It is also easy to see that activity 4 will not start until 5 months have passed (i.e., the sum of activity 1 and 2 times). The start of activity 4 is dependent on the two activities leading into node 4. Activity 2 is completed after 5 months, but activity 3 is completed at the end of 4 months. Thus, we have two

possible start times for activity 4: 5 months and 4 months. However, because the activity on node 4 cannot start until all preceding activities have been finished, the soonest node 4 can be realized is 5 months.

[Page 333]

Figure 8.9. Activity start time

[\[View full size image\]](#)



Now let us consider the activity following node 4. Using the same logic as before, we can see that activity 7 cannot start until after 8 months (5 months

at node 4 plus the 3 months required by activity 4) or after 7 months. Because all activities preceding node 7 must be completed before activity 7 can start, the soonest this can occur is 8 months. Adding 1 month for activity 7 to the time at node 7 gives a project duration of 9 months. Recall that this is the time of the longest path in the network, or the critical path.

This brief analysis demonstrates the concept of a critical path and the determination of the minimum

completion time of a project. However, this is a cumbersome method for determining a critical path. Next, we will discuss a mathematical approach for scheduling the project activities and determining the critical path.

Activity Scheduling

In our analysis of the critical path, we determined the soonest time that each activity could be finished. For example, we found that the earliest time

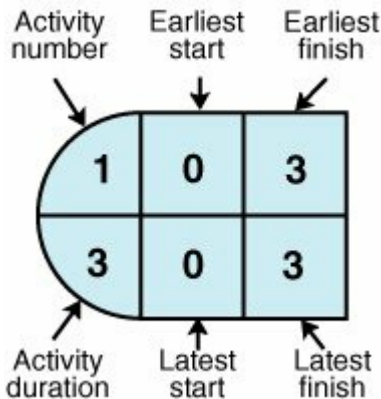
activity 4 could start was at 5 months. This time is referred to as the *earliest start time*, and it is expressed symbolically as *ES*.

ES is the earliest time an activity can start.

In order to show the earliest start time on the network, as well as some other activity times we will develop in the

scheduling process, we will alter our node structure a little. [Figure 8.10](#) shows the structure for node 1, the first activity in our example network for designing a house and obtaining financing.

Figure 8.10. Activity-on-node configuration



To determine the earliest start time for every activity, we make a *forward pass* through the network. That is, we start at the first node and move forward through the network.

The earliest time for an activity is the maximum time that all preceding activities have been completed the time when the activity start node is realized.

[Page 334]

EF is the earliest start time plus the activity time estimate.

The *earliest finish time, EF*, for

an activity is the earliest start time plus the activity time estimate. For example, if the earliest start time for activity 1 is at time 0, then the earliest finish time is 3 months. In general, the earliest start and finish times for an activity are computed according to the following mathematical formulas:

$ES = \text{Maximum } (EF \text{ immediate predecessors})$

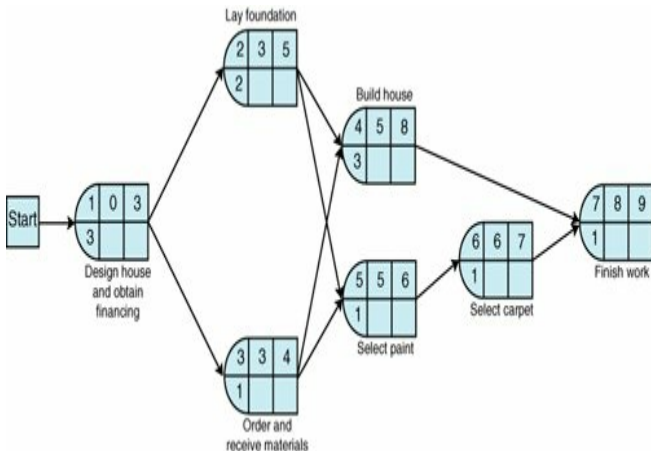
$$EF = ES + t$$

The earliest start and earliest

finish times for all the activities in our project network are shown in [Figure 8.11](#).

Figure 8.11. Earliest activity start and finish times

[\[View full size image\]](#)



The earliest start time for the first activity in the network (for which there are no predecessor activities) is always zero, or $ES = 0$. This enables us to

compute the earliest finish time for activity 1 as

$$EF = ES + t = 0 + 3 = 3$$

months

The earliest start time for activity 2 is,

$$ES = \text{Max (} EF \text{ immediate predecessors)}$$

$$= 3 \text{ months}$$

The corresponding earliest finish time is

$$EF = ES + t = 3 + 2 = 5$$

months

For activity 3 the earliest start time is 3 months, and the earliest finish time is 4 months.

Now consider activity 4, which has two predecessor activities. The earliest start time is computed as

$$ES = \text{Max (} EF \text{ immediate predecessors)}$$

$$= \text{Max} (5, 4) = 5 \text{ months}$$

[Page 335]

and the earliest finish time is

$$EF = ES + t = 5 + 3 = 8$$

months

All the remaining earliest start and finish times are computed similarly. Notice in [Figure 8.11](#) that the earliest finish time for activity 7, the last activity in the network, is 9 months,

which is the total project duration, or critical path time.

Companions to the earliest start and finish are the *latest start (LS)* and *finish (LF)* times. The latest start time is the latest time an activity can start without delaying the completion of the project beyond the project critical path time. For our example, the project completion time (and earliest finish time) at node 7 is 9 months. Thus, the objective of determining latest times is to see how long each activity can be delayed without

the project exceeding 9 months.

LS is the latest time an activity can start without delaying critical path time.

In general, the latest start and finish times for an activity are computed according to the following formulas:

$$LS = LF - t$$

$LF =$ Minimum (LS immediately following predecessors)

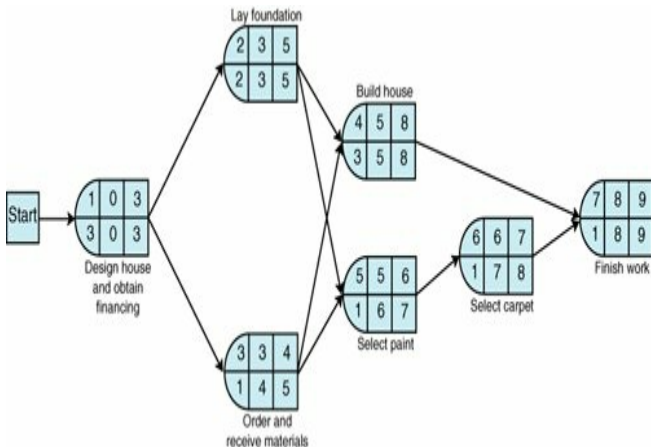
Whereas a **forward pass** through the network is made to determine the earliest times, the latest times are computed using a **backward pass**. We start at the end of the network at node 7 and work backward, computing the latest time for each activity. Because we want to determine how long each

activity in the network can be delayed without extending the project time, the latest finish time at node 7 cannot exceed the earliest finish time.

Therefore, the latest finish time at node 7 is 9 months. This and all other latest times are shown in [Figure 8.12](#).

Figure 8.12. Latest activity start and finish times

[\[View full size image\]](#)



Starting at the end of the network, the critical path time, which is also equal to the earliest finish time of activity 7, is 9 months. This automatically

becomes the latest finish time for activity 7, or

$$LF = 9 \text{ months}$$

Using this value, the latest start time for activity 7 can be computed:

$$LS = LF - t = 9 - 1 = 8 \text{ months}$$

[Page 336]

The latest finish time for activity 6 is the minimum of the latest start times for the activities following node 6. Because activity 7 follows node

6, the latest start time is computed as follows:

$$LF = \text{Min } (LS \text{ following activities})$$

$$= 8 \text{ months}$$

The latest start time for activity 6 is

$$LS = LF - t = 8 - 1 = 7 \text{ months}$$

For activity 4 the latest finish time is 8 months, and the

latest start time is 5 months;
for activity 5, the latest finish
time is 7 months, and the
latest start time is 6 months.

Now consider activity 3, which
has two activities following it.
The latest finish time is
computed as follows:

$$LF = \text{Min} (LS \text{ following activities})$$

$$= \text{Min} (5, 6) = 5 \text{ months}$$

The latest start time is

$$LS = LF - t = 5 - 1 = 4 \text{ months}$$

All the remaining latest start and latest finish times are computed similarly. [Figure 8.12](#) includes the earliest and latest start times and earliest and latest finish times for all activities.

Activity Slack

The project network in [Figure 8.12](#), with all activity start and finish times, highlights the

critical path ($1 \rightarrow 2 \rightarrow 4 \rightarrow 7$) we determined earlier by inspection. Notice that for the activities on the critical path, the earliest start times and latest start times are equal. This means that these activities on the critical path must start exactly on time and cannot be delayed at all. If the start of any activity on the critical path is delayed, then the overall project time will be increased. As a result, we now have an alternative way to determine the critical path besides simply inspecting the network. The

activities on the critical path can be determined by seeing for which activities $ES = LS$ or $EF = LF$. In [Figure 8.12](#), the activities 1, 2, 4, and 7 all have earliest start times and latest start times that are equal (and $EF = LF$); thus, they are on the critical path.

For those activities not on the critical path, the earliest and latest start times (or earliest and latest finish times) are not equal, and [**slack**](#) time exists. Slack is the amount of time an activity can be delayed without affecting the overall project

duration. In effect, it is extra time available for completing an activity.

Slack is the amount of time an activity can be delayed without delaying the project.

Slack, S , is computed using either of the following formulas:

$$S = LS - ES$$

or

$$S = LF - EF$$

For example, the slack for activity 3 is computed as follows:

$$S = LS - ES = 4 - 3 = 1 \text{ month}$$

If the start of activity 3 were delayed for 1 month, the activity could still be completed by month 5 without delaying the project completion time. The slack for each activity in

our example project network is shown in [Table 8.2](#) and in [Figure 8.13](#).

[Page 337]

Table 8.2. Activity slack

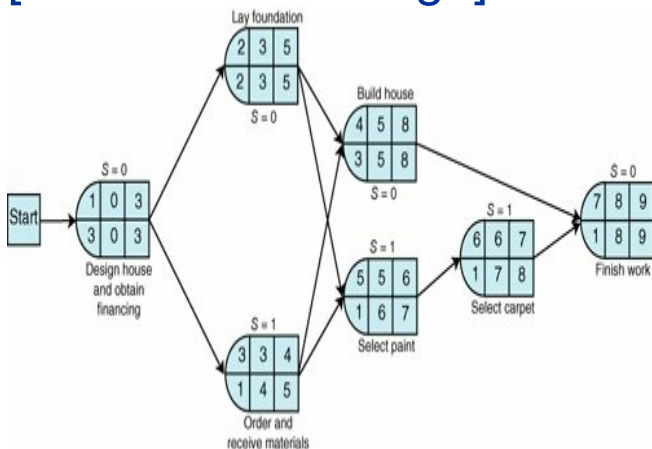
Activity	<i>LS</i>	<i>ES</i>	<i>LF</i>	<i>EF</i>	Slack, <i>S</i>
[*] ₁	0	0	3	3	0
[*] ₂	3	3	5	5	0
3	4	3	5	4	1

[*]4	5	5	8	8	0
5	6	5	7	6	1
6	7	6	8	7	1
[*]7	8	8	9	9	0

[*] Critical path activities

Figure 8.13. Activity slack

[\[View full size image\]](#)



Notice in [Figure 8.13](#) that activity 3 can be delayed 1 month and activity 5, which follows it, can be delayed 1

more month, but then activity 6 cannot be delayed at all, even though it has 1 month of slack. If activity 3 starts late, at month 4 instead of month 3, then it will be completed at month 5, which will not allow activity 5 to start until month 5. If the start of activity 5 is delayed 1 month, then it will be completed at month 7, and activity 6 cannot be delayed at all without exceeding the critical path time. The slack on these three activities is called **shared slack**. This means that the sequence of activities 3 →

5 → 6 can be delayed 2 months jointly without delaying the project, but not 3 months.

Slack is obviously beneficial to a project manager because it enables resources to be temporarily pulled away from activities with slack and used for other activities that might be delayed for various reasons or for which the time estimate has proven to be inaccurate.

The times for the network activities are simply estimates for which there is usually not a lot of historical basis (because

projects tend to be unique undertakings). As such, activity time estimates are subject to quite a bit of uncertainty. However, the uncertainty inherent in activity time estimates can be reflected to a certain extent by using probabilistic time estimates instead of the single, deterministic estimates we have used so far.

Probabilistic Activity Times

In the project network for building a house presented in the previous section, all the activity time estimates were single values. By using only a single activity time estimate, we in effect assume that activity times are known with certainty (i.e., they are deterministic). For example, in [Figure 8.8](#), the time estimate for activity 2 (laying the

foundation) is shown to be 2 months. Because only this one value is given, we must assume that the activity time does not vary (or varies very little) from 2 months. In reality, however, it is rare that activity time estimates can be made with certainty. Project activities are likely to be unique, and thus there is little historical evidence that can be used as a basis to predict actual times. However, we earlier indicated that one of the primary differences between CPM and PERT was that PERT used

probabilistic activity times. It is this approach to estimating activity times for a project network that we discuss in this section.

Probabilistic Time Estimates

To demonstrate the use of probabilistic activity times, we will employ a new example. (We could use the house-building network from the previous section; however, a network that is a little larger

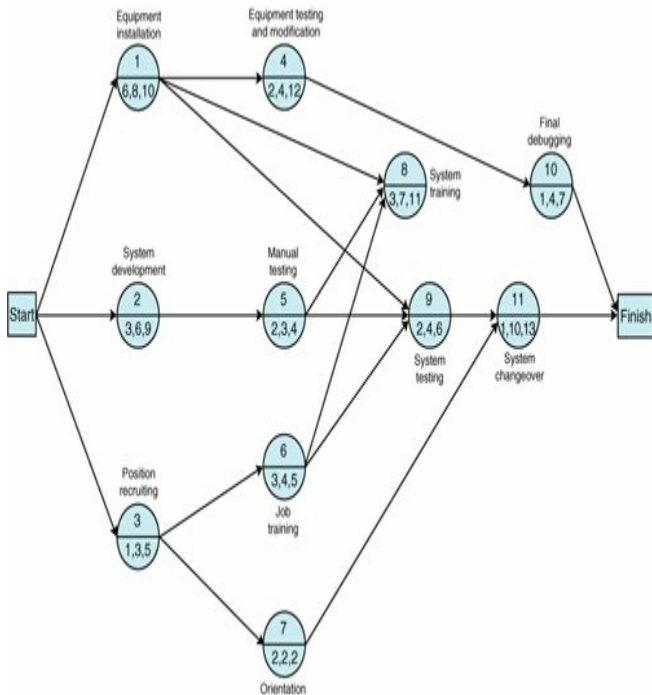
and more complex will provide more experience with different types of projects.) The Southern Textile Company has decided to install a new computerized order processing system that will link the company with customers and suppliers online. In the past, orders for the cloth the company produces were processed manually, which contributed to delays in delivering orders and resulted in lost sales. The company wants to know how long it will take to install the new system.

The network for the installation of the new order processing system is shown in [Figure 8.14](#). We will briefly describe the activities.

Figure 8.14. Network for order processing system installation

(This item is displayed on page 339 in the print version)

[\[View full size image\]](#)



The network begins with three concurrent activities: The new computer equipment is installed (activity 1); the computerized order processing system is developed (activity 2); and people are recruited to operate the system (activity 3). Once people are hired, they are trained for the job (activity 6), and other personnel in the company, such as marketing, accounting, and production personnel, are introduced to the new system (activity 7). Once the system is developed (activity 2), it is tested

manually to make sure that it is logical (activity 5). Following activity 1, the new equipment is tested, and any necessary modifications are made (activity 4), and the newly trained personnel begin training on the computerized system (activity 8). Also, node 9 begins the testing of the system on the computer to check for errors (activity 9). The final activities include a trial run and changeover to the system (activity 11) and final debugging of the computer system (activity 10).

At this stage in a project network, we previously assigned a single time estimate to each network activity. In a PERT project network, however, we determine *three time estimates* for each activity, which will enable us to estimate the mean and variance for a **beta distribution** of the activity times. We assume that the activity times can be described by a beta distribution for several reasons. First, the beta distribution mean and variance can be approximated with three

estimates. Second, the beta distribution is continuous, but it has no predetermined shape (such as the bell shape of the normal curve). It will take on the shape indicated that is, be skewed by the time estimates given. This is beneficial because typically we have no prior knowledge of the shapes of the distributions of activity times in a unique project network. Third, although other types of distributions have been shown to be no more or less accurate than the beta, it has become traditional to use

the beta distribution for probabilistic network analysis.

*Three time estimates for each activity a **most likely**, an **optimistic**, and a **pessimistic** provide an estimate of the mean and variance of a beta distribution.*

The three time estimates for each activity are the most

likely time, the optimistic time, and the pessimistic time. The **most likely time** is the time that would most frequently occur if the activity were repeated many times. The **optimistic time** is the shortest possible time within which the activity could be completed if everything went right. The **pessimistic time** is the longest possible time the activity would require to be completed, assuming that everything went wrong. In general, the person most familiar with an activity makes

these estimates to the best of his or her knowledge and ability. In other words, the estimate is subjective. (See [Chapter 11](#), on subjective probability.)

[Page 340]

These three time estimates can subsequently be used to estimate the mean and variance of a beta distribution. If

a = optimistic time estimate

m = most likely time estimate

b = pessimistic time estimate

the mean and variance are computed as follows:

mean (expected time): $t = \frac{a + 4m + b}{6}$

variance: $v = \left(\frac{b - a}{6} \right)^2$

These formulas provide a reasonable estimate of the mean and variance of the beta distribution, a distribution that is continuous and can take on various shapes that is, exhibit

skewness.

The three time estimates for each activity are shown in [Figure 8.14](#) and in [Table 8.3](#). The mean and the variance for all the activities in the network shown in [Figure 8.14](#) are also given in [Table 8.3](#).

Table 8.3. Activity time estimates for [Figure 8.14](#)

Time Estimates (weeks)						
Activity	a	m	b	Time	Mean	Variance

				<i>t</i>	<i>v</i>
1	6	8	10	8	4/9
2	3	6	9	6	1
3	1	3	5	3	4/9
4	2	4	12	5	25/9
5	2	3	4	3	1/9
6	3	4	5	4	1/9
7	2	2	2	2	0

8	3	7	11	7	16/9
9	2	4	6	4	4/9
10	1	4	7	4	1
11	1	10	13	9	4

As an example of the computation of the individual activity mean times and variances, consider activity 1. The three time estimates ($a = 6$, $m = 8$, $b = 10$) are

substituted in our beta distribution formulas as follows:

$$t = \frac{a + 4m + b}{6} = \frac{6 + 4(8) + 10}{6} = 8 \text{ weeks}$$
$$v = \left(\frac{b - a}{6} \right)^2 = \left(\frac{10 - 6}{6} \right)^2 = \frac{4}{9} \text{ weeks}$$

The other values for the mean and variance in [Table 8.3](#) are computed similarly.

[Page 341]

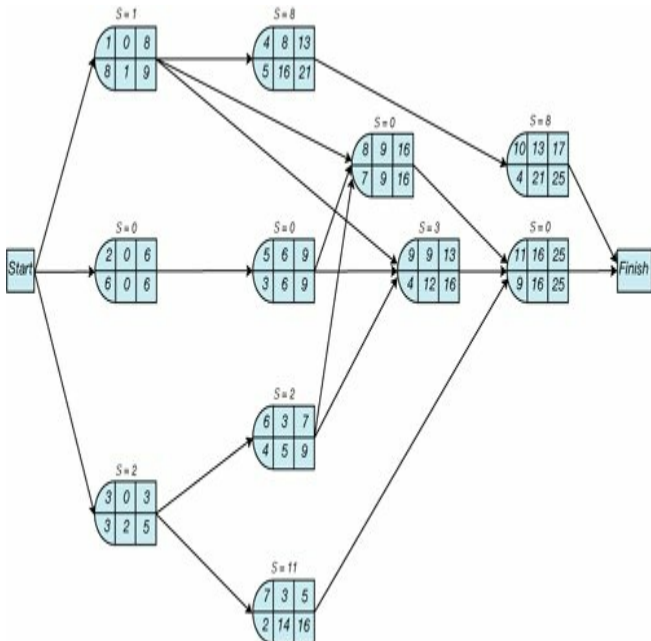
Once the expected activity times have been computed for

each activity, we can determine the critical path the same way we did previously, except that we use the expected activity times, t . Recall that in the project network, we identified the critical path as the one containing those activities with zero slack. This requires the determination of earliest and latest event times, as shown in [Figure 8.15](#).

Figure 8.15. Earliest and latest activity times

(This item is displayed on page
342 in the print version)

[\[View full size image\]](#)



The critical path has

no slack.

Observing [Figure 8.15](#), we can see that the critical path encompasses activities $2 \rightarrow 5 \rightarrow 8 \rightarrow 11$, because these activities have no available slack. We can also see that the *expected* project completion time (t_p) is 25 weeks. However, it is possible to compute the variance for project completion time. To determine the project variance, we *sum the variances*

for those activities on the critical path. Using the variances computed in [Table 8.3](#) and the critical path activities shown in [Figure 8.15](#), we can compute the variance for project duration (v_p) as follows:

Critical Path Activity	Variance
2	1
5	1/9

8

16/9

11

4

62/9

*The project variance
is the sum of the
variance of the
critical path activities.*

$$v_p = 62/9 = 6.9 \text{ weeks}$$

The CPM/PERT method assumes that the activity times are statistically independent, which allows us to sum the individual expected activity times and variances to get an expected *project* time and variance. It is further assumed that the network mean and variance are normally distributed. This assumption is based on the central limit theorem of probability, which for CPM/PERT analysis and our purposes states that if the number of

activities is large enough and the activities are statistically independent, then the sum of the means of the activities along the critical path will approach the mean of a normal distribution. For the small examples in this chapter, it is questionable whether there are sufficient activities to guarantee that the mean project completion time and variance are normally distributed. Although it has become conventional in CPM/PERT analysis to employ probability analysis by using

the normal distribution, regardless of the network size, the prudent user of CPM/PERT analysis should bear this limitation in mind.

The expected project time is assumed to be normally distributed based on the central limit theorem.

Given these assumptions, we can interpret the expected

project time (t_p) and variance (v_p) as the mean (μ) and variance (σ^2) of a normal distribution:

$$\mu = 25 \text{ weeks}$$

$$\sigma^2 = 6.9 \text{ weeks}$$

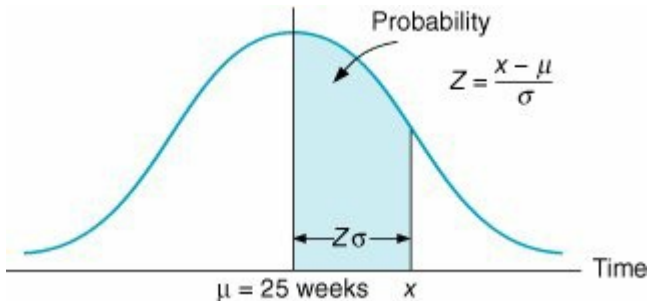
In turn, we can use these statistical parameters to make probabilistic statements about the project.

Probability Analysis of the Project Network

Using the normal distribution, probabilities can be determined by computing the number of standard deviations (Z) a value is from the mean, as illustrated in [Figure 8.16](#).

Figure 8.16. Normal distribution of network duration

(This item is displayed on page 343 in the print version)



The Z value is computed using the following formula:

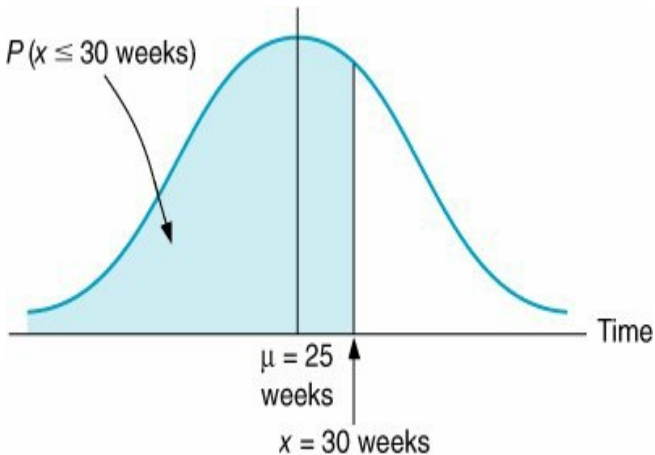
$$Z = \frac{x - \mu}{\sigma}$$

This value is then used to find the corresponding probability in [Table A.1](#) of [Appendix A](#).

For example, suppose the textile company manager told customers that the new order processing system would be completely installed in 30 weeks. What is the probability that it will, in fact, be ready by that time? This probability is illustrated as the shaded area in [Figure 8.17](#).

Figure 8.17.
Probability that the

**network will be
completed in 30
weeks or less**



To compute the Z value for a time of 30 weeks, we must first compute the standard deviation (σ) from the variance (σ^2):

$$\sigma^2 = 6.9$$
$$\sigma = \sqrt{6.9} = 2.63$$

Next, we substitute this value for the standard deviation, along with the value for the mean and our proposed project completion time (30 weeks), into the following formula:

$$Z = \frac{x - \mu}{\sigma}$$
$$= \frac{30 - 25}{2.63} = 1.90$$

A Z value of 1.90 corresponds to a probability of .4713 in [Table A.1](#) in [Appendix A](#). This means that there is a .9713 (.5000 + .4713) probability of completing the project in 30 weeks or less.

Suppose one customer, frustrated with delayed orders, has told the textile company that if the new ordering system is not working within 22 weeks,

she will trade elsewhere. The probability of the project's being completed within 22 weeks is computed as follows:

$$\begin{aligned} Z &= \frac{22 - 25}{2.63} \\ &= \frac{-3}{2.63} = -1.14 \end{aligned}$$

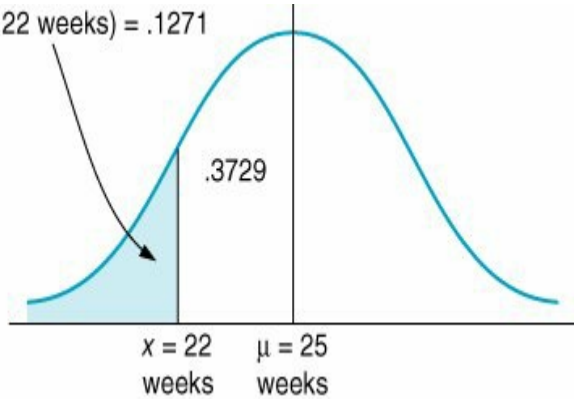
[Page 344]

A Z value of 1.14 (the negative is ignored) corresponds to a probability of .3729 in [Table A.1](#) of [Appendix A](#). Thus, there is only a .1271 probability that the customer will be retained,

as illustrated in [Figure 8.18](#).

**Figure 8.18.
Probability that the
network will be
completed in 22
weeks or less**

$$P(x \leq 22 \text{ weeks}) = .1271$$



Management Science

Application: The 2002 Winter Olympic Games in Salt Lake City

The 2002 Winter Olympic Games was a massive project, encompassing a 5-year planning period and costing \$1.9 billion. In previous Olympic Games, project management had been used to differing degrees and at different levels, primarily for large construction-related projects. However, project management was the cornerstone of the Salt Lake Organizing Committee (SLOC). Classic project management tools were used extensively at all levels and were responsible for turning a

projected \$400 million deficit into a \$100 million surplus. Four years prior to the start of the games, the SLOC employed a Primavera project management software system, with a cascading color-coded work breakdown structure to integrate planning across 78 Olympic events, 57 SLOC functions, 37,000 project tasks, and 22 venues. The SLOC awarded 3,780 contracts, valued at almost \$1 billion, and issued 7,000 purchase orders. The development of the project and work plans took 1 year to complete. SLOC employed an "Executive Roadmap," a list of the 100 Games-wide activities, to keep executives and managers apprised of progress. The Office of Operations Planning and Management's detailed attention to the project schedule allowed the SLOC to quickly know when it was off schedule and where

to apply additional resources. Even though the unexpected terrorist attack on September 11, 2001, required the project team to revisit security provisions, significantly change transportation and parking plans, and increase the number of people associated with security requirements, the project maintained its schedule and budget. Following the games, a \$76 million grant was awarded to fund on-going operations at three of the world-class Olympic facilities as training venues for future Olympians. The SLOC also built Olympic Legacy Parks as a permanent remembrance of the Games for Salt Lake City residents and visitors. All suppliers and vendors were paid within 9 months of the completion of the games, whereas in other games it had taken as long as 6 years. By all measures the 2001

Winter Olympic Games were a huge success, due in large part to project management.



Source: R. Foti, "The Best Winter Olympics, Period," *PM Network* 18, no. 1 (January 2004): 2228.

CPM/PERT Analysis with QM for Windows and Excel QM

The capability to perform CPM/PERT network analysis is a standard feature of most management science software packages for the personal computer. To illustrate the application of QM for Windows and Excel QM we will use our example of installing an order processing system at Southern

Textile Company. The QM for Windows solution output is shown in [Exhibit 8.1](#), and the Excel QM solution is shown in [Exhibit 8.2](#).

Exhibit 8.1.

[\[View full size image\]](#)

Click on
"Precedence list."

Project Management (PMT) Results

Activity	Early Start	Early Finish	Late Start	Late Finish	Slack	Precedence
1	0	10	0	10	0	1
2	10	20	10	20	0	1
3	10	20	10	20	0	1
4	10	20	10	20	0	1
5	10	20	10	20	0	1
6	10	20	10	20	0	1
7	10	20	10	20	0	1
8	10	20	10	20	0	1
9	10	20	10	20	0	1
10	10	20	10	20	0	1

Exhibit 8.2.

[\[View full size image\]](#)



Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

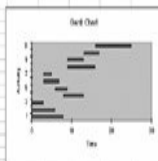
Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management



Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Enter the data in the shaded area

Order Processing System Example

Project Management

Microsoft Project

Microsoft Project is a very popular and widely used software package for project management and CPM/PERT analysis. It is also relatively easy to use. We will demonstrate how to use Microsoft Project, using the project network for building a house shown in [Figure 8.13](#). (Note that the Microsoft Project file for this example can be downloaded from the text Web

site.)

[Page 346]

When you open Microsoft Project, a screen comes up for a new project. Click on the "Tasks" button, and the screen like the one shown in [Exhibit 8.3](#) appears. Notice the set of steps on the left side of the screen, starting with "Set a date to schedule from." If you click on this step, you can set a start date. For our example, we will set the start date for our house-building project as

August 31, 2005.

Exhibit 8.3.

[\[View full size image\]](#)

Click on "Tasks" to start.

Microsoft Project - Project1

File Edit View Gantt Format Tools Project Window Help

Tasks

Task Name Duration Start Finish Predecessors

1

2

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

18

19

20

21

22

23

24

25

26

27

28

29

30

31

32

33

34

35

36

37

38

39

40

41

42

43

44

45

46

47

48

49

50

51

52

53

54

55

56

57

58

59

60

61

62

63

64

65

66

67

68

69

70

71

72

73

74

75

76

77

78

79

80

81

82

83

84

85

86

87

88

89

90

91

92

93

94

95

96

97

98

99

100

101

102

103

104

105

106

107

108

109

110

111

112

113

114

115

116

117

118

119

120

121

122

123

124

125

126

127

128

129

130

131

132

133

134

135

136

137

138

139

140

141

142

143

144

145

146

147

148

149

150

151

152

153

154

155

156

157

158

159

160

161

162

163

164

165

166

167

168

169

170

171

172

173

174

175

176

177

178

179

180

181

182

183

184

185

186

187

188

189

190

191

192

193

194

195

196

197

198

199

200

201

202

203

204

205

206

207

208

209

210

211

212

213

214

215

216

217

218

219

220

221

222

223

224

225

226

227

228

229

230

231

232

233

234

235

236

237

238

239

240

241

242

243

244

245

246

247

248

249

250

251

252

253

254

255

256

257

258

259

260

261

262

263

264

265

266

267

268

269

270

271

272

273

274

275

276

277

278

279

280

281

282

283

284

285

286

287

288

289

290

291

292

293

294

295

296

297

298

299

300

301

302

303

304

305

306

307

308

309

310

311

312

313

314

315

316

317

318

319

320

321

322

323

324

325

326

327

328

329

330

331

332

333

334

335

336

337

338

339

340

341

342

343

344

345

346

347

348

349

350

351

352

353

354

355

356

357

358

359

360

361

362

363

364

365

366

367

368

369

370

371

372

373

374

375

376

377

378

379

380

381

382

383

384

385

386

387

388

389

390

391

392

393

394

395

396

397

398

399

400

401

402

403

404

405

406

407

408

409

410

411

412

413

414

415

416

417

418

419

420

421

422

423

424

425

426

427

428

429

430

431

432

433

434

435

436

437

438

439

440

441

442

443

444

445

446

447

448

449

450

451

452

453

454

455

456

457

458

459

460

461

462

463

464

465

466

467

468

469

470

471

472

473

474

475

476

477

478

479

480

481

482

483

484

485

486

487

488

489

490

491

492

493

494

495

496

497

498

499

500

501

502

503

504

505

506

507

508

509

510

511

512

513

514

515

516

517

518

519

520

521

522

523

524

525

526

527

528

529

530

531

532

533

534

535

536

537

538

539

540

541

542

543

544

545

546

547

548

549

550

551

552

553

554

555

556

557

558

559

560

561

562

563

564

565

566

567

568

569

570

571

572

573

574

575

576

577

578

579

580

581

582

583

584

585

586

587

588

589

590

591

592

593

594

595

596

597

598

599

600

601

602

603

604

605

606

607

608

609

610

611

612

613

614

615

616

617

618

619

620

621

622

623

624

625

626

627

628

629

630

631

632

633

634

635

636

637

638

639

640

641

642

643

644

645

646

647

648

649

650

651

652

653

654

655

656

657

658

659

660

661

662

663

664

665

666

667

668

669

670

671

672

673

674

675

676

677

678

679

680

681

682

683

684

685

686

687

688

689

690

691

692

693

694

695

696

697

698

699

700

701

702

703

704

705

706

707

708

709

710

711

712

713

714

715

716

717

718

719

720

721

722

723

724

725

726

727

728

729

730

731

732

733

734

735

736

737

738

739

740

741

742

743

744

745

746

747

748

749

750

751

752

753

754

755

756

757

758

759

760

761

762

763

764

765

766

767

768

769

770

771

772

773

774

775

776

777

778

779

780

781

782

783

784

785

786

787

788

789

790

791

792

793

794

795

796

797

798

799

800

801

802

803

804

805

806

807

808

809

810

811

812

813

814

815

816

817

818

819

820

821

822

823

824

825

826

827

828

829

830

831

832

833

834

835

836

837

838

839

840

841

842

843

844

845

846

847

848

849

850

851

852

853

854

855

856

857

858

859

860

861

862

863

864

865

866

867

868

869

870

871

872

873

874

875

876

877

878

879

880

881

882

883

884

885

886

887

888

889

890

891

892

893

894

895

896

897

898

899

900

901

902

903

904

905

906

907

908

909

910

911

912

913

914

915

916

917

918

919

920

921

922

923

924

925

926

927

928

929

930

931

932

933

934

935

936

937

938

939

940

941

942

943

944

945

946

947

948

949

950

951

952

953

954

955

956

957

958

959

960

961

962

963

964

965

966

967

968

969

970

971

972

973

974

975

976

977

978

979

980

981

982

983

984

985

986

987

988

989

990

991

992

993

994

995

996

997

998

999

1000

1001

1002

1003

1004

1005

1006

1007

1008

1009

1010

1011

1012

1013

1014

1015

1016

1017

1018

1019

1020

1021

1022

1023

1024

1025

1026

1027

1028

1029

1030

1031

1032

1033

1034

1035

1036

1037

1038

1039

1040

1041

1042

1043

1044

1045

1046

1047

1048

1049

1050

1051

1052

1053

1054

1055

1056

1057

1058

1059

1060

1061

1062

1063

1064

1065

1066

1067

1068

1069

1070

1071

1072

1073

1074

1075

1076

1077

1078

1079

1080

1081

1082

1083

1084

1085

1086

1087

1088

1089

1090

1091

1092

1093

1094

1095

1096

1097

1098

1099

1100

1101

1102

1103

1104

1105

1106

1107

1108

1109

1110

1111

1112

1113

1114

1115

1116

1117

1118

1119

1120

1121

1122

1123

1124

1125

1126

1127

1128

1129

1130

1131

1132

1133

1134

1135

1136

1137

1138

1139

1140

1141

1142

1143

1144

1145

1146

1147

1148

1149

1150

1151

1152

1153

1154

1155

1156

1157

1158

1159

1160

1161

1162

1163

1164

1165

1166

1167

1168

1169

1170

1171

1172

1173

1174

1175

1176

1177

1178

1179

1180

1181

1182

1183

1184

1185

1186

1187

1188

1189

1190

1191

1192

1193

1194

1195

1196

1197

1198

1199

1200

1201

1202

1203

1204

1205

1206

1207

1208

1209

1210

1211

1212

1213

1214

1215

1216

1217

1218

1219

1220

1221

1222

1223

1224

1225

1226

1227

1228

1229

1230

1231

1232

1233

1234

1235

1236

1237

1238

1239

1240

1241

1242

1243

1244

1245

1246

1247

1248

1249

1250

1251

1252

1253

1254

1255

1256

1257

1258

1259

1260

1261

1262

1263

1264

1265

1266

1267

1268

1269

1270

1271

1272

1273

1274

1275

1276

1277

1278

1279

1280

1281

1282

1283

1284

1285

1286

1287

1288

1289

1290

1291

1292

1293

1294

1295

1296

1297

1298

1299

1300

1301

1302

1303

1304

1305

1306

1307

1308

1309

1310

1311

1312

1313

1314

1315

1316

1317

1318

1319

1320

1321

1322

1323

1324

1325

1326

1327

1328

1329

1330

1331

1332

1333

1334

1335

1336

1337

1338

1339

1340

1341

1342

1343

1344

1345

1346

1347

1348

1349

1350

1351

1352

1353

1354

1355

1356

1357

1358

1359

1360

1361

1362

1363

1364

1365

1366

1367

1368

1369

1370

1371

1372

1373

1374

1375

1376

1377

1378

1379

1380

1381

1382

1383

1384

1385

1386

1387

1388

1389

1390

1391

1392

1393

1394

1395

1396

1397

1398

1399

1400

1401

1402

1403

1404

1405

1406

1407

1408

1409

1410

1411

1412

1413

1414

1415

1416

1417

1418

1419

1420

1421

1422

1423

1424

1425

1426

1427

1428

1429

1430

1431

1432

1433

1434

1435

1436

1437

1438

1439

1440

1441

1442

1443

1444

1445

1446

1447

1448

1449

1450

1451

1452

1453

1454

1455

1456

1457

1458

1459

1460

1461

1462

1463

1464

1465

1466

1467

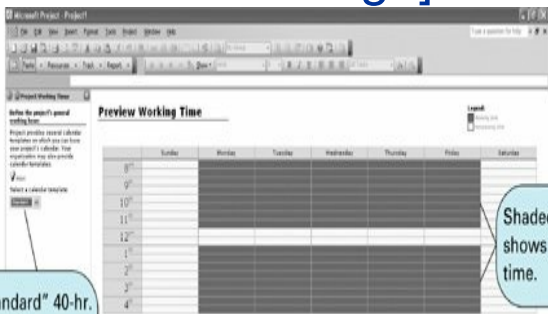
1468

146

The second step in the "Tasks" menu, "Define general working times," allows the user to specify general work rules and a work calendar, including such things as working hours per day, holidays, and weekend days off. For example, [Exhibit 8.4](#) shows the screen for selecting the work days and times; in this case, we have selected a "standard" workweek (Monday through Friday) with an 8-hour work day, as shown by the shaded area.

Exhibit 8.4.

[\[View full size image\]](#)



A "standard" 40-hr. workweek

Shaded area shows work time.

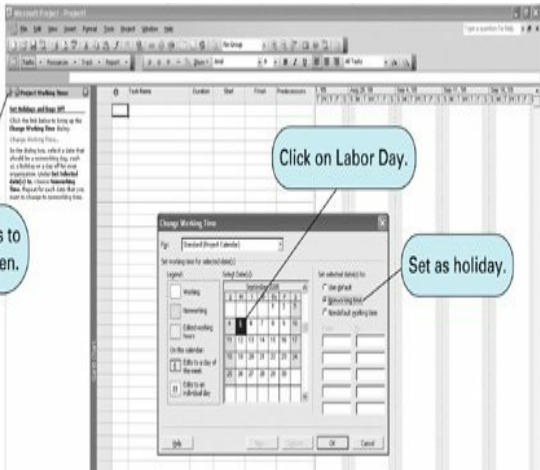
The screen in [Exhibit 8.5](#) shows how to set holidays and days

off. Notice that in this screen for September, we have selected Labor Day as a nonworking time.

Exhibit 8.5.

(This item is displayed on page
347 in the print version)

[\[View full size image\]](#)



The third step in the "Tasks" menu is "List the tasks in the project." The tasks for our house-building project are

shown in [Exhibit 8.6](#). Notice that we have indicated the duration of each task in the "Duration" column. For example, to enter the duration of the first activity, you would type "3 months" in the duration column. Notice that all the tasks show a start date of August 31, 2005, because we haven't set the task sequencing (i.e., precedence relationships) yet. The bars on the far right are for a Gantt chart, but because we haven't established the task sequencing yet, the chart shows nothing but bars

across the sheet.

[Page 347]

Exhibit 8.6.

[\[View full size image\]](#)

Click on "View" to
access Gantt chart.



The next step is "Schedule tasks," which we do by specifying the predecessor and successor activities in our

network. This is done by using the buttons under "Link dependent tasks" in [Exhibit 8.7](#). For example, to show that activity 1 precedes activity 2, we put the cursor on activity 1 and then hold down the "Ctrl" key while clicking on activity 2. These two activities should be shaded. Next, we click on the icon on the left side of the spreadsheet to create a finish-to-start link. This makes a finish-to-start link between the two activities, which means that activity 2 cannot start until activity 1 is finished. It

creates the precedence relationship between these two activities, as shown under the "Predecessor" column in [Exhibit 8.7](#).

Exhibit 8.7.

(This item is displayed on page 348 in the print version)

[\[View full size image\]](#)

Click on "Format," then "Timescale" to scale Gantt chart to fit page.

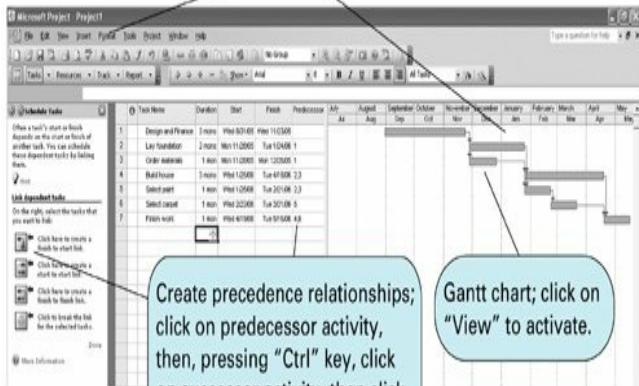


Exhibit 8.7 also shows the

completed Gantt chart for our network. The Gantt chart is accessed by clicking on the "View" button on the toolbar at the top of the screen and then clicking on "Gantt chart." You may also need to alter the time frame to get all of the Gantt chart on your screen, as shown in [Exhibit 8.7](#). You can do this by clicking on the "Format" button on the toolbar at the top of the spreadsheet and then clicking on the "Timescale" option. This results in a window from which you can adjust the timescale from days to months,

as shown in [Exhibit 8.7](#). You can show the critical path (which will appear in red) by again clicking on "Format" on the toolbar and then activating "Gantt Chart Wizard," which allows you to highlight the critical path, among other options.

[Page 348]

To see the project network, click on the "View" button again on the toolbar and then click on "Network Diagram." [Exhibit 8.8](#) shows the project

network, with the critical path highlighted in red. We used the "Zoom" option under the "View" menu to get the entire network on the screen. (We also removed the "Tasks" menu on the left side of the screen for the moment to show the complete network; it can be restored by clicking on the "Tasks" button again.)

Exhibit 8.8.

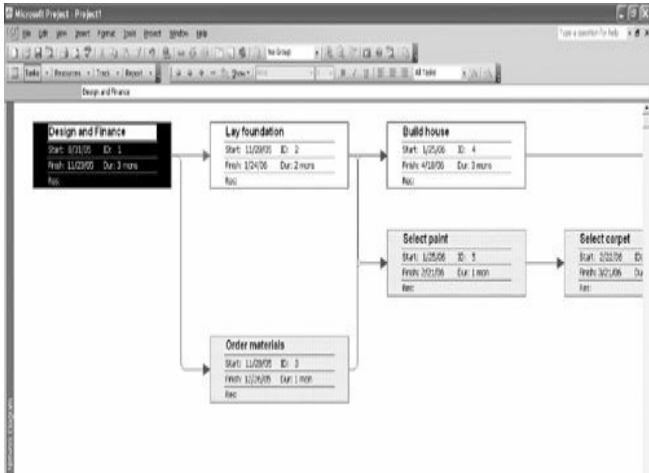
[\[View full size image\]](#)

the start and finish dates and the activity number.

[Page 349]

Exhibit 8.9.

[\[View full size image\]](#)



Microsoft Project has many additional tools and features for project updating and resource management. As an example,

we will demonstrate one feature that updates the project schedule. First, we double-click on the first task, "Design and finance," which results in the window labeled Task Information, shown in [Exhibit 8.10](#). On the "General" tab, we have entered 100% in the "Percent complete" window, meaning this activity has been completed. We have also indicated that activities 2 and 3 have been completed, while activity 4 is 60% complete and activity 5 is 20% complete, by double-clicking on each of

these activities and repeating the process. The resulting screen is shown in [Exhibit 8.11](#). Notice that the dark line through each Gantt chart bar indicates the degree of completion.

Exhibit 8.10.

[\[View full size image\]](#)

Task Information

General | Predecessors | Resources | Advanced | Notes | Custom Fields

Name: Design and Finance Duration: 3mo ☐ Estimated

Percent complete: 100% Priority: 500

Dates

Start: Wed 8/31/05 Finish: Wed 11/23/05

☐ Hide task bar
☐ Roll up Gantt bar to summary

Help OK Cancel

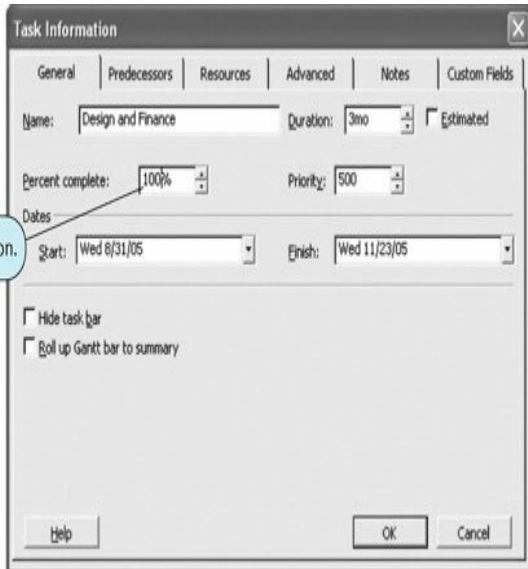
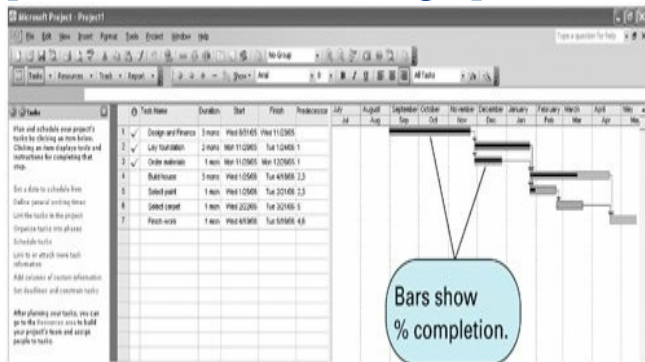


Exhibit 8.11.

[\[View full size image\]](#)



PERT Analysis with Microsoft Project

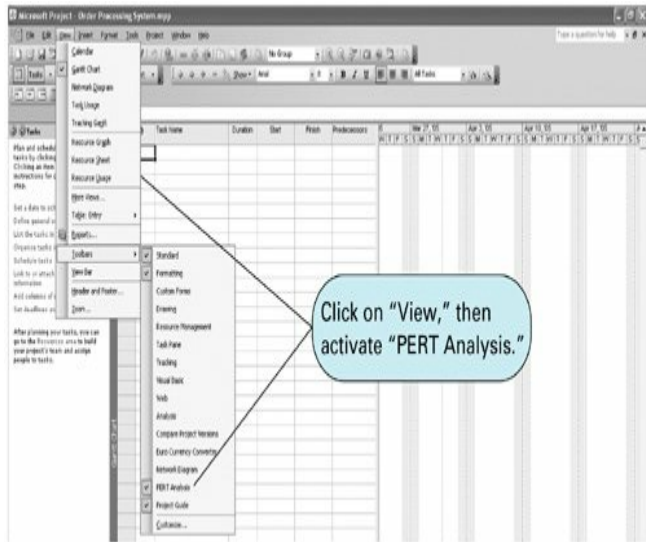
A PERT network with three time estimates can also be developed by using Microsoft Project. We will demonstrate this capability by using our order processing system project network for the Southern Textile Company, which is shown in [Figure 8.15](#). After all the project tasks are listed, we enter the three activity time estimates by clicking on the "PERT Entry Sheet" button on

the toolbar. If this button is not on your toolbar, you must add it by clicking on "View" and then from the "Toolbars" options selecting "PERT Analysis," as shown in [Exhibit 8.12](#). (If "PERT Analysis" is still not available to you, you must add it through "COM Add-ins.")

Exhibit 8.12.

(This item is displayed on page 350 in the print version)

[\[View full size image\]](#)



[Exhibit 8.13](#) shows the PERT time estimate entries for the activities in our order

processing system project example, and it shows the activity durations based on the three time estimates for each activity. [Exhibit 8.14](#) shows the precedence relationships and the project Gantt chart. [Exhibit 8.15](#) shows the project network.

Exhibit 8.13.

(This item is displayed on page 350 in the print version)

[\[View full size image\]](#)

Click on the PERT entry sheet to enter three time estimates.

PERT
add-ins

Microsoft Project - Order Processing System.mpp

Task PERT Add-ins

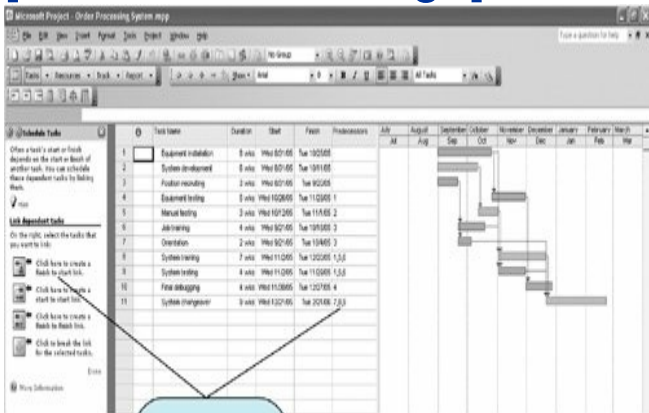
Task Name Duration Optimistic (o) Expected (e) Pessimistic (p)

1	Equipment installation	8 wks	8 wks	8 wks	10 wks
2	System development	8 wks	3 wks	8 wks	8 wks
3	Problem-solving	3 wks	1 wk	3 wks	5 wks
4	Equipment testing	5 wks	2 wks	4 wks	12 wks
5	System testing	3 wks	2 wks	3 wks	4 wks
6	Job training	4 wks	2 wks	4 wks	5 wks
7	Operation	2 wks	2 wks	2 wks	2 wks
8	System testing	7 wks	3 wks	7 wks	11 wks
9	System testing	4 wks	2 wks	4 wks	6 wks
10	Final debugging	4 wks	1 wk	4 wks	7 wks
11	System changeover	3 wks	1 wk	10 wks	12 wks

Click on the PERT
calculator to compute
activity duration.

Exhibit 8.14.

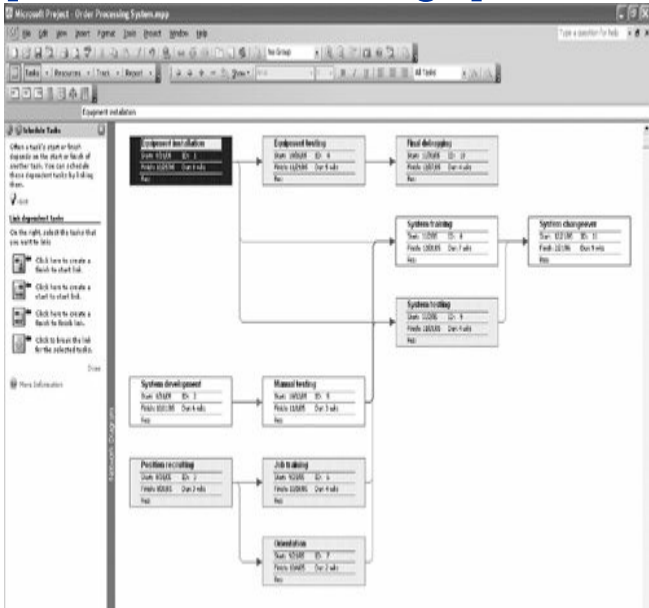
[View full size image]



Enter precedence relationships.

Exhibit 8.15.

[\[View full size image\]](#)



Project Crashing and TimeCost Trade-off

To this point we have demonstrated the use of CPM and PERT network analysis for determining project time schedules. This in itself is valuable to a manager planning a project. However, in addition to scheduling projects, a project manager is frequently confronted with the problem of having to reduce the scheduled completion time of a project to

meet a deadline. In other words, the manager must finish the project sooner than indicated by the CPM or PERT network analysis.

Project duration can be reduced by assigning more labor to project activities, often in the form of overtime, and by assigning more resources (material, equipment, etc.). However, additional labor and resources cost money and hence increase the overall project cost. Thus, the decision to reduce the project duration must be based on an analysis

of the *trade-off* between time and cost.

Project crashing is a method for shortening project duration by reducing the time of one or more of the critical project activities to a time that is less than the normal activity time. This reduction in the normal activity times is referred to as *crashing*. Crashing is achieved by devoting more resources, measured in terms of dollars, to the activities to be crashed.

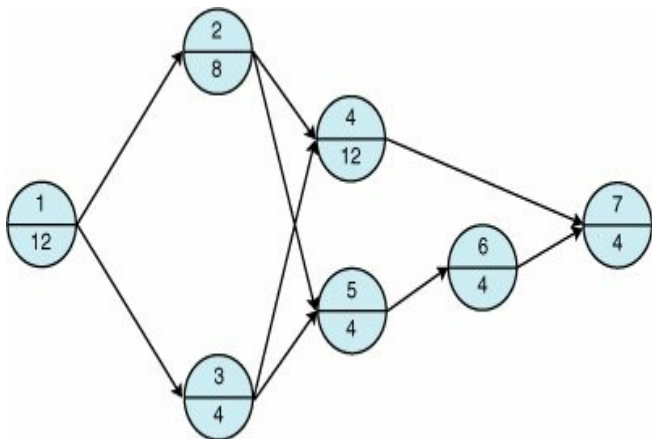
Project crashing

shortens the project time by reducing critical activity times at a cost.

To demonstrate how project crashing works, we will employ the network for constructing a house shown in [Figure 8.8](#). This network is repeated in [Figure 8.19](#), except that the activity times previously shown as months have been converted to weeks. Although this example

network encompasses only single-activity time estimates, the project crashing procedure can be applied in the same manner to PERT networks with probabilistic activity time estimates.

Figure 8.19. The project network for building a house



In [Figure 8.19](#), we will assume that the times (in weeks) shown on the network activities are the *normal activity times*. For example, normally 12

weeks are required to complete activity 1. Furthermore, we will assume that the cost required to complete this activity in the time indicated is \$3,000. This cost is referred to as the *normal activity cost*. Next, we will assume that the building contractor has estimated that activity 1 can be completed in 7 weeks, but it will cost \$5,000 to complete the activity instead of \$3,000. This new estimated activity time is known as the *crash time*, and the revised cost is referred to as the *crash cost*.

Activity 1 can be crashed a total of 5 weeks (normal time crash time = $12 - 7 = 5$ weeks), at a total crash cost of \$2,000 (crash cost normal cost = $\$5,000 - \$3,000 = \$2,000$).

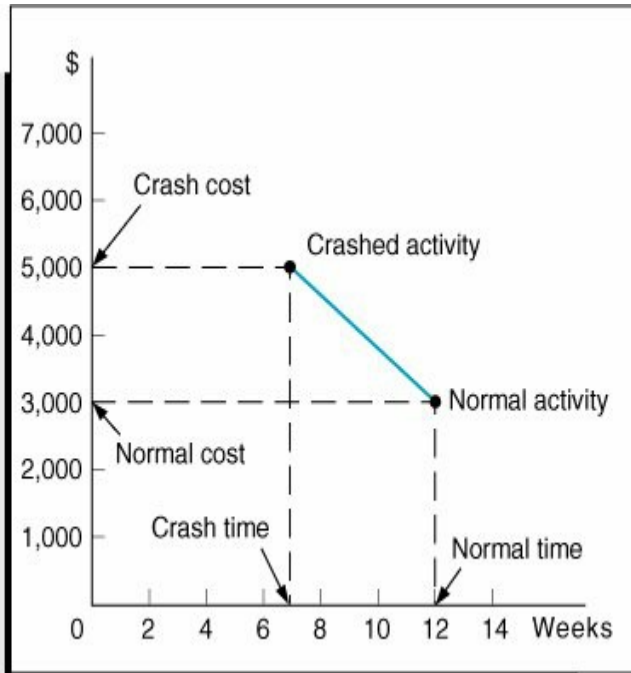
Dividing the total crash cost by the total allowable crash time yields the crash cost per week:

$$\frac{\text{total crash cost}}{\text{total crash time}} = \frac{\$2,000}{5} = \$400 \text{ per week}$$

If we assume that the relationship between crash cost and crash time is linear, then

activity 1 can be crashed by any amount of time (not exceeding the maximum allowable crash time) at a rate of \$400 per week. For example, if the contractor decided to crash activity 1 by only 2 weeks (for an activity time of 10 weeks), the crash cost would be \$800 (\$400 per week x 2 weeks). The linear relationships between crash cost and crash time and between normal cost and normal time are illustrated in [Figure 8.20](#).

Figure 8.20. Timecost relationship for crashing activity 1.



Crash cost and crash time have a linear relationship.

The normal times and costs, the crash times and costs, the total allowable crash times, and the crash cost per week for each activity in the network in [Figure 8.19](#) are summarized in [Table 8.4](#).

**Table 8.4. Normal activity a
for the network in [Figu](#)**

(This item is displayed on page 254 in th

Activity	Normal Time (weeks)	Crash Time (weeks)	Normal Cost	Crash Cost
1	12	7	\$ 3,000	\$ 5,000
2	8	5	2,000	3,500
3	4	3	4,000	7,000
4	12	9	50,000	71,000
5	4	1	500	1,100
6	4	1	500	1,100

7	4	3	15,000	22,000
			\$75,000	\$110,750

Recall that the critical path for the house-building network encompassed activities $1 \rightarrow 2 \rightarrow 4 \rightarrow 7$, and the project duration was 9 months, or 36 weeks. Suppose that the home builder needed the house in 30 weeks and wanted to know how much extra cost would be

incurred to complete the house in this time. To analyze this situation, the contractor would crash the project network to 30 weeks, using the information in [Table 8.4](#).

As activities are crashed, the critical path may change, and several paths may become critical.

The objective of project

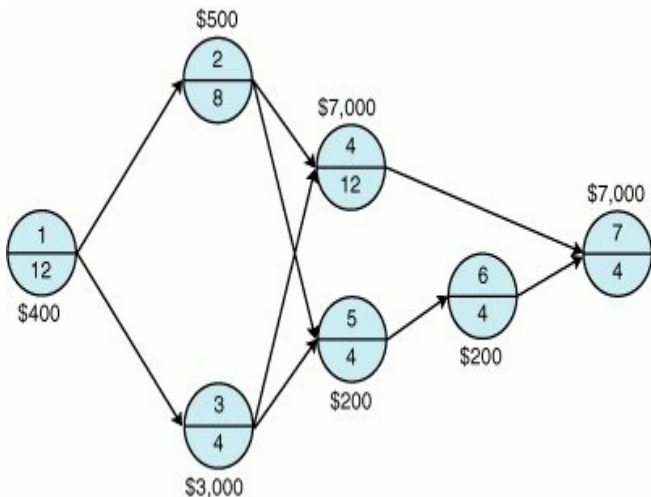
crashing is to reduce the project duration while minimizing the cost of crashing. Because the project completion time can be shortened only by crashing activities on the critical path, it may turn out that not all activities have to be crashed. However, as activities are crashed, the critical path may change, requiring crashing of previously noncritical activities to further reduce the project completion time.

We start the crashing process by looking at the critical path and seeing which activity has the minimum crash cost per week. Observing [Table 8.4](#) and [Figure 8.21](#), we see that on the critical path, activity 1 has the minimum crash cost of \$400. Thus, activity 1 will be reduced as much as possible. [Table 8.4](#) shows that the maximum allowable reduction for activity 1 is 5 weeks, *but we can reduce activity 1 only to the point where another path becomes critical*. When two paths simultaneously become

critical, activities on both must be reduced by the same amount. (If we reduce the activity time beyond the point where another path becomes critical, we may incur an unnecessary cost.) This last stipulation means that we must keep up with all the network paths as we reduce individual activities, a condition that makes manual crashing very cumbersome. Later we will demonstrate an alternative method for project crashing, using linear programming; however, for the moment we

will pursue this example in order to demonstrate the logic of project crashing.

Figure 8.21. Network with normal activity times and weekly activity crashing costs

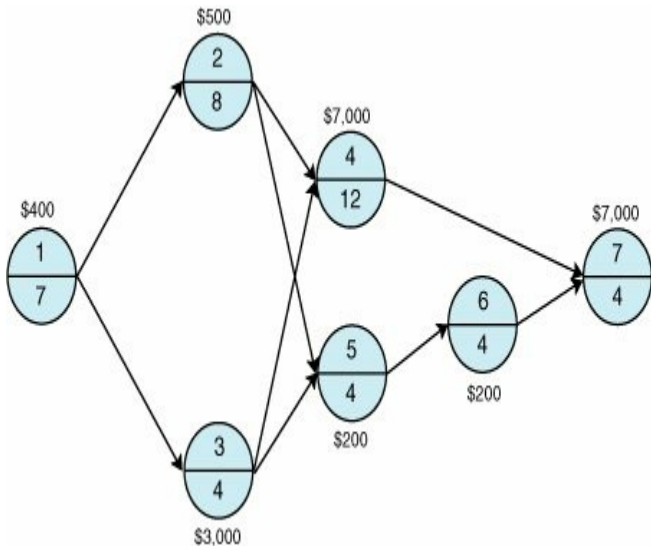


It turns out that activity 1 can be crashed by the total amount of 5 weeks without another path's becoming critical

because activity 1 is included in all four paths in the network. Crashing this activity results in a revised project duration of 31 weeks, at a crash cost of \$2,000. The revised network is shown in [Figure 8.22](#).

[Page 355]

Figure 8.22. Revised network with activity 1 crashed



This process must now be repeated. The critical path in [Figure 8.22](#) remains the same,

and the new minimum activity crash cost on the critical path is \$500 for activity 2. Activity 2 can be crashed a total of 3 weeks, but because the contractor desires to crash the network to only 30 weeks, we need to crash activity 2 by only 1 week. Crashing activity 2 by 1 week does not result in any other path's becoming critical, so we can safely make this reduction. Crashing activity 2 to 7 weeks (i.e., a 1-week reduction) costs \$500 and reduces the project duration to 30 weeks.

The extra cost of crashing the project to 30 weeks is \$2,500. Thus, the contractor could inform the customer that an additional cost of only \$2,500 would be incurred to finish the house in 30 weeks.

As indicated earlier, the manual procedure for crashing a network is very cumbersome and generally unacceptable for project crashing. It is basically a trial-and-error approach that is useful for demonstrating the logic of crashing; however, it quickly becomes unmanageable for larger networks. This

approach would have become difficult if we had pursued even the house-building example to a crash time greater than 30 weeks.

Project Crashing with QM for Windows

QM for Windows also has the capability to crash a network *completely*. In other words, it crashes the network by the maximum amount possible. In our house-building example in the previous section, we

crashed the network to only 30 weeks, and we did not consider by how much the network could have actually been crashed. Alternatively, QM for Windows crashes a network by the maximum amount possible. The QM for Windows solution for our house-building example is shown in [Exhibit 8.16](#). Notice that the network has been crashed to 24 weeks, at a total crash cost of \$31,500.

Exhibit 8.16.

[\[View full size image\]](#)

Project Management (PERT/CPM) Results							
House Building Example Solution							
	Normal time	Crash time	Normal Cost	Crash Cost	Crash cost/wk	Crash by	Crashing cost
Project	36	24					
1	12	7	3000	5000	400	5	2000
2	8	5	2000	3500	500	3	1500
3	4	3	4000	7000	3000	0	0
4	12	9	50000	71000	7000	3	21000
5	4	1	500	1100	200	0	0
6	4	1	500	1100	200	0	0
7	4	3	15000	22000	7000	1	7000
TOTALS			75000				31500

The General Relationship of Time and Cost

In our discussion of project crashing, we demonstrated how the project critical path time could be reduced by increasing expenditures for labor and direct resources. The implicit objective of crashing was to reduce the scheduled completion time for its own sake that is, to reap the results of the project sooner. However, there may be other important reasons for reducing project

time. As projects continue over time, they consume various *indirect costs*, including the cost of facilities, equipment, and machinery; interest on investment; utilities, labor, and personnel costs; and the loss of skills and labor from members of the project team who are not working at their regular jobs. There also may be direct financial penalties for not completing a project on time. For example, many construction contracts and government contracts have penalty clauses for exceeding

the project completion date.

[Page 357]

Management Science

Application: Reconstructing the Pentagon After 9/11

(This item is displayed on page 356 in
the print version)

On September 11, 2001, at 9:37 A.M., American Airlines flight 77, which had been hijacked by terrorists, was flown into the west face of the Pentagon in Arlington, Virginia. More than 400,000 square feet of office space was destroyed, and an additional 1.6 million square feet was damaged. Almost immediately, the "Phoenix Project" to restore the Pentagon was initiated. A deadline of 1 year was established for project completion, which required the demolition and removal of the

destroyed portion of the building, followed by the building restoration, including restoration of the limestone facade. The Pentagon consists of five rings of offices (housing 25,000 employees) that emanate from the center of the building; ring A is the innermost ring, and ring E is the outermost. Ten corridors radiate out from the building's hub, bisecting the ring, and forming the Pentagon's five distinctive wedges. At the time of the attack, the Pentagon was undergoing a 20-year, \$1.2 billion renovation program, and the renovation of Wedge 1, which was demolished in the 9/11 attack, was nearing completion. As a result, the Phoenix Project leaders were able to use the Wedge 1 renovation project structure and plans as a basis for their own reconstruction plan and schedule, saving much time in the process.

Project leaders were in place and able to assign resources on the very day of the attack. The project included more than 30,000 activities and a 3,000-member project team and required 3 million person-hours of work. Over 56,000 tons of contaminated debris was removed from the site, 2.5 million pounds of limestone was used to reconstruct the facade (using the original drawings from 1941), 21,000 cubic yards of concrete was poured, and 3,800 tons of reinforcing steel was placed. The Phoenix Project was completed almost a month ahead of schedule and nearly \$194 million under the original budget estimate of \$700 million.



Source: N. Bauer, "Rising from the Ashes," *PM Network* 18, no. 5 (May 2004): 2432.

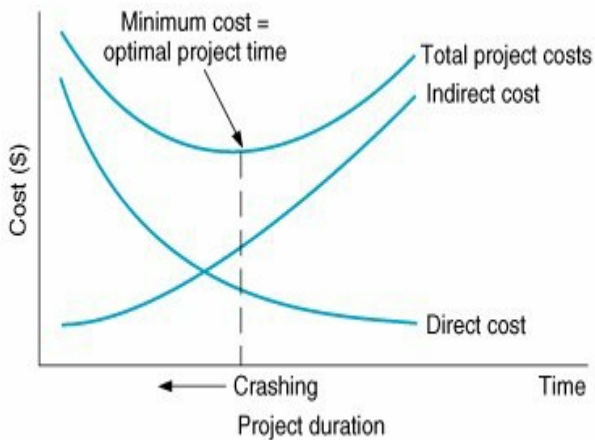
*Crash costs increase
as project time
decreases; indirect*

*costs increase as
project time
increases.*

In general, project crash costs and indirect costs have an inverse relationship; crash costs are highest when the project is shortened, whereas indirect costs increase as the project duration increases. This timecost relationship is illustrated in [Figure 8.23](#). The best, or optimal, project time is

at the minimum point on the total cost curve.

Figure 8.23. The timecost trade-off



Formulating the CPM/PERT Network as a Linear Programming Model

First we will look at the linear programming formulation of the general CPM/PERT network model and then at the formulation of the project crashing network.

We will formulate the linear programming model of a

CPM/PERT network, using the AOA convention. As the first step in formulating the linear programming model, we will define the decision variables. In our discussion of CPM/PERT networks using the AOA convention, we designated an activity by its start and ending node numbers. Thus, an activity starting at node 1 and ending at node 2 was referred to as activity $1 \rightarrow 2$. We will use a similar designation to define the decision variables of our linear programming model.

We will use a different

scheduling convention. Instead of determining the earliest activity start time for each activity, we will use the *earliest event time* at each node. This is the earliest time that a node (i or j) can be realized. In other words, it is the earliest time that the event the node represents, either the completion of all the activities leading into it or the start of all activities leaving it, can occur. Thus, for an activity $i \rightarrow j$, the earliest event time of node i will be x_i , and the earliest event time of node j will be x_j .

The objective function for the linear programming model is to minimize project duration.

The objective of the project network is to determine the earliest time the project can be completed (i.e., the critical path time). We have already determined from our discussion of CPM/PERT network analysis that the earliest event time of

the last node in the network equals the critical path time. If we let x_i equal the earliest event times of the nodes in the network, then the objective function can be expressed as

$$\text{minimize } Z = \sum_i x_i$$

[Page 358]

Because the value of Z is the sum of all the earliest event times, it has no real meaning; however, it will ensure the earliest event time at each

node.

Next, we must develop the model constraints. We will define the time for activity $i \rightarrow j$ as t_{ij} (as we did earlier in this chapter). From our previous discussion of CPM/PERT network analysis, we know that the difference between the earliest event time at node j and the earliest event time at node i must be at least as great as the activity time t_{ij} . A set of constraints that expresses this condition is defined as

$$x_j - x_i \geq t_{ij}$$

The general linear programming model of formulation of a CPM/PERT network can be summarized as

$$\text{minimize } Z = \sum_i x_i$$

subject to

$$x_j - x_i \geq t_{ij}, \text{ for all activities } i \rightarrow j$$

$$x_i, x_j \geq 0$$

where

x_i = earliest event time of node i

x_j = earliest event time of node j

t_{ij} = time of activity $i \rightarrow j$

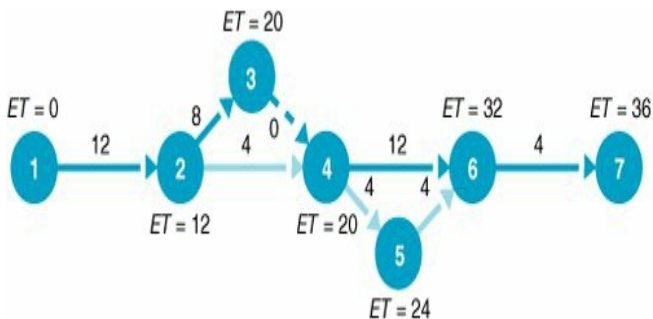
The solution of this linear programming model will indicate the earliest event time of each node in the network and the project duration.

As an example of the linear programming model formulation and solution of a project network, we will use the house-building network that we used to demonstrate project crashing. This network,

with activity times in weeks and earliest event times, is shown in [Figure 8.24](#).

Figure 8.24.

**CPM/PERT network
for the house-
building project, with
earliest event times**



The linear programming model for the network in [Figure 8.24](#) is

minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
subject to

$$x_2 - x_1 \geq 12$$

$$x_3 - x_2 \geq 8$$

$$x_4 - x_2 \geq 4$$

$$x_4 - x_3 \geq 0$$

$$x_5 - x_4 \geq 4$$

$$x_6 - x_4 \geq 12$$

$$x_6 - x_5 \geq 4$$

$$x_7 - x_6 \geq 4$$

$$x_i, x_j \geq 0$$

Notice in this model that there is a constraint for each activity in the network.

Solution of the CPM/PERT Linear Programming Model with Excel

The linear programming model of the CPM/PERT network in the preceding section enables us to use Excel to schedule the project. [Exhibit 8.17](#) shows an Excel spreadsheet set up to determine the earliest event times for each node that is, the x_i and x_j values in our linear programming model for our house-building example. The

earliest start times are in cells **B6:B12**. Cells **F6:F13** contain the model constraints. For example, cell F6 contains the constraint formula **=B7B6**, and cell F7 contains the formula **=B8B7**. These constraint formulas will be set $\geq$ to the activity times in column G when we access Solver. (Also, because activity 3 $\rightarrow$ 4 is a dummy, a constraint for **F9=0** must be added to Solver.)

Exhibit 8.17.

[\[View full size image\]](#)

Microsoft Excel - Libro18.15.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

100%

Libro18.15.xls

1 House Building Project

Node	Earliest Event Time	Activity Node	Activity Time	Network Activity Time
1		1 2	0	12
2		2 3	0	8
3		2 4	0	4
4		3 4	0	0
5		4 5	0	4
6		4 6	0	12
7		5 6	0	4
8	0	6 7	0	4

= B7-B6

Decision variables,
B6:B12

Solver is accessed from the "Tools" menu located at the top of the spreadsheet. The Solver Parameters window with the

model data is shown in [Exhibit 8.18](#). Notice that the objective is to minimize the project duration in cell B13, which actually contains the earliest event time for node 7.

Exhibit 8.18.

Solver Parameters [X]

Set Target Cell: [icon]

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells: [icon]

Subject to the Constraints:

\$B\$6:\$B\$12 >= 0

\$F\$6:\$F\$13 >= \$G\$6:\$G\$13

\$F\$9 = 0

^

▼

The solution is shown in [Exhibit 8.19](#). Notice that the earliest time at each node is given in

cells **B6:B12**, and the total project duration is 36 weeks. However, this output does not indicate the critical path. The critical path can be determined by accessing the sensitivity report for this problem. Recall that when you click on "Solve" from Solver, a screen comes up, indicating that Solver has reached a solution. This screen also provides the opportunity to generate several different kinds of reports, including an answer report and a sensitivity report. When we click on "Sensitivity" under the report options, the

information shown in [Exhibit 8.20](#) is provided.

[Page 360]

Exhibit 8.19.

[\[View full size image\]](#)

Microsoft Excel - Exhibit8.17.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

100% 400

Worksheet: Exhibit8.17.xls

8 *9.A616.612

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	House Building Project																	
2																		
3																		
4	Earliest		Activity		Actual	Network												
5	Node	Event Time	Node	Node	Time	Activity												
6	1	0	1	2	12	12												
7	2	12	2	3	8	8												
8	3	20	2	4	8	4												
9	4	20	3	4	0	0												
10	5	24	4	5	4	4												
11	6	32	4	6	12	12												
12	7	28	5	6	0	4												
13		344	6	7	4	4												
14																		
15																		

Exhibit 8.20.

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
\$F\$6	Time	12	1	12	1E+30	12
\$F\$7	Time	8	1	8	1E+30	4
\$F\$8	Time	8	0	4	4	1E+30
\$F\$9	Time	0	1	0	1E+30	4
\$F\$10	Time	8	0	4	4	1E+30
\$F\$11	Time	12	1	12	1E+30	4
\$F\$12	Time	4	0	4	4	1E+30
\$F\$13	weeks Time	4	1	4	1E+30	36

Indicates critical path

The information in which we are interested is the shadow price for each of the activity

constraints. (Remember that to get the shadow prices on the sensitivity report, you must click on "Options" from Solver and then select "Assume Linear Model." This is covered in [Chapter 3](#), on solving linear programming problems with Excel.) The shadow price for each activity will be either 1 or 0. A positive shadow price of 1 for an activity means that you can reduce the overall project duration by an amount with a corresponding decrease by the same amount in the activity duration. Alternatively, a

shadow price of 0 means that the project duration will not change, even if you change the activity duration by some amount. This means that those activities with a shadow price of 1 are on the critical path. Cells F6, F7, F9, F11, and F13 have shadow prices of 1, and referring back to [Exhibit 8.19](#), we see that these cells correspond to activities $1 \rightarrow 2$, $2 \rightarrow 3$, $3 \rightarrow 4$, $4 \rightarrow 6$, and $6 \rightarrow 7$, which are the activities on the critical path.

Project Crashing with

Linear Programming

The linear programming model required to perform project crashing analysis differs from the linear programming model formulated for the general CPM/PERT network in the previous section. The linear programming model for project crashing is somewhat longer and more complex.

The objective of the project crashing model is to minimize the cost of crashing.

The objective for our general linear programming model was to minimize project duration; the objective of project crashing is to minimize the cost of crashing, given the limits on how much individual activities can be crashed. As a result, the general linear programming model formulation must be expanded to include crash times and cost. We will continue to define the earliest event times for activity $i \rightarrow j$ as x_i and x_j . In addition, we will

define the amount of time each activity $i \rightarrow j$ is crashed as y_{ij} . Thus, the decision variables are defined as

[Page 361]

x_i = earliest event time of node i

x_j = earliest event time of node j

y_{ij} = amount of time by which activity $i \rightarrow j$ is crashed (i.e., reduced)

The objective of project crashing is to reduce the project duration at the minimum possible crash cost. For our house-building network, the objective function is written as

$$\text{minimize } Z = \$400y_{12} + 500y_{23} + 3,000y_{24} + 200y_{45} + 7,000y_{46} + 200y_{56} + 7,000y_{67}$$

The objective function coefficients are the activity crash costs per week from [Table 8.4](#); the variables y_{ij} indicate the number of weeks each

activity will be reduced. For example, if activity $1 \rightarrow 2$ is crashed by 2 weeks, then $y_{12} = 2$, and a cost of \$800 is incurred.

The model constraints must specify the limits on the amount of time each activity can be crashed. Using the allowable crash times for each activity from [Table 8.4](#) enables us to develop the following set of constraints:

$$y_{12} \leq 5$$

$$y_{23} \leq 3$$

$$y_{24} \leq 1$$

$$y_{34} \leq 0$$

$$y_{45} \leq 3$$

$$y_{46} \leq 3$$

$$y_{56} \leq 3$$

$$y_{67} \leq 1$$

For example, the first constraint, $y_{12} \leq 5$, specifies that the amount of time by which activity $1 \rightarrow 2$ is reduced cannot exceed 5 weeks.

The next group of constraints must mathematically represent the relationship between earliest event times for each activity in the network, as the constraint $x_j - x_i \geq t_{ij}$ did in our original linear programming model. However, we must now reflect the fact that activity

times can be crashed by an amount y_{ij} . Recall the formulation of the activity 1 → 2 constraint for the general linear programming model formulation in the previous section:

$$x_2 - x_1 \geq 12$$

This constraint can also be written as

$$x_1 + 12 \leq x_2$$

This latter constraint indicates that the earliest event time at

node 1 (x_1), plus the normal activity time (12 weeks) cannot exceed the earliest event time at node 2 (x_2). To reflect the fact that this activity can be crashed, it is necessary only to subtract the amount by which it can be crashed from the left-hand side of the preceding constraint:

amount activity 1 $\rightarrow$ 2 can be crashed


$$x_1 + 12 - y_{12} \leq x_2$$

This revised constraint now indicates that the earliest event time at node 2 (x_2) is

determined not only by the earliest event time at node 1 plus the activity time but also by the amount the activity is crashed. Each activity in the network must have a similar constraint, as follows:

[Page 362]

$$x_1 + 12 - y_{12} \leq x_2$$

$$x_2 + 8 - y_{23} \leq x_3$$

$$x_2 + 4 - y_{24} \leq x_4$$

$$x_3 + 0 - y_{34} \leq x_4$$

$$x_4 + 4 - y_{45} \leq x_5$$

$$x_4 + 12 - y_{46} \leq x_6$$

$$x_5 + 4 - y_{56} \leq x_6$$

$$x_6 + 4 - y_{67} \leq x_7$$

Finally, we must indicate the project duration we are seeking (i.e., the crashed project time). Because the housing contractor wants to crash the project from the 36-week normal critical path time to 30 weeks, our final model constraint specifies that the earliest event time at node 7 should not exceed 30 weeks:

$$x_7 \leq 30$$

The complete linear programming model formulation is summarized as follows:

minimize $Z = \$400y_{12} + 500y_{23} + 3,000y_{24} + 200y_{45} + 7,000y_{46} + 200y_{56} + 7,000y_{67}$
subject to

$$y_{12} \leq 5$$

$$y_{23} \leq 3$$

$$y_{24} \leq 1$$

$$y_{34} \leq 0$$

$$y_{45} \leq 3$$

$$y_{46} \leq 3$$

$$y_{56} \leq 3$$

$$y_{67} \leq 1$$

$$y_{12} + x_2 - x_1 \geq 12$$

$$y_{23} + x_3 - x_2 \geq 8$$

$$y_{24} + x_4 - x_2 \geq 4$$

$$y_{34} + x_4 - x_3 \geq 0$$

$$y_{45} + x_5 - x_4 \geq 4$$

$$y_{46} + x_6 - x_4 \geq 12$$

$$y_{56} + x_6 - x_5 \geq 4$$

$$y_{67} + x_7 - x_6 \geq 4$$

$$x_7 \leq 30$$

$$x_i, y_{ij} \geq 0$$

Project Crashing with Excel

Because we have been able to develop a linear programming model for project crashing, we can also solve this model by using Excel. [Exhibit 8.21](#) shows a modified version of the Excel spreadsheet we developed earlier, in [Exhibit 8.16](#), to determine the earliest event times for our CPM/PERT network for the house-building project. We have added columns H, I, and J for the

activity crash costs, the activity crash times, and the actual activity crash times. Cells **J6:J13** correspond to the y_{ij} variables in the linear programming model. The constraint formulas for each activity are included in cells **F6:F13**. For example, cell F6 contains the formula **=J6+B7B6**, and cell F7 contains the formula **=J7+B8B7**. These constraints and the others in column F must be set $\geq$ to the activity times in column G. The objective function formula in

cell B16 is shown on the formula bar at the top of the spreadsheet. The crashing goal of 30 weeks is included in cell B15.

[Page 363]

Exhibit 8.21.

[\[View full size image\]](#)

Microsoft Excel - Exhibit8.20.xls

File Edit View Insert Format Tools Data Window Help Solver

Formulas: =SUMPRODUCT(\$B\$13:\$F\$13)

8%

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	House Building Project																	
2																		
3																		
4		Earliest			Activity	Actual	Network	Activity	Activity	Activity								
5	Node	Event Time		Node	Node	Time	Time	Crash	Crash	Crash								
6	1			1	2	0	12	400	5	0								
7	2			2	3	0	8	600	3	0								
8	3			2	4	0	4	3000	1	0								
9	4			3	4	0	0	0	0	0								
10	5			4	5	0	4	200	3	0								
11	6			4	6	0	12	7000	3	0								
12	7			5	6	0	4	200	3	0								
13	Project time =	0	weeks	6	7	0	4	7000	1	0								
14																		
15	Project goal =	20	weeks															
16	Project cost =	0	dollars															
17																		

The problem in [Exhibit 8.21](#) is solved by using Solver, as shown in [Exhibit 8.22](#). Notice that there are two sets of variables in cells **B6:B12** and **J6:J13**. The project crashing

solution is shown in [Exhibit 8.23](#).

Exhibit 8.22.

Solver Parameters

Set Target Cell:

\$B\$16



Solve

Equal To:



Max



Min



Value of:

0

Close

By Changing Cells:

\$B\$6:\$B\$12,\$J\$6:\$J\$13



Guess

Subject to the Constraints:

\$B\$12 <= \$B\$15

\$B\$6:\$B\$12 >= 0

\$F\$6:\$F\$13 >= \$G\$6:\$G\$13

\$J\$6:\$J\$13 <= \$I\$6:\$I\$13

\$J\$6:\$J\$13 >= 0



Add

Change

Delete

Options

Reset All

Help

Exhibit 8.23.

[\[View full size image\]](#)

Microsoft Excel - Tableau08-22.xls

Type a question for help

100%

1

2

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

House Building Project

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	
1	House Building Project																		
2																			
3						Actual	Network	Activity	Activity	Actual									
4		Earliest		Activity	Activity	Time	Time	Cost	Crash	Crash	Crash								
5	Node	Event Time		Node	Node	Time	Time	Cost	Crash	Crash	Crash								
6	1	0		1	2	12	12	400	5	5									
7	2	7		2	3	8	8	600	3	1									
8	3	14		3	4	7	4	3000	1	0									
9	4	14		3	4	0	0	0	0	0									
10	5	22		4	5	8	4	200	3	0									
11	6	26		4	6	12	12	7000	3	0									
12	7	30		5	6	4	4	200	3	0									
13	Project time =		30	weeks	6	7	4	4	7000	1	0								
14																			
15	Project goal =		30	weeks															
16	Project cost =		2500	dollars															
17																			

Summary

In this chapter we discussed two of the most popular management science techniques CPM and PERT networks. Their popularity is due primarily to the fact that a network forms a picture of the system under analysis that is easy for a manager to interpret. Sometimes it is difficult to explain a set of mathematical equations to a manager, but a network often

can be easily explained. CPM/PERT has been applied in a variety of government agencies concerned with project control, including various military agencies, the National Aeronautics and Space Administration (NASA), the Federal Aviation Administration (FAA), and the General Services Administration (GSA). These agencies are frequently involved in large-scale projects involving millions of dollars and many subcontractors. Examples of such governmental projects include the development of

weapons systems, aircraft, and NASA space exploration projects. It has become a common practice for these agencies to require subcontractors to develop and use a CPM/PERT analysis to maintain management control of the myriad project components and subprojects.

CPM/PERT has also been widely applied in the private sector. Two of the major areas of application of CPM/PERT in the private sector have been research and development (R&D) and construction.

CPM/PERT has been applied to various R&D projects, such as developing new drugs, planning and introducing new products, and developing new and more powerful computer systems. CPM/PERT analysis has been particularly applicable to construction projects. Almost every type of construction project from building a house to constructing a major sports stadium to building a ship to constructing an oil pipeline has been subjected to network analysis.

Network analysis is also

applicable to the planning and scheduling of major events, such as summit conferences, sports festivals, basketball tournaments, football bowl games, parades, political conventions, school registrations, and rock concerts. The availability of powerful, user-friendly project management software packages for the personal computer will serve to increase the use of this technique.

References

Levy, F., Thompson, G., and Wiest, J. "The ABC's of the Critical Path Method." *Harvard Business Review* 41, no. 5 (October 1963).

Moder, J., Phillips, C. R., and Davis, E. W. *Project Management with CPM and PERT and Precedence Diagramming*, 3rd ed. New York: Van Nostrand Reinhold, 1983.

O'Brian, J. *CPM in Construction Management*. New York: McGraw-Hill, 1965.

Wiest, J. D., and Levy, F. K. *A Management Guide to PERT/CPM*, 2nd ed. Upper Saddle River, NJ: Prentice Hall, 1977.

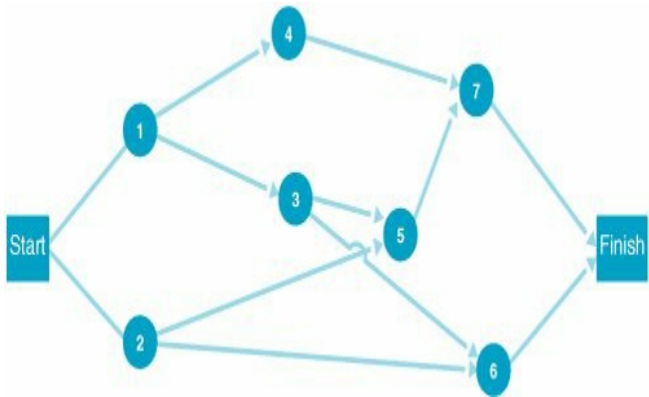
Example Problem Solution

The following example will illustrate CPM/PERT network analysis and probability analysis.

Problem Statement

Given the following AON network and activity time estimates, determine the expected project completion

time and variance, as well as the probability that the project will be completed in 28 days or less:



Time Estimates (weeks)

Activity	a	m	b
1	5	8	17
2	7	10	13
3	3	5	7
4	1	3	5
5	4	6	8
6	3	3	3
7	3	4	5

Solution

Compute the Expected Activity Times and Variances

Using the following formulas, compute the expected time and variance for each activity:

$$t = \frac{a + 4m + b}{6}$$
$$v = \left(\frac{b - a}{6} \right)^2$$

[Page 366]

For example, the expected time and variance for activity 1 is

$$t = \frac{5 + 4(8) + 17}{6}$$
$$= 9$$
$$v = \left(\frac{17 - 5}{6} \right)^2$$
$$= 4$$

Step 1. These values and the remaining expected times and variances for each activity follow:

Activity	t	v
1	9	4
2	10	1
3	5	4/9

4

3

 $4/9$

5

6

 $4/9$

6

3

0

7

4

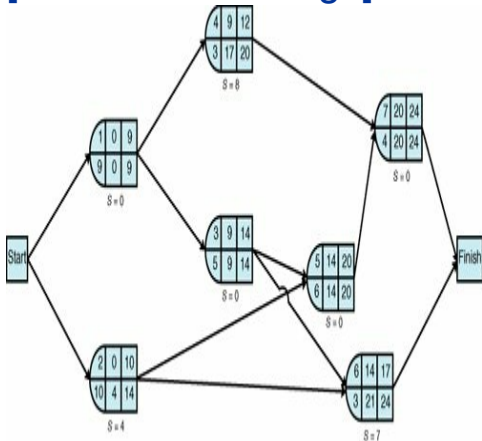
 $1/9$

**Determine the
Earliest and Latest
Times at Each Node**

The earliest and latest activity times and the activity slack are shown on the following network:

Step 2.

[\[View full size image\]](#)



Identify the Critical Path and Compute Expected Project Completion Time and Variance

After observing the foregoing network and those activities with no slack (i.e., $S = 0$), we can identify the critical path as $1 \rightarrow 3 \rightarrow 5 \rightarrow$

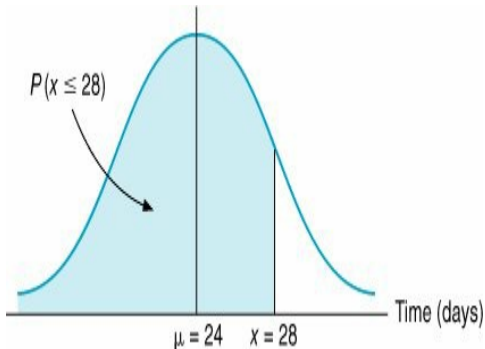
Step 3. 7. The expected project

completion time (t_p) is 24 days. The variance is computed by summing the variances for the activities in the critical path:

$$\begin{aligned}v_p &= 4 + 4/9 + 4/9 + 1/9 \\&= 5 \text{ days}\end{aligned}$$

Determine the Probability That the Project Will Be Completed in 28 Days or Less

The following normal probability distribution describes the probability analysis:



Step 4.

Compute Z by using the following formula:

$$\begin{aligned}
 Z &= \frac{x - \mu}{\sigma} \\
 &= \frac{28 - 24}{\sqrt{5}} \\
 &= 1.79
 \end{aligned}$$

The corresponding probability from [Table A.1](#) in [Appendix A](#) is .4633; thus,

$$P(x \leq 28) = .9633$$

Problems

Construct a Gantt chart for the following activities and indicate the project completion date.

Activity	Activity Predecessor
1	
2	
3	1

1.

Construct a Gantt chart for the following activities and indicate the project completion slack for each activity:

[Page 368]

Activity

**Activity
Predecessor**

1

2

2.

3

1

4

2

5

2

6

4

7

5

Construct a Gantt chart and project network for the following set of activities, compute the critical path in the network, and indicate the critical activities.

Activity**Activity
Predecessor**

1

1

2

3

1

3.

4

1

5

2

6

3

7

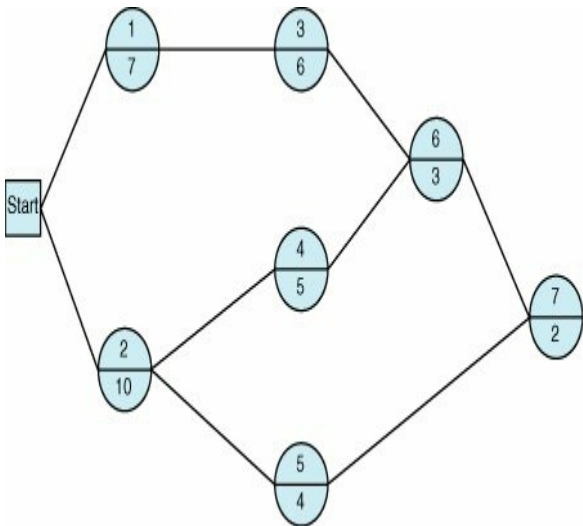
3

8

4, 5, 6

Identify all the paths in the following network, the length of each, and indicate the critical path (activity times are in weeks):

4.



For the network in [Problem 4](#), determine the earliest activity times and the latest activity times and the slack for each activity. Indicate how the critical path would be determined from this information.

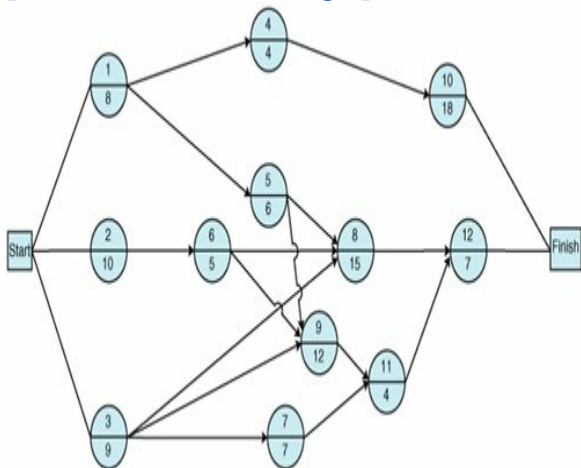
Given the following network, with activity

months, determine the earliest and late and slack for each activity. Indicate the the project duration.

[Page 369]

[\[View full size image\]](#)

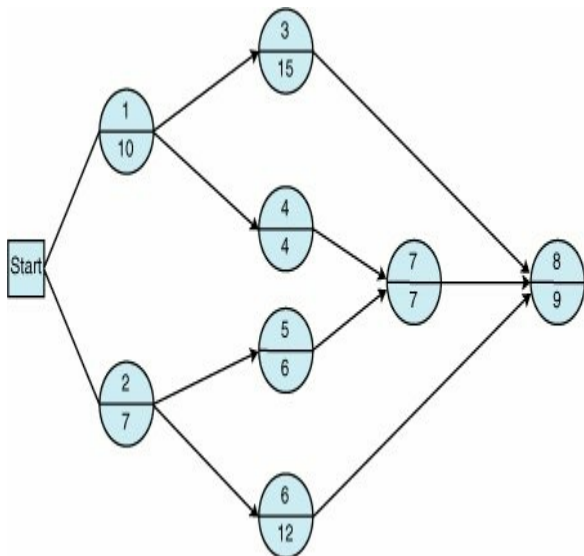
6.



[Page 370]

Given the following network, with activity weeks, determine the earliest and latest and the slack for each activity. Indicate and the project duration:

7.

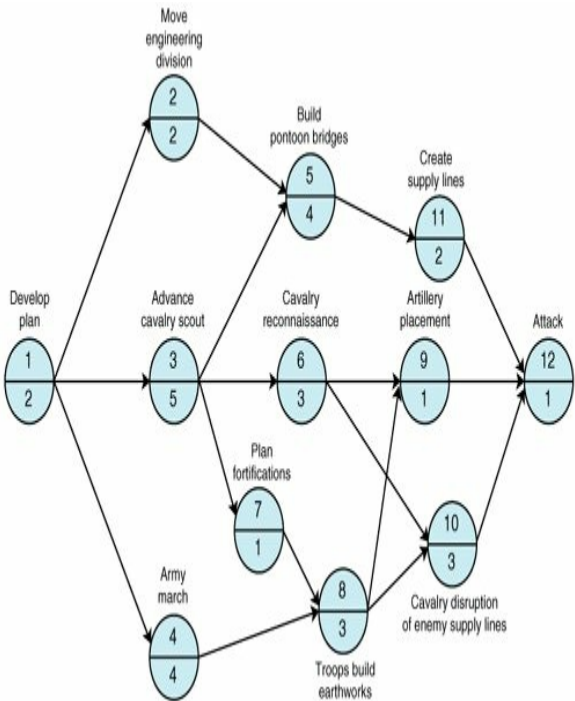


[Page 371]

In one of the little-known battles of the General Tecumseh Beauregard lost the Bull Run because his preparations were when the enemy attacked. If the critica

had been available, the general could have done better. Suppose that the following plan had been available with activity times in days, had been available.

[\[View full size image\]](#)



Determine the earliest and latest activity activity slack for the network. Indicate and the time between the general's rec

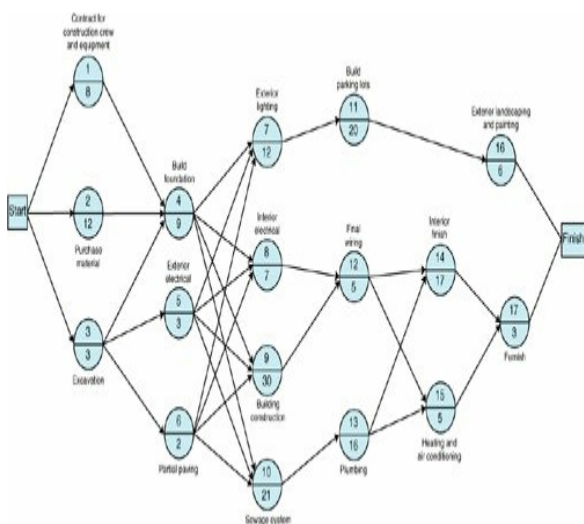
orders and the onset of battle.

A group of developers is building a new
A consultant for the developers has co
following project network and assigned
in weeks.

[Page 372]

[\[View full size image\]](#)

9.



[Page 373]

Determine the earliest and latest activity slack, critical path, and duration for the

A farm owner is going to erect a maintenance building with a connecting electrical generator and a

The activities, activity descriptions, and durations are given in the following table

Activity	Activity Description	Activity Predecessors
a	Excavate	
b	Erect building	a
c	Install generator	a
d	Install tank	a
e	Install maintenance	b

10.

equipment

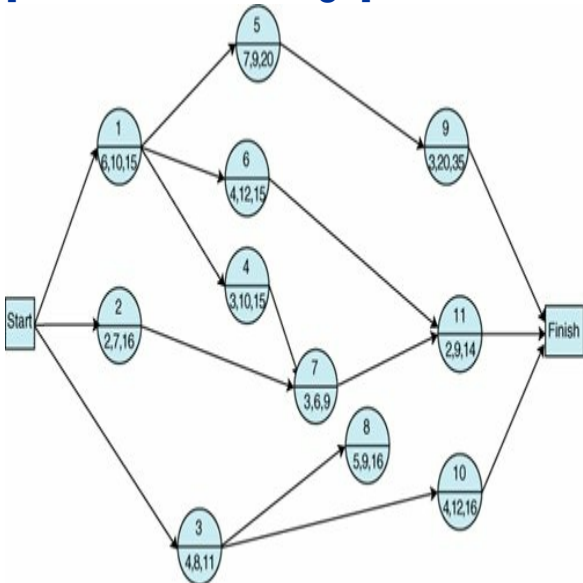
f	Connect generator and tank to building	b, c, d
g	Paint finish	b
h	Check out facility	e, f

(Notice that the activities are defined not by numbers but by activity descriptions. This form of expressing activities and precedence relationships is often used in CPM/PERT. For the network for this project, identify the critical path and determine the project duration time.

Given the following network and activity estimates, determine the expected time for each activity and indicate the critical

[\[View full size image\]](#)

11.



The Farmer's American Bank of Leesbu install a new computerized accounts sy management has determined the activ complete the project, the precedence r the activities, and activity time estimat the following table:

Activity	Activity Description	Activit Predeces
a	Position recruiting	
b	System development	

c	System training	a
---	-----------------	---

d	Equipment training	a
---	--------------------	---

12.

e	Manual system test	b, c
---	--------------------	------

f	Preliminary system changeover	b, c
---	----------------------------------	------

g	Computerpersonnel interface	d, e
---	--------------------------------	------

h	Equipment modification	d, e
---	---------------------------	------

i	Equipment testing	h
---	-------------------	---

j	System debugging and installation	f, g
k	Equipment changeover	g, i

Determine the expected project completion time, standard deviation and variance and determine the probability the project will be completed in 40 weeks or less.

The following activity time estimates are shown in the network in [Problem 6](#):

Time Estimates (m

13.	Activity	a	m
	1	4	8
	2	6	10
	3	2	10
	4	1	4
	5	3	6
	6	3	6
	7	2	8
	8	0	15

8	9	15
9	5	12
10	7	20
11	5	6
12	3	8

Determine the following:

- 1.** Expected activity times
- 2.** Earliest activity times
- 3.** Latest activity times

4. Activity slack
5. Critical path
6. Expected project duration and variance

[Page 375]

The following activity time estimates are given for the network in [Problem 8](#):

Time Estimates (days)		
Activity	a	m
1	1	2
2	1	3

3	3	5
4	3	6
5	2	4
6	2	3
7	1	1.5
8	1	3
9	1	1

14.

10

2

4

11

1

2

12

1

1

Determine the following:

- 1.** Expected activity times
- 2.** Earliest activity times
- 3.** Latest activity times
- 4.** Activity slack
- 5.** Critical path

6. Expected project duration and var

15. For the CPM/PERT network in [Problem](#) the probability that the network duratic months.

The Center for Information Technology University has outgrown its office in Ba is moving to Allen (A) Hall, which has r move will take place during the 3-week the end of summer semester and the t semester. Movers will be hired from the physical plant to move the furniture, bc and files that the faculty will pack. The (a local retail computer firm to move its computers so they won't be damaged. list of activities, their precedence relatic probabilistic time estimates for this pro.

Activity	Activity Description	Activity Predecessor
----------	----------------------	----------------------

a	Pack A offices	
---	----------------	--

b	Network A offices	
---	-------------------	--

c	Pack B offices	
---	----------------	--

d	Movers move A offices	a
---	-----------------------	---

e	Paint and clean A offices	d
---	---------------------------	---

16.

f	Move computers	b, e
g	Movers move B offices	b, c, e
h	Computer installation	f
i	Faculty move and unpack	g
j	Faculty set up computers and offices	h, i

Determine the earliest and latest start times for each activity, the critical path, and the expected project completion time. What is the probability that the center will move before the start of the fall semester?

[Page 376]

Jane and Jim Smith are going to give a party on Friday evening at 7:00 P.M. Their two children, Judy and John, are going to help them get ready. All four of them get home from work and school at 4:00 P.M. and Jim and Judy have developed a project network to help them schedule their dinner preparations. The network lists the activities, the precedence relationships, and the activity times involved in the project.

Activity

Activity

Activity

T
(

	Description	Predecessor
a	Prepare salad	
b	Prepare appetizer	
c	Dust/clean	
d	Vacuum	c
e	Prepare dessert	a, b
f	Set table	d
g	Get ice from market	f

17.

h	Prepare/start tenderloin	e
i	Cut and prepare vegetables	e
j	Jim shower and dress	g
k	Jane shower and dress	h, i, j
l	Uncork wine/water carafes	h, i, j

m	Prepare bread	k, l
---	---------------	------

n	Prepare and set out dishes	k, l
---	-------------------------------	------

o	Cut meat	k, l
---	----------	------

p	Heat vegetable dish	k, l
---	------------------------	------

q	Put out appetizers	m
---	-----------------------	---

r	Pour champagne	o
---	-------------------	---

Develop a project network and determine earliest start and finish times, activity critical path. Compute the probability that the project will be ready by 7:00 P.M.

The Stone River Textile Mill was inspected and found to be in violation of a number of OSHA regulations. The OSHA inspectors ordered the mill to (1) alter some existing machinery to make it safer (e.g., add safety guards, etc.); purchase some new machinery to replace older, dangerous machinery; and (2) alter the mill's machinery to make safer passages and entrances and exits. OSHA gave the mill 30 days to make the changes; if the changes were not made by then, the mill would be fined \$300,000.

[Page 377]

The mill determined the activities in a CPM network that would have to be completed to modernize the mill. The estimated activity times, in weeks, are given in the following table:

Activity	Activity Description	Activity Predecessor
-----------------	-----------------------------	-----------------------------

a Order new
machinery

b Plan new
physical layout

c Determine
safety changes
in existing
machinery

18.

d	Receive equipment	a
e	Hire new employees	a
f	Make plant alterations	b
g	Make changes in existing machinery	c
h	Train new employees	d, e
i	Install new machinery	d, e, f

j	Relocate old machinery	d, e, f, g
---	---------------------------	------------

k	Conduct employee safety orientation	h, i, j
---	--	---------

Construct the project network for this determine the following:

- 1.** Expected activity times
- 2.** Earliest and latest activity times a
- 3.** Critical path
- 4.** Expected project duration and vai

5. The probability that the mill will be

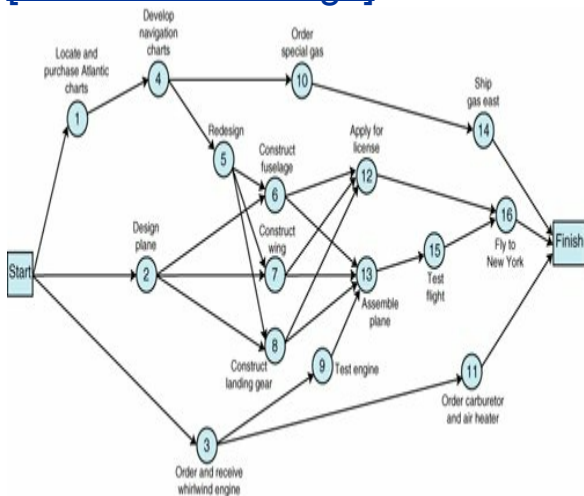
19. In the Third Battle of Bull Run, for which network was developed in [Problem 14](#), Beauregard would have won if his preparations had been completed in 15 days. What would be the probability of General Beauregard's winning the battle?

On May 21, 1927, Charles Lindbergh left Long Beach, California, for Bourget Field in Paris, completing his first transatlantic solo flight. The preparation for his flight was quite hectic, and time was wasted because several other famous pilots of the era were also planning transatlantic flights. Lindbergh had been contracted to build the *Spirit of St. Louis* only a little over 2.5 months to construct and fly it to New York for the takeoff. If CPM had been available to Charles Lindbergh, it would have been useful in helping him plan this project. Assume that a CPM/PERT chart for the construction of the *Spirit of St. Louis* has the following estimated activity times and

for the flight.

[Page 378]

[\[View full size image\]](#)



[Page 379]

		Time Estimates	
		<i>a</i>	<i>m</i>
20.	1	1	3
	2	4	6
	3	20	35
	4	4	7
	5	2	3
	6	8	12

7

10

16

8

5

9

9

1

2

10

6

8

11

5

8

12

5

10

13

4

7

14

5

7

15

5

9

16

1

3

Determine the expected project duration and the probability of completing the project in 16 days.

RusTech Tooling is a large job shop operating machine tools and dies to manufacture specialized items. The company primarily fulfills government-related contracts to produce such items as military aircraft and weapons, as well as the space program. The company was on a contract to produce a component of the shuttle fuselage assembly in a new space shuttle program. The criterion for selecting the winning bid, based on cost, is the time required to produce the part. When the company is awarded the contract, it will be required to complete the part by a specific date. Delays to the completion date specified in the contract will result in severe financial penalties. To determine the project completion time, the company has identified the project activities, their precedence relationships, and activity times. The following table:

		Time Estimate
Activity	Activity	<i>a</i>

Predecessor

a		3
b	a	2
c	a	1
d	a	4
e	b	2
f	b	5
g	c	4

21.

h	c	6
---	---	---

i	d	3
---	---	---

j	d	8
---	---	---

k	f, g	9
---	------	---

l	h, i	6
---	------	---

m	e	4
---	---	---

n	j	3
---	---	---

o	n	5
---	---	---

If RusTech wants to be 90% certain that it will win the part without incurring a penalty, what should it specify in the bid?

PM Computers is an international manufacturer of computer equipment and software. It is planning to introduce a number of new products in the next year, and it wants to develop a marketing program to accompany the product introductions. The program includes the preparation of promotional materials distributed directly by the company and by the company's marketing personnel, vendors, and sales representatives; print advertising in regional trade journals, and newspapers; and television commercials. The program also includes training programs for marketing personnel and representatives about the new products and the management team with members from

department and manufacturing areas have the following list of activities for the development of a marketing program:

7

Activity	Activity Description	Activity Predecessor
-----------------	-----------------------------	-----------------------------

a	Preliminary budget and plan approval	
---	--------------------------------------	--

b	Select marketing personnel for training	
---	---	--

c	Develop overall media plan	a
---	----------------------------	---

d	Prepare separate media plans	c
---	------------------------------	---

e	Develop training plan	c
---	-----------------------	---

f	Design training course	e
---	------------------------	---

g	Train marketing personnel	b, f
---	---------------------------	------

22.

h	Plan TV commercials with agency	d
---	---------------------------------	---

i	Draft in-house print materials	d
---	--------------------------------	---

j	Develop print advertising layouts with agency	d
---	---	---

k	Review print advertising layouts	j
---	----------------------------------	---

l	Review TV commercials	h
---	-----------------------	---

m	Review and print in-house materials	i
n	Release advertising to print media	g, i, k
o	Release TV commercials to networks	l
p	Final marketing personnel review	g, i, k
q	Run media advertising,	m, n, o

Construct the network for this project the activity schedule. Identify the critical determine the expected project duration variance. What is the probability that the be completed within 4 months?

A marketing firm is planning to conduct segment of the potential product audience its customers. The planning process for conduct the survey consists of six activities precedence relationships and activity times follows:

Activity	Activity Description	Activity Predecessors
----------	----------------------	-----------------------

-
- a Determine survey objectives
 - b Select and hire personnel
 - c Design questionnaire
 - d Train personnel
 - e Select target audience
 - f Make personnel assignments
-

23.

- 1.** Determine all paths through the network from node a to node f and the duration of each path. The path with the longest duration indicates the critical path.
- 2.** Determine the earliest and latest start and finish times for each activity.
- 3.** Determine the slack for each activity.

Lakeland-Bering Aircraft Company is preparing a contract proposal to submit to the defense department for a new military aircraft, the X-300J. The proposal is a development and production schedule for completion of the first aircraft. The project consists of three primary categories: engineering development, development and production of the aircraft airframe (e.g., the aircraft body), and development of the aircraft avionics (e.g., the electronic equipment).

electronic systems, equipment, and other components to operate and control the aircraft). For each of the project activities, with time estimates in months):

Activity	Activity Description	Activity Predecessors
1	General design	
2	Engine design	
3	Airframe design	1

24.

4	Avionics design	1
5	Develop test engine	2
6	Develop test airframe	3
7	Develop interim avionics	4
8	Develop engine	2
9	Assemble test aircraft	5, 6, 7
10	Test avionics	4

11	Conduct engine/airframe flight trials	9
12	Conduct avionics flight trials	10
13	Produce engine	8
14	Produce airframe	11
15	Produce avionics	11, 12
16	Final assembly/finish	13, 14, 15

Develop the project network and determine the expected project duration, and the probability that the project is completed within 8 years?

The Valley United Soccer Club is planning a tournament for the weekend of April 29. The club's officers know that by March 30 they must send out acceptances to teams that have applied, and that by April 15 they must send out the tournament game schedule to teams that have been selected to play. Their tentative plan is to start the initial activities for tournament preparation by sending out the application forms to prospective teams, on January 20. Following is a list of the activities, their precedence relationships, and their duration, in days:

Activity	Activity Description	Activity Predecessors
----------	----------------------	-----------------------

a	Send application forms	
---	------------------------	--

b	Get volunteer workers	
---	-----------------------	--

c	Line up referees	
---	------------------	--

d	Reserve fields	
---	----------------	--

e	Receive and	a
---	-------------	---

process forms

f	Determine age divisions	b, c, d, e
---	-------------------------	------------

g	Assign fields to divisions	f
---	----------------------------	---

25.

h	Sell program ads	b
---	------------------	---

i	Acquire donated items for team gift bags	b
---	--	---

j	Schedule games	g
---	----------------	---

k	Design ads	h
---	------------	---

l	Fill gift bags	i
m	Process team T-shirt orders	e
n	Send acceptance letters	f
o	Design and print programs	j, k, l, n
p	Put together registration boxes (programs, gifts, etc.)	
q	Send out game	s, t, u, v

q	-	j, k, l, n
	schedules	
r	Assign referees to games	j, k, l, n
s	Get trophies	f
t	Silk-screen T-shirts	m
u	Package team T-shirt orders	t

Develop a project network for the club preparation process and determine the will meet its schedule milestones and c

process according to the scheduled tour on April 29.

During a violent thunderstorm with very strong gusts in the third week of March, the building for the public radio station WVPR, atop Mount Airy in Roanoke, collapsed. This greatly reduced the strength of the station's signal in the area. The management immediately began plans to build a new tower. Following is a list of the requirements for building the new tower with optimistic (m), and pessimistic (b) time estimates. However, the sequence of the activities is designated:

Activity

Activity Description

-
- a Removal of debris
 - b Survey new tower site
 - c Procure structural steel
 - d Procure electrical/broadcasting equipment
 - e Grade tower site

26.

- f Pour concrete footings and anchors
- g Deliver and unload steel

- h Deliver and unload electrical/broadcast equipment
- i Erect tower
- j Connect electrical cables between tower and building
- k Construct storm drains and tiles
- l Backfill and grade tower site
- m Clean up
- n Obtain inspection approval

Using your best judgment, develop a C network for this project and determine project completion time. Also determine that the station signal will be back at full 3 months.

The following table contains the activities, wedding and the activity time estimates, precedence relationships between activities included:

Activity	Activity Description	
----------	----------------------	--

a	Determine date	1
b	Obtain marriage license	1
c	Select bridal attendants	3
d	Order dresses	1
e	Fit dresses	5
f	Select groomsmen	1
g	Order tuxedos	3
h	Find and rent church	6

i	Hire florist	3
j	Develop/print programs	1
k	Hire photographer	3
<u>27.</u> l	Develop guest list	1
m	Order invitations	7
n	Address and mail invitations	1
o	Compile RSVP list	3
p	Reserve reception hall	3

q	Hire caterer	2
r	Determine reception menu	1
s	Make final order	2
t	Hire band	1
u	Decorate reception hall	1
v	Wedding ceremony	0.
w	Wedding reception	0.

Using your best judgment, determine the project network, critical path, and expected project completion time. If it is January 1 and a couple is planning a wedding, what is the probability that it will be completed on time?

The following table provides the information needed to construct a project network and project completion time.

		Activity Time (weeks)	
Activity	(i, j) Predecessor	<i>Normal</i>	<i>Crash</i>
a	(1, 2)	20	8

b	$(1, 4)$		24	20
c	$(1, 3)$		14	7
d	$(2, 4)$	a	10	6
e	$(3, 4)$	c	11	5

28.

1. Construct the project network.
 2. Compute the total allowable crash activity and the crash cost per week activity.
-

3. Determine the maximum possible the network and manually crash the maximum amount.
4. Compute the normal project cost the crashed project.
5. Formulate the general linear program for this network and solve it.
6. Formulate the linear programming that would crash this network by amount and solve it.

The following table provides the information to construct a project network and pro

		(weeks)		
Activity	(i, j)	Activity Predecessor	Normal	Crash

a	$(1, 2)$		16	8
---	----------	--	----	---

b	$(1, 3)$		14	9
---	----------	--	----	---

c	$(2, 4)$	a	8	6
---	----------	---	---	---

d	$(2, 5)$	a	5	4
---	----------	---	---	---

e	$(3, 5)$	b	4	2
---	----------	---	---	---

29.

	3)			
f	(3, 6)	b	6	4
g	(4, 6)	c	10	7
h	(5, 6)	d, e	15	10

- 1.** Construct the project network.
- 2.** Manually crash the network to 28
- 3.** Formulate the general linear program for this network.
- 4.** Formulate the linear programming

that would crash this model by the amount.

- 30.** Formulate the general linear programming [Problem 4](#), and solve it.

- 31.** Formulate the general linear programming the project network for installing an operating system shown in [Figure 8.14](#) and solve

Reconstruct the example problem at the chapter as an AOA network. Assume the likely times (m) are the normal activity the optimistic times (a) are the activity Further assume that the activities have normal and crash costs:

Activity

(i, j)

(normal

32.

1	$(1, 2)$	$(\$10)$
2	$(1, 3)$	$(\$20)$
3	$(2, 3)$	$(\$40)$
4	$(2, 4)$	$(\$20)$
5	$(3, 4)$	$(\$10)$
6	$(3, 5)$	$(\$10)$
7	$(4, 5)$	$(\$30)$

1. Formulate the general linear programming model for this project network, using expected times (t).

[Page 385]

2. Formulate the linear programming model that would crash this network by a given amount.
3. Solve this model by using the computer.

The following table provides the crash cost data for the network project described in [Problem 1](#).

Activity Time (weeks)			
Activity	Normal	Crash	Normal Cost

a	9	7	\$4
b	11	9	9,
c	7	5	3,
d	10	8	3,
e	1	1	
f	5	3	1,
g	6	5	1,
h	3	3	

33.

i	1	1	
j	2	2	
k	8	6	5,

The normal activity times are considered deterministic and not probabilistic. Using crash the network to 26 weeks. Indicate how much would cost the bank and then indicate the

The following table provides the crash cost for the network project described in [Problem 6](#)

**Activity Time
(months)**

Activity	<i>Normal</i>	<i>Crash</i>	<i>Normal Cost</i>
1	8	5	\$1,000
2	10	9	1,000
3	9	7	900
4	4	2	500
5	6	3	500
6	5	4	500

34.

7	7	5	7
8	15	12	1,
9	12	10	1,
10	18	14	1,
11	4	3	5
12	7	6	8

Using the computer, crash the network
 Indicate the first critical path activities and
 of crashing the network.

Case Problem

THE BLOODLESS COUP CONCERT

John Aaron called the meeting of the Programs and Arts Committee of the Student Government Association to order. "Okay, okay, everybody, quiet down. I have an important announcement to make," he shouted above the noise. The room got quiet, and John started again, "Well, you guys, we can have the Coup."

His audience looked puzzled, and Randy Jones asked, "What coup have we scored this time, John?"

"The Coup, the Coup! You know, the rock group, the Bloodless Coup!"

Everyone in the room cheered and started talking excitedly. John stood up and waved his arms and shouted, "Hey, calm down, everybody, and listen up." The room quieted again, and everyone focused on John. "The good news is that they can come." He paused a

moment. "The bad news is that they will be here in 18 days."

The students groaned and seemed to share Jim Hastings' feelings: "No way, man. It can't be done. Why can't we put it off for a couple of weeks?"

John answered, "They're just starting their new tour and are looking for some warm-up concerts. They'll be traveling near here for their first concert date in DC and saw they had a letter from us, so they said they could come now. But that's it now or never." He

looked around the room at the solemn faces. "Look, you guys, we can handle this. Let's think of what we have to do. Come on, perk up. Let's make a list of everything we have to do to get ready and figure out how long it will take. So somebody tell me what we have to do first!"

Anna Mendoza shouted from the back of the room, "We have to find a place; you know, get an auditorium somewhere. I've done that before, and it should take anywhere from 2 days up to 7 days, most likely about 4

days."

"Okay, that's great," John said, as he wrote down the activity "Secure auditorium" on the blackboard, with the times out to the side. "What's next?"

"We need to print tickets and quick," Tracey Shea called. "It could only take 1 day if the printer isn't busy, but it could take up to 4 days if he is. It should probably take about 2 days."

"But we can't print tickets until we know where the concert will

be because of the security arrangement," Andy Taylor noted.

"Right," said John. "Get the auditorium first; then print the tickets. What else?"

"We need to make hotel and transportation arrangements for the Coup and their entourage while they're here," Jim Hastings proposed. "But we better not do that until we get the auditorium. If we can't find a place for the concert, everything falls through."

"How long do you think it will take to make the arrangements?" John asked.

"Oh, between 3 and 10 days, probably about 5, most likely," Jim answered.

"We also have to negotiate with the local union for concert employees, stagehands, and whoever else we need to hire," Reggie Wilkes interjected.

"That could take 1 day or up to 8 days, but 3 days would be my best guess."

"We should probably also hold

off on talking to the union until we get the auditorium," John added. "That will probably be a factor in the negotiations."

"After we work things out with the union, we can hire some stagehands," Reggie continued. "That could take as few as 2 days but as long as 7. I imagine it'll take about 4 days. We should also be able to get some student ushers at the same time, once we get union approval. That could take only 1 day, but it has taken 5 days in the past; 3 days is probably the most likely."

"We need to arrange a press conference," said Art Cohen, who was leaning against a wall. "This is a heavy group, big time."

"But doesn't a press conference usually take place at the hotel?" John asked.

"Yeah, that's right," Art answered. "We can't make arrangements for the press conference until we work things out with the hotel. When we do that, it should take about 3 days to set up a press conference 2 days if we're lucky,

and 4 at the most."

The room got quiet as everyone thought.

"What else?" John asked.

"Hey, I know," Annie Roark spoke up. "Once we hire the stagehands, they have to set up the stage. I think that could be done in a couple of days, but it could take up to 6 days, with 3 most likely." She paused for a moment before adding, "And we can also assign the ushers to their jobs once we hire them. That shouldn't take long,

maybe only 1 day3 days worst, probably 2 days would be a good time to put down."

"We also have to do some advertising and promotion if we want people to show up for this thing," mentioned Art nonchalantly. "I guess we need to wait until we print the tickets, so we'll have something to sell. That depends on the media, the paper, and radio stations. I've worked with this before. It could get done really quick, like 2 days, if we can make the right contacts. But it could take a lot longer, like 12

days, if we hit any snags. We probably ought to count on 6 days as our best estimate."

"Hey, if we're going to promote this, shouldn't we also have a preliminary act, some other group?" Annie asked.

"Wow, I forgot all about that!" John exclaimed. "Hiring another act will take me between 4 and 8 days; I can probably do it in 5. I can start on that right away, at the same time you guys are arranging for an auditorium." He thought for a moment. "But we really can't

begin to work on the promotion until I get the lead-in group. So what's left?"

"Sell the tickets," shouted several people at once.

"Right," said John. "We have to wait until they're printed; but I don't think we have to wait for the advertising and promotion to start, do we?"

[Page 387]

"No," Jim replied. "But we should hire the preliminary act

first so people will know what they're buying a ticket for."

"Agreed," said John. "The tickets could go quick; I suppose in the first day."

"Or," interrupted Mike Eggleston, "it could take longer. I remember 2 years ago, it took 12 days to sell out for the Cosmic Modem."

"Okay, so it's between 1 and 12 days to sell the tickets," said John, "but I think about 5 days is more likely. Everybody agree?"

The group nodded in unison, and they all turned at once to the list of activities and times John had written on the blackboard.

Use PERT analysis to determine the probability that the concert preparations will be completed in time.

Case Problem

MOORE HOUSING CONTRACTORS

Moore Housing Contractors is negotiating a deal with Countryside Realtors to build six houses in a new development. Countryside wants Moore Contractors to start in late winter or early spring, when the weather begins to moderate, and build on through the summer and into the fall. The summer

months are an especially busy time for the realty company, and it believes it can sell the houses almost as soon as they are ready, and maybe even before. The houses all have similar floor plans and are of approximately equal size; only the exteriors are noticeably different. The completion time is so critical for Countryside Realtors that it is insisting that a project management network accompany the contractor's bid for the job, with an estimate of the completion time for a house. The realtor also needs

to be able to plan its offerings and marketing for the summer. It wants each house to be completed within 45 days after it is started. If a house is not completed within this time frame, it wants to be able to charge the contractor a penalty. Mary and Sandy Moore, the president and vice president, respectively, of Moore Contractors, are concerned about the prospect of a penalty charge. They want to be very confident that they can meet the deadline for a house before they enter into

any kind of agreement with a penalty involved. (If there is a reasonable likelihood that they cannot finish a house within 45 days, they want to increase their bid to cover potential penalty charges.)

The Moores are experienced home builders, so it was not difficult for them to list the activities involved in building a house or to estimate activity times. However, they made their estimates conservatively and tended to increase their pessimistic estimates to compensate for the possibility

of bad weather and variations in their workforce. Following is a list of the activities involved in building a house and the activity time estimates:

[Page 388]

			Time (days)
Activity	Activity Description	Predecessors	<i>a m l</i>
a	Excavation, pour footers		3 4 6

b	Lay foundation	a	2	3	!
c	Frame and roof	b	2	4	!
d	Lay drain tiles	b	1	2	4
e	Sewer (floor) drains	b	1	2	!
f	Install insulation	c	2	4	!
g	Pour basement floor	e	2	3	!

h	Rough plumbing, pipes	e	2 4 7
i	Install windows	f	1 3 4
j	Rough electrical wiring	f	1 2 4
k	Install furnace, air conditioner	c, g	3 5 8
l	Exterior brickwork	i	5 6 11
	Install		

m	plasterboard, mud, plaster	j, h, k	6 8 1
n	Roof shingles, flashing	l	2 3 6
o	Attach gutter, downspouts	n	1 2 3
p	Grading	d, o	2 3 7
q	Lay subflooring	m	3 4 6
r	Lay driveway, walks,	p	4 6 1

landscape

s	Finish carpentry	q	3 5 1
---	---------------------	---	-------

t	Kitchen cabinetry, sink, and appliances	q	2 4 8
---	--	---	-------

u	Bathroom cabinetry, fixtures	q	2 3 6
---	------------------------------------	---	-------

v	Painting (interior and exterior)	t, u	4 6 1
---	--	------	-------

Finish wood

w	floors, lay carpet	v, s	2 5 8
---	-----------------------	------	-------

x	Final electrical, light fixtures	v	1 3 4
---	--	---	-------

- 1.** Develop a CPM/PERT network for Moore Housing Contractors and determine the probability that the company can complete a house within 45 days. Does it appear that the Moores might need to increase

their bid to compensate for potential penalties?

- 2.** Indicate which project activities Moore Housing Contractors should be particularly diligent to keep on schedule by making sure workers and materials are always available. Also indicate which activities the company might shift workers from as the need arises.

Chapter 9.

Multicriteria

Decision Making

[Page 390]

In all the linear programming models presented in [Chapters 2](#) through [8](#), a single objective was either maximized or minimized. However, a company or an organization often has more than one objective, which may relate to something other than profit or

cost. In fact, a company may have several criteria, that is, *multiple criteria*, that it will consider in making a decision instead of just a single objective. For example, in addition to maximizing profit, a company in danger of a labor strike might want to avoid employee layoffs, or a company about to be fined for pollution infractions might want to minimize the emission of pollutants. A company deciding between several potential research and development projects might want to consider

the probability of success of each of the projects, the cost and time required for each, and potential profitability in making a selection.

In this chapter we discuss three techniques that can be used to solve problems when they have multiple objectives: *goal programming*, the *analytical hierarchy process*, and *scoring models*. Goal programming is a variation of linear programming in that it considers more than one objective (called goals) in the objective function. Goal programming models are set up

in the same general format as linear programming models, with an objective function and linear constraints. The model solutions are very much like the solutions to linear programming models. The format for the analytical hierarchy process and scoring models, however, is quite different from that of linear programming. These methods are based on a comparison of decision alternatives for different criteria that reflects the decision maker's preferences. The result is a

mathematical "score" for each alternative that helps the decision maker rank the alternatives in terms of preferability.

Goal Programming

As we mentioned, a **goal programming** model is very similar to a linear programming model, with an objective function, decision variables, and constraints. Like linear programming, goal programming models with two decision variables can be solved graphically and by using QM for Windows and Excel. We begin our presentation of goal programming as we did linear

programming, demonstrating through an example of how to formulate a model. This will illustrate the main differences between goal and linear programming.

Model Formulation

We will use the Beaver Creek Pottery Company example again to illustrate the way a goal programming model is formulated and the differences between a linear programming model and a goal programming

model. Recall that this model was originally formulated in [Chapter 2](#) as follows:

$$\text{maximize } Z = \$40x_1 + 50x_2$$

subject to

$$x_1 + 2x_2 \leq 40 \text{ hr. of labor}$$

$$4x_1 + 3x_2 \leq 120 \text{ lb. of clay}$$

$$x_1, x_2 \geq 0$$

where

x_1 = number of bowls produced

x_2 = number of mugs produced

The objective function, Z , represents the total profit to be made from bowls and mugs, given that \$40 is the profit per bowl and \$50 is the profit per

mug. The first constraint is for available labor. It shows that a bowl requires 1 hour of labor, a mug requires 2 hours, and 40 hours of labor are available daily. The second constraint is for clay, and it shows that each bowl requires 4 pounds of clay, each mug requires 3 pounds, and the daily limit of clay is 120 pounds.

[Page 391]

This is a standard linear programming model; as such, it has a single objective function

for profit. However, let us suppose that instead of having one objective, the pottery company has several objectives, listed here *in order of importance*:

- 1.** To avoid layoffs, the company does not want to use fewer than 40 hours of labor per day.
- 2.** The company would like to achieve a *satisfactory* profit level of \$1,600 per day.
- 3.** Because the clay must be stored in a special place so

that it does not dry out, the company prefers not to keep more than 120 pounds on hand each day.

4. Because high overhead costs result when the plant is kept open past normal hours, the company would like to minimize the amount of overtime.

These different objectives are referred to as goals in the context of the goal programming technique. The company would, naturally, like to come as close as possible to

achieving each of these goals. Because the regular form of the linear programming model presented in previous chapters considers only one objective, we must develop an alternative form of the model to reflect these multiple goals. Our first step in formulating a goal programming model is to transform the linear programming model constraints into goals.

*The different
objectives in a goal
programming*

problem are referred to as goals.

Labor Goal

The first goal of the pottery company is to avoid *underutilization* of labor that is, using fewer than 40 hours of labor each day. To represent the possibility of underutilizing labor, the linear programming constraint for labor, $x_1 + 2x_2 \leq$

40 hours of labor, is reformulated as

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40 \text{ hr.}$$

This reformulated equation is referred to as a **goal constraint**. The two new variables, d_1 and d_1^+ , are called **deviational variables**. They represent the number of labor hours less than 40 (d_1) and the number of labor hours exceeding 40 (d_1^+). More specifically, d_1 represents labor underutilization, and d_1^+ represents *overtime*. For

example, if $x_1 = 5$ bowls and $x_2 = 10$ mugs, then a total of 25 hours of labor have been expended. Substituting these values into our goal constraint gives

$$(5) + 2(10) + d_1^- - d_1^+ = 40$$
$$25 + d_1^- - d_1^+ = 40$$

All goal constraints are equalities that include deviational variables, d and d^+ .

Because only 25 hours were used in production, labor was underutilized by 15 hours ($40 - 25 = 15$). Thus, if we let $d_1^- = 15$ hours and $d_1^+ = 0$ (because there is obviously no overtime), we have

$$25 + d_1^- - d_1^+ = 40$$

$$25 + 15 - 0 = 40$$

$$40 = 40$$

A positive deviational variable (d^+) is the amount by which a goal level is exceeded.

A negative deviational variable (d) is the amount by which a goal level is underachieved.

Now consider the case where $x_1 = 10$ bowls and $x_2 = 20$ mugs. This means that a total of 50 hours have been used for production, or 10 hours above

the goal level of 40 hours. This extra 10 hours is overtime. Thus, $d_1 = 0$ (because there is no underutilization) and $d_1^+ = 10$ hours.

In each of these two brief examples, at least one of the deviational variables equaled zero. In the first example, $d_1^+ = 0$, and in the second example, $d_1 = 0$. This is because it is impossible to use fewer than 40 hours of labor and more than 40 hours of labor *at the same time*. Of course, both deviational

variables, d_1 and d_1^+ , could have equaled zero if exactly 40 hours were used in production. These examples illustrate one of the fundamental characteristics of goal programming: At least one or both of the deviational variables in a goal constraint must equal zero.

[Page 392]

*At least one or both
deviational variables
in a goal constraint
must equal zero.*

The next step in formulating our goal programming model is to represent the goal of not using fewer than 40 hours of labor. We do this by creating a new form of objective function:

$$\text{minimize } P_1 d_1$$

The objective function in all goal programming models is to *minimize* deviation from the goal constraint levels. In this objective function, the goal is to minimize d_1 , the

underutilization of labor. If d_1 equaled zero, then we would not be using fewer than 40 hours of labor. Thus, it is our objective to make d_1 equal zero or the minimum amount possible. The symbol P_1 in the objective function designates the minimization of d_1 as the *first-priority* goal. This means that when this model is solved, the first step will be to minimize the value of d_1 before any other goal is addressed.

The objective

function in a goal programming model seeks to minimize the deviation from goals in order of the goal priorities.

The fourth-priority goal in this problem is also associated with the labor constraint. The fourth goal, P_4 , reflects a desire to minimize overtime. Recall that hours of overtime are represented by d_1^+ ; the

objective function, therefore, becomes

$$\text{minimize } P_1 d_1^-, P_4 d_1^+$$

As before, the objective is to minimize the deviational variable d_1^+ . In other words, if d_1^+ equaled zero, there would be no overtime at all. In solving this model, the achievement of this fourth-ranked goal will not be attempted until goals one, two, and three have been considered.

Profit Goal

The second goal in our goal programming model is to achieve a daily profit of \$1,600. Recall that the original linear programming objective function was

$$Z = 40x_1 + 50x_2$$

Now we reformulate this objective function as a goal constraint with the following goal level:

$$40x_1 + 50x_2 + d_2 - d_2^+ = \$1,600$$

The deviational variables d_2 and d_2^+ represent the amount of profit less than \$1,600 (d_2) and the amount of profit exceeding \$1,600 (d_2^+). The pottery company's goal of achieving \$1,600 in profit is represented in the objective function as

$$\text{minimize } P_1 d_1^-, P_2 d_2^-, P_4 d_1^+$$

Notice that only d_2 is being minimized, not d_2^+ , because it is logical to assume that the pottery company would be

willing to accept all profits in excess of \$1,600 (i.e., it does not desire to minimize d_2^+ , excess profit). By minimizing d_2 at the second-priority level, the pottery company hopes that d_2 will equal zero, which will result in at least \$1,600 in profit.

Material Goal

The third goal of the company is to avoid keeping more than 120 pounds of clay on hand each day. The goal constraint is

$$4x_1 + 3x_2 + d_3 d_3^+ = 120 \text{ lb.}$$

[Page 393]

Because the deviational variable d_3 represents the amount of clay less than 120 pounds, and d_3^+ represents the amount in excess of 120 pounds, this goal can be reflected in the objective function as

$$\text{minimize } P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^+$$

The term $P_3 d_3^+$ represents the company's desire to minimize

d_3^+ , the amount of clay in excess of 120 pounds. The P_3 designation indicates that it is the pottery company's third most important goal.

The complete goal programming model can now be summarized as follows:

minimize $P_1 d_1^-, P_2 d_2^-, P_3 d_3^+, P_4 d_1^-$
subject to

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1,600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

The one basic difference between this model and the

standard linear programming model is that the objective function terms *are not summed* to equal a total value, Z . This is because the deviational variables in the objective function represent different units of measure. For example, d_1 and d_1^+ represent hours of labor, d_2 represents dollars, and d_3^+ represents pounds of clay. It would be illogical to sum hours, dollars, and pounds. The objective function in a goal programming model specifies only that the deviations from the goals represented in the

objective function be minimized *individually*, in order of their priority.

Terms are not summed in the objective function because the deviational variables often have different units of measure.

Alternative Forms of Goal

Constraints

Let us now alter the preceding goal programming model so that our fourth-priority goal limits overtime to 10 hours instead of minimizing overtime. Recall that the goal constraint for labor is

$$x_1 + 2x_2 + d_1 - d_1^+ = 40$$

In this goal constraint, d_1^+ represents overtime. Because the new fourth-priority goal is to limit overtime to 10 hours, the following goal constraint is

developed:

$$d_1^+ + d_4^- d_4^+ = 10$$

Although this goal constraint looks unusual, it is acceptable in goal programming to have an equation with *all deviational variables*. In this equation, d_4^- represents the amount of overtime less than 10 hours, and d_4^+ represents the amount of overtime greater than 10 hours. Because the company desires to *limit overtime* to 10 hours, d_4^+ is minimized in the objective function:

minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+$

Goal constraints can include all deviational variables.

Now let us consider the addition of a fifth-priority goal to this example. Assume that the pottery company has limited warehouse space, so it can produce no more than 30 bowls and 20 mugs daily. If possible, the company would

like to produce these numbers. However, because the profit for mugs is greater than the profit for bowls (i.e., \$50 rather than \$40), it is more important to achieve the goal for mugs. This fifth goal requires that two new goal constraints be formulated, as follows:

$$x_1 + d_5 = 30 \text{ bowls}$$

$$x_2 + d_6 = 20 \text{ mugs}$$

[Page 394]

Notice that the positive

deviation variables d_5^+ and d_6^+ have been deleted from these goal constraints. This is because the statement of the fifth goal specifies that "no more than 30 bowls and 20 mugs" can be produced. In other words, positive deviation, or overproduction, is not possible.

Because the actual goal of the company is to achieve the levels of production shown in these two goal constraints, the negative deviation variables d_5 and d_6 are minimized in the

objective function. However, recall that it is *more important* to the company to achieve the goal for mugs because mugs generate greater profit. This condition is reflected in the objective function, as follows:

$$\text{minimize } P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$$

Because the goal for mugs is more important than the goal for bowls, the *degree of importance* should be in proportion to the amount of profit (i.e., \$50 for each mug and \$40 for each bowl). Thus, the goal for mugs is more

important than the goal for bowls by a ratio of 5 to 4.

The coefficients of 5 for P_5d_6 and 4 for P_5d_5 are referred to as *weights*. In other words, the minimization of d_6 is "weighted" higher than the minimization of d_5 at the fifth priority level. When this model is solved, the achievement of the goal for minimizing d_6 (bowls) is more important, even though both goals are at the same priority level.

Two or more goals at

the same priority level can be assigned weights to indicate their relative importance.

Notice, however, that these two weighted goals have been summed because they are at the same priority level. Their sum represents achievement of the desired goal at this particular priority level. The complete goal programming

model, with the new goals for both overtime and production, is

minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$
subject to

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1,600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$d_1^+ + d_4^- - d_4^+ = 10$$

$$x_1 + d_5^- = 30$$

$$x_2 + d_6^- = 20$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

Graphical Interpretation of Goal Programming

In [Chapter 2](#) we analyzed the solution of linear programming models by using graphical analysis. Because goal programming models are linear, they can also be analyzed graphically. The original goal programming model for Beaver Creek Pottery Company, formulated at the beginning of this chapter, will be used as an example:

minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$
subject to

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

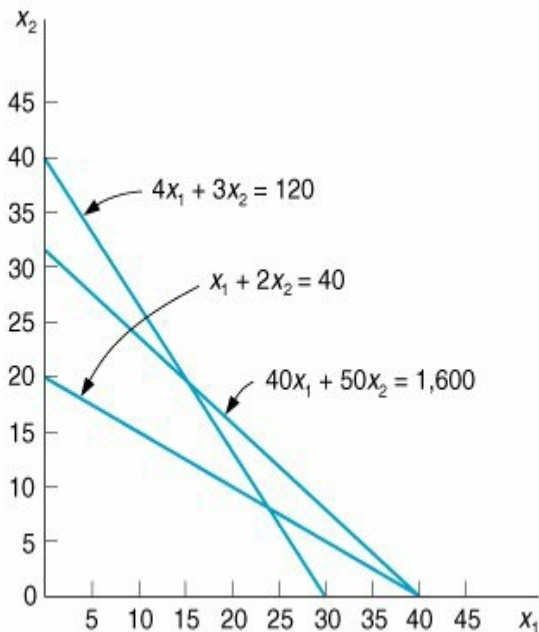
$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1,600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

To graph this model, the deviational variables in each goal constraint are set equal to zero, and we graph each subsequent equation on a set of coordinates, just as in [Chapter 2. Figure 9.1](#) is a graph of the three goal constraints for this model.

Figure 9.1. Goal constraints



Notice that in [Figure 9.1](#) there is no feasible solution space indicated, as in a regular linear programming model. This is because all three goal constraints are *equations*; thus, all solution points are on the constraint lines.

The solution logic in a goal programming model is to attempt to achieve the goals in the objective function, in order of their priorities. As a goal is achieved, the next highest-ranked goal is then considered. However, a higher-ranked goal that has been achieved is never

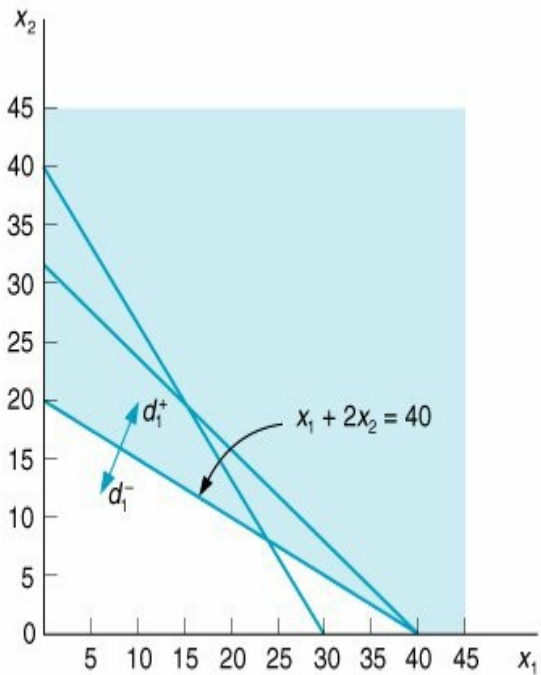
given up in order to achieve a lower-ranked goal.

Graphical solution illustrates the goal programming solution logic seeking to achieve goals by minimizing deviation in order of their priority.

In this example we first consider the first-priority goal,

minimizing d_1 . The relationship of d_1 and d_1^+ to the goal constraint is shown in [Figure 9.2](#). The area below the goal constraint line $x_1 + 2x_2 = 40$ represents possible values for d_1 , and the area above the line represents values for d_1^+ . In order to achieve the goal of minimizing d_1 , the area below the constraint line corresponding to d_1 is eliminated, leaving the shaded area as a possible solution area.

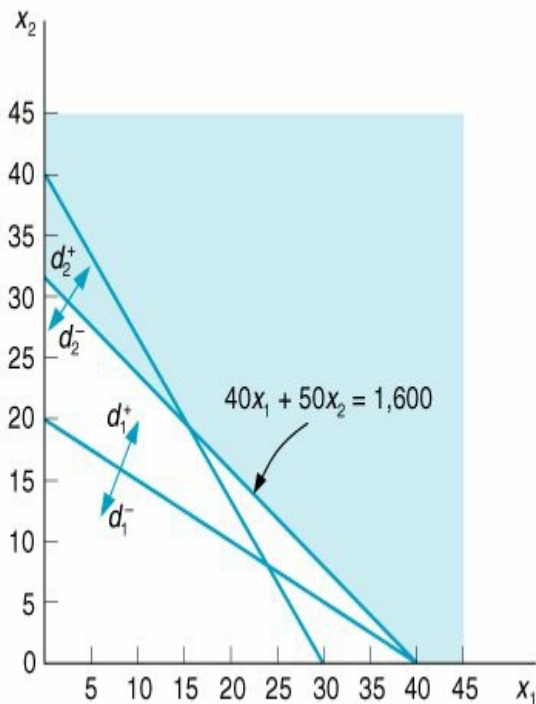
**Figure 9.2. The first-
priority goal:
minimize d_1**



Next, we consider the second-priority goal, minimizing d_2 . In [Figure 9.3](#), the area below the constraint line $40x_1 + 50x_2 = 1,600$ represents the values for d_2 , and the area above the line represents the values for d_2^+ . To minimize d_2 , the area below the constraint line corresponding to d_2 is eliminated. Notice that by eliminating the area for d_2 , we

do not affect the first-priority goal of minimizing d_1 .

Figure 9.3. The second-priority goal: minimize d_2

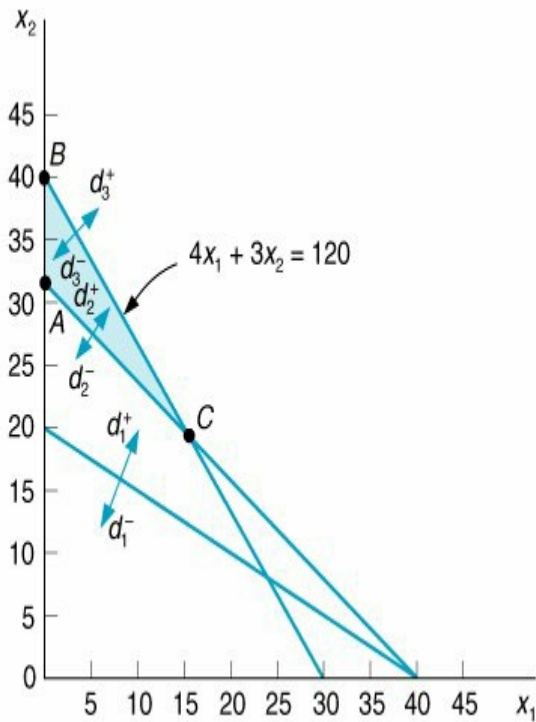


One goal is never achieved at the expense of another higher-priority goal.

Next, the third-priority goal, minimizing d_3^+ , is considered. [Figure 9.4](#) shows the areas corresponding to d_3 and d_3^+ . To minimize d_3^+ , the area above the constraint line $4x_1 + 3x_2 =$

120 is eliminated. After considering the first three goals, we are left with the area between the line segments AC and BC , which contains possible solution points that satisfy the first three goals.

**Figure 9.4. The third-priority goal:
minimize d_3^+**



Finally, we must consider the fourth-priority goal, minimizing d_1^+ . To achieve this final goal, the area above the constraint line $x_1 + 2x_2 = 40$ must be eliminated. However, if we eliminate this area, then both d_2 and d_3 must take on values. In other words, we cannot minimize d_1^+ totally without violating the first- and second-priority goals. Therefore, we want to find a solution point that satisfies the first three

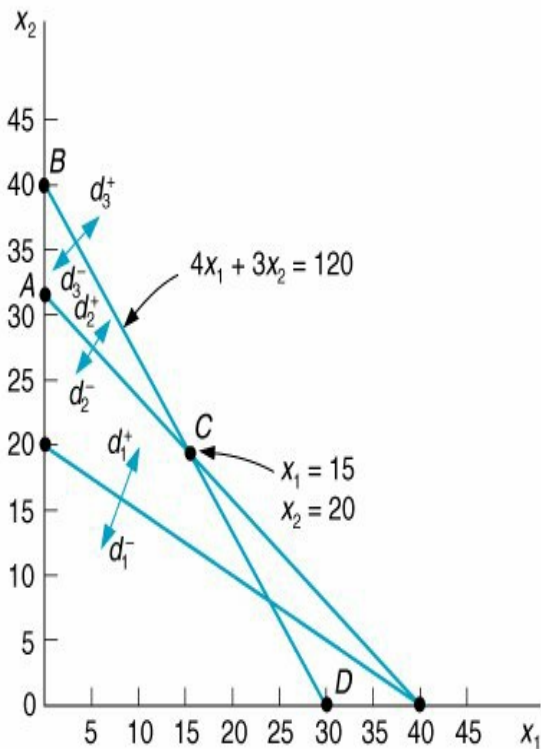
goals but achieves as much of the fourth-priority goal as possible.

[Page 397]

Point C in [Figure 9.5](#) is a solution that *satisfies* these conditions. Notice that if we move down the goal constraint line $4x_1 + 3x_2 = 120$ toward point D , d_1^+ is further minimized; however, d_2 takes on a value as we move past point C . Thus, the minimization of d_1^+ would be accomplished

only at the expense of a higher-ranked goal.

Figure 9.5. The fourth-priority goal (minimize d_1^+) and the solution



The solution at point C is determined by simultaneously solving the two equations that intersect at this point. Doing so results in the following solution:

$$x_1 = 15 \text{ bowls}$$

$$x_2 = 20 \text{ mugs}$$

$$d_1^+ = 15 \text{ hours}$$

Because the deviational variables d_1 , d_2 , and d_3^+ all equal zero, they have been minimized, and the first three

goals have been achieved. Because $d_1^+ = 15$ hours of overtime, the fourth-priority goal has not been achieved. The solution to a goal programming model such as this one is referred to as the *most satisfactory* solution rather than the optimal solution because it satisfies the specified goals as well as possible.

Goal programming solutions do not always achieve all goals, and they are

*not optional; they
achieve the best or
most satisfactory
solution possible.*

Computer Solution of Goal Programming Problems with QM for Windows and Excel

Goal programming problems can be solved using QM for Windows and Excel spreadsheets. In this section we will demonstrate how to use both computing alternatives to solve the Beaver Creek Pottery Company example we solved graphically in the previous

section.

[Page 398]

Management Science

Application: Developing Television Advertising Sales Plans at NBC

The National Broadcasting Company (NBC), a subsidiary of General Electric, generated more than \$4 billion in revenue in 2000 from its television network. In May the network announces its programming schedule for the broadcast year, which begins in the third week in September. Soon after that initial announcement, the network begins the sale of its inventory of advertising slots on its TV shows to advertisers. During this time advertising agencies approach the network with requests to purchase time for their clients for

the year. The requests generally include a budget, the demographic audience in which the client is interested, and a desired program mix. NBC subsequently develops a sales plan consisting of a schedule of television commercials that will meet the client's requirements. The most popular commercial durations are 30 and 15 seconds, although 60- and 120-second slots are not uncommon.

After the network finalizes its programming schedule, it develops a rating forecast that projects the audience size for several demographic groupings for each airing of a show. These ratings projections are based on such factors as the strength of the show, historical ratings for a time slot, competing shows on other networks, and the performance of adjacent

shows. The network then develops advertising rate cards that set the prices of advertising slots for each airing of a show. The prices vary according to the time of the year the show airs. For example, prices are high during the sweeps months of November, February, and May and are lower in January and the summer; thus prices are weighted for different weeks of the year. The sales management staff prioritizes sales requests based on the client's importance to NBC. NBC also desires to make the most profitable use of its limited inventory. It is optimal for NBC to use as little premium inventory as possible in meeting each client's request.

NBC uses a 01 integer goal programming model to develop annual advertising sales plans for individual clients. The model

minimizes the amount of premium inventory assigned to a sales plan while also minimizing the penalty incurred in not meeting other goals. Goals are for inventory (i.e., airtime availability), product conflict (i.e., no two similar products should advertise on the same show), client budget constraints, showmix goals, unitmix (i.e., commercial length) goals, and weekly weighting constraints. The decision variables are x_{ij} and indicate the number of commercials of each length aired on shows during weeks included in the sales plan. The model has enabled NBC to save millions of dollars of premium inventory while meeting customer requirements, and it has reduced the time required to develop a sales plan from 3 to 4 hours to 20 minutes. It is estimated that this model and associated planning systems increase NBC's

revenue by at least \$50 million per year.



Source: S. Bollapragada, et al., "NBC Optimization Systems Increase Revenues and Productivity," *Interfaces* 32, no. 1 (January/February 2002): 4760.

QM for Windows

We will demonstrate how to solve a goal programming model by using our Beaver Creek Pottery Company example, which was formulated as follows:

minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_1^+$
subject to

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1,600$$

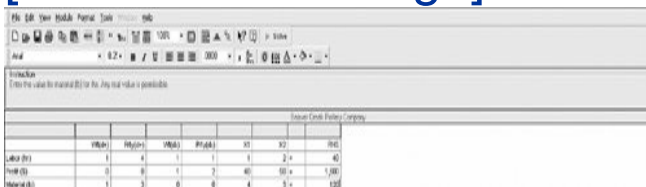
$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+ \geq 0$$

QM for Windows includes a goal programming module that can be accessed by clicking on the "Module" button at the top of the screen. The model parameters are entered onto the data input screen as shown in [Exhibit 9.1](#). Notice that each prioritized deviational variable must be assigned a weight, which for this problem is always one. The solution is obtained by clicking on the "Solve" button at the top of the screen. The solution summary for our model is shown in [Exhibit 9.2](#).

Exhibit 9.1.

[\[View full size image\]](#)



The screenshot shows a spreadsheet application with a menu bar (File, Edit, View, Format, Tools, Window, Help) and a toolbar. The main area contains a table with the following data:

	Value	My (b)	Value	My (b)	X1	X2	Y0
Value (b)	1	4	1	1	1	2	40
My (b)	0	0	1	2	40	10	1,000
Value (b)	1	2	0	0	4	5	120

Exhibit 9.2.

[\[View full size image\]](#)

Summary				
Beaver Creek Pottery Company Solution				
Item				
Decision variable analysis	Value			
X1	15			
X2	20			
Priority analysis	Nonachievement			
Priority 1	0			
Priority 2	0			
Priority 3	0			
Priority 4	15			
Constraint Analysis	RHS	d+ (row i)	d- (row i)	
Labor (hr)	40	15	0	
Profit (\$)	1,600	0	0	
Material (lb)	120	0	0	

QM for Windows will also provide a graphical analysis of a goal programming model. We click on the "Windows" button

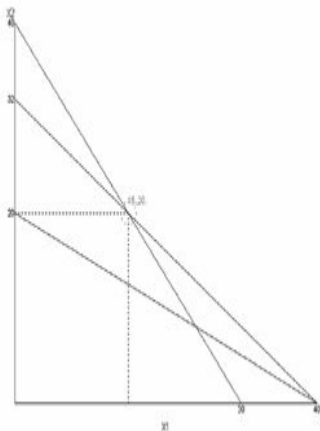
and then select "Graph" from the menu. The graph for our example is shown in [Exhibit 9.3](#).

Exhibit 9.3.

(This item is displayed on page 400 in the print version)

[\[View full size image\]](#)

Beaver Creek Pottery Company



Constraint Display

- ☐ $0X_1 + 20X_2 = 40$
- ☐ $40X_1 + 80X_2 = 160$
- ☐ $60X_1 + 30X_2 = 120$
- ☒ none

Solution

$X_1 = 15$
 $X_2 = 20$
 $Z = 0$

Excel Spreadsheets

Solving a goal programming problem by using Excel is similar to solving a linear programming model, although not quite as straightforward.

[Exhibit 9.4](#) shows the spreadsheet format for our Beaver Creek Pottery Company example. Cells G5, G6, and G7, under the heading "Constraint Total," contain the formulas for our goal constraints, including deviational variables. The formula for the labor constraint is shown on the formula bar at

the top of the screen in [Exhibit 9.4](#). The model decision variables are in cells B10 and B11, and the deviational variables are in cells **E5:F7**. The goals are established by setting the constraint formulas in G5 to G7 equal to the goal levels in cells I5 to I7.

Exhibit 9.4.

(This item is displayed on page 400 in the print version)

[\[View full size image\]](#)

Goal constraint
for cell G5

Microsoft Excel - Table1.xls

File Edit View Insert Format Tools Data Window Help

Formulas: =SUM(B5:D5)+G5+F5

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Goal Programming Example: The Beaver Creek Pottery Company														
2															
3	Products:	Bowl	Mug				Constraint								
4	Goal constraints				d-	d+	Total	Constraint	Goal						
5	labor (hr/unit)	1	2	0	0	0	0	=	40						
6	profit (\$/unit)	40	50	0	0	0	0	=	1000						
7	material (lb/unit)	4	3	0	0	0	0	=	120						
8															
9	Production:														
10	Bowls =														
11	Mugs =														
12															

Decision variables—
B10:B11

Deviational variables—
E5:F7

When using a spreadsheet (or any regular linear programming program) to solve a goal programming problem, it

must be solved *sequentially*. In this procedure, a new problem is formulated and solved for each priority goal in the objective function, beginning with the highest priority. In other words, the minimization of the deviational variable at the highest priority is the initial objective. Once a solution for this formulation is achieved, the value of the deviational variable that is the objective is added to the model as a constraint, and the second-priority deviational variable becomes the new objective. A

new solution is achieved for each new objective sequentially until all the priorities are exhausted or it is clear that a better solution cannot be reached. For our purposes, this means editing Excel's Solver for each new solution.

[Page 400]

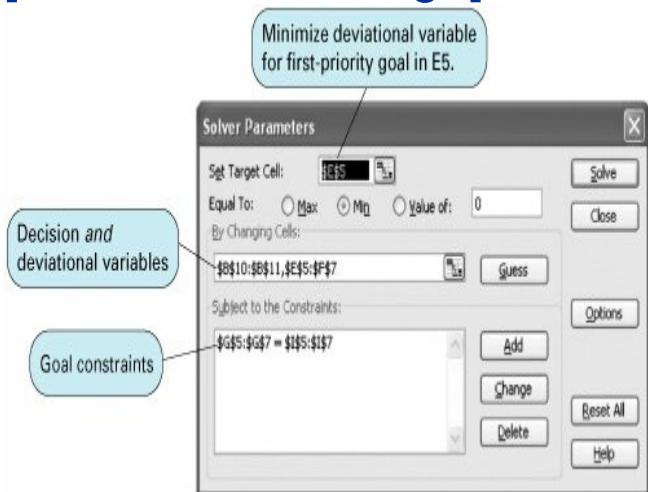
[Exhibit 9.5](#) shows the Solver Parameters window for our spreadsheet example. Recall that the first-priority goal for our model is the minimization of the negative deviational

variable (d_1) for our labor goal. This deviational variable is located in cell E5; thus, we start Solver by minimizing cell E5. We identify **B10:B11** (the decision variables) as well as **E5:F7** (the deviational variables) as variables in the model. The model constraints are for our goals (i.e., **G5:G7 = I5:I7**).

[Page 401]

Exhibit 9.5.

[\[View full size image\]](#)



The spreadsheet with the solution to our example problem is shown in [Exhibit](#)

9.6. We know this is the most satisfactory solution we can achieve from the QM for Windows solution, so it will not be necessary to perform any additional sequential steps. From the spreadsheet we can see that we have achieved all the goals except the fourth-priority goal, to minimize d_1^+ . However, if we had not achieved all our top goals so readily, the next step would be to include **E5 = 0** as a constraint in Solver and then minimize our next-priority goal, which would be E6 (i.e.,

d_2).

[Page 402]

Exhibit 9.6.

(This item is displayed on page
401 in the print version)

[\[View full size image\]](#)

Time Out: For Abraham Charnes and William W. Cooper

(This item is displayed on page 401 in
the print version)

The concept of goal programming was first introduced in 1955 by Abraham Charnes and William Cooper of the Carnegie Institute of Technology, with R. O. Ferguson, a consultant for the Methods Engineering Council. They developed a model for a plan for executive compensation for a division of General Electric (using competing salary offers for executives as multiple goals), and they solved it by reducing it to an equivalent linear programming formulation. However,

it was not until 1961 that Charnes and Cooper coined the name *goal programming*. Yuji Ijiri of Stanford (and a former doctoral student of Cooper's at Carnegie Tech) in 1965 developed the concept of preemptive priorities for treating multiple goals and the general solution approach.

Next we will use Excel to solve a slightly more complicated goal programming model that is, the altered Beaver Creek Pottery Company example we developed at the beginning of this chapter, with goals for

overtime and maximum storage levels for bowls and mugs:

minimize $P_1d_1^-, P_2d_2^-, P_3d_3^+, P_4d_4^+, 4P_5d_5^- + 5P_5d_6^-$
subject to

$$x_1 + 2x_2 + d_1^- - d_1^+ = 40$$

$$40x_1 + 50x_2 + d_2^- - d_2^+ = 1,600$$

$$4x_1 + 3x_2 + d_3^- - d_3^+ = 120$$

$$d_1^+ + d_4^- - d_4^+ = 10$$

$$x_1 + d_5^- = 30$$

$$x_2 + d_6^- = 20$$

$$x_1, x_2, d_1^-, d_1^+, d_2^-, d_2^+, d_3^-, d_3^+, d_4^-, d_4^+, d_5^-, d_6^- \geq 0$$

The spreadsheet for this modified version of our example is shown in [Exhibit 9.7](#). The spreadsheet is set up much the same as the original version of this example, with

the exception of the goal constraint for overtime. The formula for this goal constraint is included in cell G8 as **=F5 + E8 F8**. In addition, the positive deviational variables for the last two goal constraints are now included in the formulas embedded in cells G9 and G10. For example, in cell G9 the constraint formula is **= C9 * B13 + E9**.

Exhibit 9.7.

[\[View full size image\]](#)

Goal constraint for labor

Microsoft Excel - Todd97.7.xls

Goal = $C9*B13+C9*B16+E5-F5$

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Goal Programming Example: The Beaver Creek Pottery Company														
2															
3	Products:	Bowl	Mug				Constraint								
4	Goal constraints:				d-	d+	Total	Constraint	Goal						
5	labor (minut)	1	2	0	0	0	0	=	40						
6	profit (\$/unit)	40	50	0	0	0	0	=	1600						
7	material (\$/unit)	4	3	0	0	0	0	=	120						
8	overtime (hr)			0	0	0	0	=	10						
9	bowl (unit)	1	0	0	0	0	0	=	30						
10	mug (unit)	0	1	0	0	0	0	=	20						
11															
12	Production:														
13	Bowls =														
14	Mugs =														
15															

Goal constraint for overtime;
 $=F5+E8-F8$

$=C9*B13+E9$

The Solver Parameters window for this spreadsheet is shown in

[Exhibit 9.8](#), and the resulting solution is shown in [Exhibit 9.9](#).

Exhibit 9.8.

(This item is displayed on page 403 in the print version)

Solver Parameters ✕

Set Target Cell: 

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells: 

Subject to the Constraints:



Exhibit 9.9.

(This item is displayed on page 403 in the print version)

[\[View full size image\]](#)

Microsoft Excel - Exhibit 13.7.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

100%

166

Goal Programming Example: The Beaver Creek Pottery Company

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Goal Programming Example: The Beaver Creek Pottery Company														
2															
3	Products:	Bowl	Mug				Constraint								
4	Goal constraints				d ₁ ⁻	d ₁ ⁺	Total	Constraint	Goal						
5	labor (hrs/unit)	1	2	0	0	0	40	=	40						
6	profit (\$/unit)	40	50	0	0	0	1600	=	1600						
7	material (lbs/unit)	4	3	0	0	24	120	=	120						
8	overtime (hrs)				10	0	10	=	10						
9	bowl (unit)	1	0	0	0	0	30	=	30						
10	mug (unit)	0	1	12	0	0	20	=	20						
11															
12	Production:														
13	Bowls =	30													
14	Mugs =	8													
15															

First two priority goals, minimizing d_1^- and d_2^- , achieved

Third-priority goal to minimize d_3^+ not achieved

You will notice that this solution achieves the first two priority goals for minimizing d_1 and d_2 ,

which are in cells E5 and E6. However, the third-priority goal to minimize d_3^+ is not achieved because its cell (F7) has a value of 24 in it. Thus, we must follow the sequential approach to attempt to obtain a better solution. We accomplish this by including **E5=0** (the achievement of our first goal) as a constraint in Solver, as well as our second-priority goal that this solution achieved, **E6=0**, and then minimizing F7. This Solver Parameters screen is shown in [Exhibit 9.10](#).

Exhibit 9.10.

[\[View full size image\]](#)

First- and second-priority goals achieved; add E5=0 and E6=0

Solver Parameters

Set Target Cell:

Equal To: ☐ Max ☒ Min ☐ Value of:

By Changing Cells:

Subject to the Constraints:

-
-

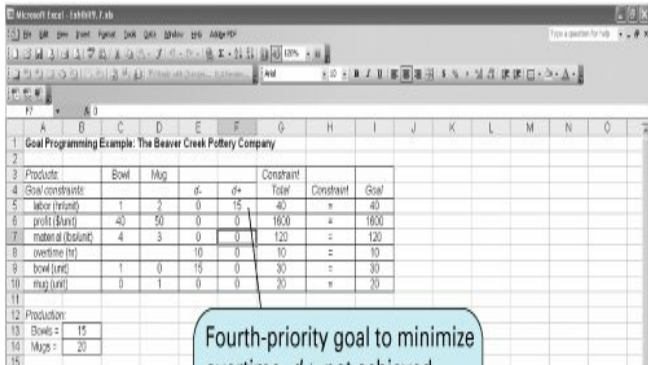
Buttons: Solve, Close, Options, Add, Change, Delete, Reset All, Help

The solution is shown in [Exhibit 9.11](#). This is the same solution that we obtained for our original version of the model, without the alterations for overtime and production levels. This solution achieves the first three priority goals. We could continue to attempt to achieve the fourth-priority goal by including **F7 = 0** as a constraint in Solver and minimizing cell F5 (i.e., d_1^+); however, doing so will not result in a better solution

without sacrificing the goal achievement at the higher-priority levels. Thus, this is the best solution we can achieve.

Exhibit 9.11.

[\[View full size image\]](#)



The Analytical Hierarchy Process

Goal programming is a method that provides us with a mathematical "quantity" for the decision variables that best achieves a set of goals. It answers the question "How much?" The **analytical hierarchy process (AHP)**, developed by Thomas Saaty, is a method for ranking decision alternatives and selecting the best one when the decision

maker has multiple objectives, or criteria, on which to base the decision. Thus, it answers the question "Which one?" A decision maker usually has several alternatives from which to choose when making a decision. For example, someone buying a house might have several houses for sale from which to choose; someone buying a new car might have several makes and styles to consider; and a prospective student might select a college to attend from a group of schools. In each of these

examples, the decision maker would typically make a decision based on how the alternatives compare, according to several criteria. For example, a home buyer might consider the cost, proximity of schools, trees, neighborhood, and public transportation when comparing several houses; a car buyer might compare different cars based on price, interior comfort, mileage per gallon, and appearance. In each case, the decision maker will select the alternative that best meets his or her decision criteria. AHP

is a process for developing a numeric score to rank each decision alternative, based on how well each alternative meets the decision maker's criteria.

AHP is a method for ranking decision alternatives and selecting the best one given multiple criteria.

We will demonstrate AHP by using an example. Southcorp Development builds and operates shopping malls around the country. The company has identified three potential sites for its latest project, near Atlanta, Birmingham, and Charlotte. The company has identified four primary criteria on which it will compare the sites (1) the customer market base (including overall market size and population at different age levels); (2) income level; (3) infrastructure (including utilities and roads); and (4)

transportation (i.e., proximity to interstate highways for supplier deliveries and customer access).

[Page 405]

Management Science

Application: Selecting Students for Graduate Studies Abroad at Dar Al- Hekma Women's College

Prior to 1999, all universities and colleges in Saudi Arabia were public institutions; however, in 1999, in response to increased demand for higher education, the Ministry of Higher Education in Saudi Arabia began allowing nonprofit, private colleges. Dar Al-Hekma, located in Jeddah, was one of the first private colleges for women instituted under this new policy.

Dar Al-Hekma offers a 4-year degree program with degrees in business

information systems, interior design, and special education. One of the initial problems the new college faced was the lack of qualified women to staff the three departments. To help alleviate this problem, the Ministry of Higher Education provided the college with nine scholarships to send women to the United States and United Kingdom for graduate studies. A committee consisting of faculty members from local public universities and members of the college's board of trustees was established to select the nine candidates in what was expected to be a highly competitive process.

Given the various criteria that could be used to select the scholarship candidates, the committee decided to use the analytical hierarchy process (AHP) for the selection process. Two overall sets of criteria

were determined: a qualitative set and a quantitative set. The qualitative set of criteria included subcriteria for personality, work experience, and other information, while the quantitative set included subcriteria for language and previous academic achievement/degrees. Various measures such as interviews, essays, work experience, grade point average, degrees awarded, and TOEFL scores were used to rank candidates in each category.



The AHP model was successful in helping the committee select the candidates. The AHP model reduced subjectivity and ensured fairness in the highly competitive selection process; it provided a systematic framework for committee interaction and decision making, and it saved time and removed potential personal and political conflicts from the process. The AHP process was so successful that it was subsequently used by the college for other decision situations, including the selection of financial aid recipients for enrolling students and internal faculty grant recipients.

Source: A. M. A. Bahurmoz, "The Analytical Hierarchy Process at Dar Al-Hekma, Saudi Arabia," *Interfaces* 33, no. 4, (July/August 2003): 7078.

The overall objective of the company is to select the best site. This goal is at the top of the *hierarchy* of the problem. At the next (second) level of the hierarchy, we determine how the four *criteria* contribute to achieving the objective. At the level of the *problem hierarchy*, we determine how each of the three alternatives (Atlanta, Birmingham, and Charlotte) contributes to each of the four criteria.

The general mathematical process involved in AHP is to establish preferences at each of these hierarchical levels. First, we mathematically determine our preferences for each site for each criterion. For example, we first determine our site preferences for the customer market base. We might decide that Atlanta has a better market base than the other two cities; that is, we prefer Atlanta for the customer market criterion. Then we determine our site preferences for income level, and so on. Next, we

mathematically determine our preferences for the criteria that is, which of the four criteria is most important, which is the next most important, and so on. For example, we might decide that the customer market is a more important criterion than the others. Finally, we combine these two sets of preferences for sites within each criterion and for the four criteria to mathematically derive a score for each site, with the highest score being the best.

In the following sections we

describe these general steps in greater detail.

[Page 406]

Pairwise Comparisons

In AHP the decision maker determines how well each alternative "scores" on a criterion by using **pairwise comparisons**. In a pairwise comparison, the decision maker compares two alternatives (i.e., a pair) according to one criterion and indicates a

preference. For example, Southcorp might compare the Atlanta (A) site with the Birmingham (B) site and decide which one it prefers according to the customer market criterion. These comparisons are made by using a **preference scale**, which assigns numeric values to different levels of preference.

In a pairwise comparison, two alternatives are compared according to a criterion, and

one is preferred.

*A preference scale
assigns numeric
values to different
levels of preference.*

The standard preference scale used for AHP is shown in [Table 9.1](#). This scale has been determined by experienced

researchers in AHP to be a reasonable basis for comparing two items or alternatives. Each rating on the scale is based on a comparison of two items. For example, if the Atlanta site is "moderately preferred" to the Birmingham site, then a value of 3 is assigned to this particular comparison. The rating of 3 is a measure of the decision maker's preference for one of the alternatives over the other.

Table 9.1. Preference scale for pairwise comparisons

Preference Level	Numeric Value
Equally preferred	1
Equally to moderately preferred	2
Moderately preferred	3
Moderately to strongly preferred	4
Strongly preferred	5
Strongly to very strongly preferred	6
Very strongly	

preferred	7
-----------	---

Very strongly to extremely preferred	8
---	---

Extremely preferred	9
---------------------	---

If Southcorp compares Atlanta to Birmingham and moderately prefers Atlanta, resulting in a comparison value of 3 for the customer market criterion, it is not necessary for Southcorp to compare Birmingham to Atlanta

to determine a separate preference value for this "opposite" comparison. The preference value of B for A is simply the reciprocal or inverse of the preference of A for B. Thus, in our example if the preference value of Atlanta for Birmingham is 3, the preference value of Birmingham for Atlanta is $1/3$.

Southcorp's pairwise comparison ratings for each of the three sites for the customer market criterion are summarized in a matrix, a rectangular array of numbers.

This **pairwise comparison matrix** will have a number of rows and columns equal to the decision alternatives:

Customer Market			
Site	<i>A</i>	<i>B</i>	<i>C</i>
A	1	3	2
B	1/3	1	1/5
C	1/2	5	1

A pairwise comparison matrix summarizes the pairwise comparisons for a criterion.

This matrix shows that the customer market in Atlanta (A) is equally to moderately preferred (2) over the Charlotte (C) customer market, but Charlotte (C) is strongly

preferred (5) over Birmingham (B). Notice that any site compared against itself, such as A compared to A, must be "equally preferred," with a preference value of 1. Thus, the values along the diagonal of our matrix must be 1s.

[Page 407]

The remaining pairwise comparison matrices for the other three criteria income level, infrastructure, and transportation have been developed by Southcorp as

follows:

<i>Income Level</i>	<i>Infrastructure</i>	<i>Transportation</i>
A $\begin{bmatrix} 1 & 6 & 1/3 \\ 1/6 & 1 & 1/9 \\ 3 & 9 & 1 \end{bmatrix}$	A $\begin{bmatrix} 1 & 1/3 & 1 \\ 3 & 1 & 7 \\ 1 & 1/7 & 1 \end{bmatrix}$	A $\begin{bmatrix} 1 & 1/3 & 1/2 \\ 3 & 1 & 4 \\ 2 & 1/4 & 1 \end{bmatrix}$
B	B	B
C	C	C

Developing Preferences Within Criteria

The next step in AHP is to prioritize the decision alternatives within each criterion. For our site selection example, this means that we want to determine which site is

the most preferred, the second most preferred, and the third most preferred for each of the four criteria. This step in AHP is referred to as **synthesization**. The mathematical procedure for synthesization is very complex and beyond the scope of this text. Instead, we will use an approximation method for synthesization that provides a reasonably good estimate of preference scores for each decision in each criterion.

*In synthesization,
decision alternatives*

are prioritized within each criterion.

The first step in developing preference scores is to sum the values in each column of the pairwise comparison matrices. The column sums for our customer market matrix are shown as follows:

Customer Market

Site	<i>A</i>	<i>B</i>	<i>C</i>
A	1	3	2
B	1/3	1	1/5
C	1/2	5	1
	11/6	9	16/5

Next, we divide each value in a column by its corresponding column sum. This results in a *normalized* matrix, as follows:

Customer Market

Site	<i>A</i>	<i>B</i>	<i>C</i>
A	6/11	3/9	5/8
B	2/11	1/9	1/16
C	3/11	5/9	5/16

Notice that the values in each column sum to 1. The next step is to average the values in each

row. At this point, we convert the fractional values in the matrix to decimals, as shown in [Table 9.2](#). The row average for each site is also shown in [Table 9.2](#).

Table 9.2. The normalized matrix with row averages

Customer Market				
Site	<i>A</i>	<i>B</i>	<i>C</i>	Row Average
A	0.5455	0.3333	0.6250	0.5012

B	0.1818	0.1111	0.0625	0.1185
C	0.2727	0.5556	0.3125	0.3803
				1.0000

[Page 408]

The row averages in [Table 9.2](#) provide us with Southcorp's preferences for the three sites for the customer market criterion. The most preferred site is Atlanta, followed by

Charlotte; the least preferred site (for this criterion) is Birmingham. We can write these preferences as a matrix with one column, which we will refer to as a *vector*:

Customer Market

$$\begin{array}{l} \text{A} \\ \text{B} \\ \text{C} \end{array} \begin{bmatrix} 0.5012 \\ 0.1185 \\ 0.3803 \end{bmatrix}$$

1.0000

The preference vectors for the other decision criteria are computed similarly:

<i>Income Level</i>	<i>Infrastructure</i>	<i>Transportation</i>
A $\begin{bmatrix} 0.2819 \end{bmatrix}$	A $\begin{bmatrix} 0.1790 \end{bmatrix}$	A $\begin{bmatrix} 0.1561 \end{bmatrix}$
B $\begin{bmatrix} 0.0598 \end{bmatrix}$	B $\begin{bmatrix} 0.6850 \end{bmatrix}$	B $\begin{bmatrix} 0.6196 \end{bmatrix}$
C $\begin{bmatrix} 0.6583 \end{bmatrix}$	C $\begin{bmatrix} 0.1360 \end{bmatrix}$	C $\begin{bmatrix} 0.2243 \end{bmatrix}$

These four preference vectors for the four criteria are summarized in a single preference matrix, shown in [Table 9.3](#).

Table 9.3. Criteria preference matrix

Criterion				
Site	MARKET	INCOME LEVEL	INFRASTRUCTURE	TRANSP

A	0.5012	0.2819	0.1790	0.1
B	0.1185	0.0598	0.6850	0.6
C	0.3803	0.6583	0.1360	0.2

Ranking the Criteria

The next step in AHP is to determine the relative importance, or weight, of the criteria that is, to rank the

criteria from most important to least important. This is accomplished the same way we ranked the sites within each criterion: using pairwise comparisons. The following pairwise comparison matrix for the four criteria in our example was developed by using the preference scale in [Table 9.1](#):

Criterion	Market	Income	Infrastructure
Market	1	1/5	3
Income	5	1	9

Infrastructure	1/3	1/9	1
Transportation	1/4	1/7	1/2

The normalized matrix converted to decimals, with the row averages for each criterion, is shown in [Table 9.4](#).

Table 9.4. Normalized matrix row averages

Criterion	Market Income Infrastruct		
Market	0.1519	0.1375	0.2222
Income	0.7595	0.6878	0.6667
Infrastructure	0.0506	0.0764	0.0741
Transportation	0.0380	0.0983	0.0370

The preference vector,

computed from the normalized matrix by computing the row averages in [Table 9.4](#), is as follows:

	<i>Criterion</i>
Market	0.1993
Income	0.6535
Infrastructure	0.0860
Transportation	0.0612

Clearly, income level is the highest-priority criterion, and customer market is second. Infrastructure and transportation appear to be relatively unimportant third- and fourth-ranked priorities in

terms of the overall objective of determining the best site for the new shopping mall. The next step in AHP is to combine the preference matrices we developed for the sites for each criterion in [Table 9.3](#) with the preceding preference vector for the four criteria.

Developing an Overall Ranking

Recall that earlier we summarized Southcorp's preferences for each site for

each criterion in a preference matrix shown in [Table 9.3](#) and repeated as follows:

		<i>Criterion</i>			
		Market	Income	Infrastructure	Transportation
<i>Site</i>	A	0.5012	0.2819	0.1790	0.1561
	B	0.1185	0.0598	0.6850	0.6196
	C	0.3803	0.6583	0.1360	0.2243

In the previous section we used pairwise comparisons to develop a preference vector for the four criteria in our example:

	<i>Criterion</i>
Market	0.1993
Income	0.6535
Infrastructure	0.0860
Transportation	0.0612

An overall score for each site is computed by multiplying the values in the criteria preference vector by the preceding criteria matrix and summing the products, as follows:

$$\begin{aligned} \text{site A score} &= 0.1993(0.5012) \\ &+ 0.6535(0.2819) + \\ &0.0860(0.1790) + \\ &0.0612(0.1561) = 0.3091 \end{aligned}$$

$$\begin{aligned}\text{site B score} &= 0.1993(0.1185) \\ &+ 0.6535(0.0598) + \\ &0.0860(0.6850) + \\ &0.0612(0.6196) = 0.1595\end{aligned}$$

$$\begin{aligned}\text{site C score} &= 0.1993(0.3803) \\ &+ 0.6535(0.6583) + \\ &0.0860(0.1360) + \\ &0.0612(0.2243) = 0.5314\end{aligned}$$

[Page 410]

The three sites, in order of the magnitude of their scores, result in the following AHP ranking:

Site	Score
Charlotte	0.5314
Atlanta	0.3091
Birmingham	0.1595
	1.0000

Based on these scores developed by AHP, Charlotte should be selected as the site for the new shopping mall, with

Atlanta ranked second and Birmingham third. To rely on this result to make its site selection decision, Southcorp must have confidence in the judgments it made in the pairwise comparisons, and it must also have confidence in AHP. However, whether the AHP-recommended decision is the one made by Southcorp or not, following this process can help identify and prioritize criteria and enlighten the company about how it makes decisions.

Following is a summary of the

mathematical steps used to arrive at the AHP-recommended decision:

Develop a pairwise comparison matrix for each

- 1.** decision alternative (site) for each criterion.

Synthesization:

- 1.** Sum the values in each column of the pairwise comparison matrices.
- 2.** Divide each value in

each column of the pairwise comparison matrices by the corresponding column sum these are the *normalized* matrices.

- 2.
3. Average the values in each row of the normalized matrices these are the *preference vectors*.
4. Combine the vectors of preferences for each criterion (from step 2c) into one preference

matrix that shows the preference for each site for each criterion.

3. Develop a pairwise comparison matrix for the criteria.

4. Compute the normalized matrix by dividing each value in each column of the matrix by the corresponding column sum.

Develop the preference vector by computing the

- 5.** row averages for the normalized matrix.

Compute an overall score for each decision alternative by multiplying the criteria

- 6.** preference vector (from step 5) by the criteria matrix (from step 2d).

Rank the decision alternatives, based on the

- 7.** magnitude of their scores computed in step 6.

AHP Consistency

AHP is based primarily on the pairwise comparisons a decision maker uses to establish preferences between decision alternatives for different criteria. The normal procedure in AHP for developing these pairwise comparisons is for an interviewer to elicit verbal preferences from the decision maker, using the preference scale in [Table 9.1](#). However, when a decision maker has to make a lot of comparisons (i.e., three or more), he or she can

lose track of previous responses. Because AHP is based on these responses, it is important that they be in some sense valid and especially that the responses be consistent. That is, a preference indicated for one set of pairwise comparisons needs to be consistent with another set of comparisons.

Management Science

Application: Analyzing

Advanced-Technology

Projects at NASA

Evaluating advanced-technology projects at the National Aeronautics and Space Administration (NASA) is a critical multicriteria decision-making process. During the past two decades, technological advances have increased the number of possible projects that NASA might undertake. At the same time, the government and public have become more demanding about cost accountability. The project evaluation and selection process is very complex, requiring the consideration and analysis of vast amounts of

information, concerning such things as safety, systems engineering, cost, reliability, implementation, and project success without a formal, structured decision-making model.

At the Shuttle Project Engineering Office at the Kennedy Space Center, a multicriteria group decision-making model called CROSS (consensus-ranking organizational-support system) was developed for the project evaluation process, based in part on the analytical hierarchy process (AHP).

Prior to the implementation of CROSS, the Shuttle Project Engineering Office used a 15-member working committee to evaluate advanced-technology projects, which are independent requests for engineering changes to the space shuttle program. Each committee

member represents a specific department at the Kennedy Space Center. Committee members assigned each an assessment score from 1 to 10, and these scores were averaged to produce an overall score. Projects were then sorted into three categories, based on these scores: superior, borderline, and inferior. This process was considered to be very intuitive and subjective and thus subject to inconsistency.

CROSS guides the committee through a systematic evaluation process, using AHP to weight the stakeholder departments that will be responsible for implementation of selected projects. The stakeholders then use AHP to identify and weight the criteria that should be used to evaluate the projects. For example, a typical evaluation scenario might include a budget of \$6 million, with

10 possible projects (costing a total of \$15 million) encompassing six stakeholder departments and 40 criteria. Individual project scores are calculated by using an Excel program. AHP is also used to evaluate the decision makers' consistency. CROSS helped NASA decision makers to think systematically about the project evaluation and selection process and improve the quality of their decisions. It also brought consistency to the decision-making process.



Source: M. Tavana, "CROSS: A Multicriteria Group-Decision-Making Model for Evaluating and Prioritizing Advanced-Technology Projects at NASA." *Interfaces* 33, no. 3, (May/June 2003): 4056.

Using our site selection example, suppose for the income level criterion, Southcorp indicates that A is "very strongly preferred" to B and that A is "moderately preferred" to C. That's fine, but then suppose Southcorp says that C is "equally preferred" to B for the same criterion. That comparison is not entirely consistent with the previous two pairwise comparisons. To say that A is strongly preferred over B and moderately preferred over C and then turn around and say C is equally

preferred to B does not reflect a consistent preference rating for the three sets of pairwise comparisons. A more logical comparison would be that C is preferred to B to some degree. This kind of inconsistency can creep into AHP when the decision maker is asked to make verbal responses for a lot of pairwise comparisons. In general, they are usually not a serious problem; some degree of slight inconsistency is expected. However, a consistency index can be computed that measures the

degree of inconsistency in the pairwise comparisons.

A consistency index measures the degree of inconsistency in pairwise comparisons.

[Page 412]

To demonstrate how to compute the consistency index (CI), we will check the consistency of the pairwise comparisons for

the four site selection criteria. This matrix, shown as follows, is multiplied by the preference vector for the criteria:

	<i>Market</i>	<i>Income</i>	<i>Infrastructure</i>	<i>Transportation</i>		<i>Criteria</i>
Market	1	1/5	3	4	×	0.1993
Income	5	1	9	7		0.6535
Infrastructure	1/3	1/9	1	2		0.0860
Transportation	1/4	1/7	1/2	1		0.0612

The product of the multiplication of this matrix and vector is computed as follows:

$$(1)(0.1993) + (1/5)(0.6535) + (3)(0.0860) + (4)(0.0612) = 0.8328$$

$$(5)(0.1993) + (1)(0.6535) + (9)(0.0860) + (7)(0.0612) = 2.8524$$

$$(1/3)(0.1993) + (1/9)(0.6535) + (1)(0.0860) + (2)(0.0612) = 0.3474$$

$$(1/4)(0.1993) + (1/7)(0.6535) + (1/2)(0.0860) + (1)(0.0612) = 0.2473$$

Next, we divide each of these values by the corresponding weights from the criteria preference vector:

$$0.8328/0.1993 = 4.1786$$

$$2.8524/0.6535 = 4.3648$$

$$0.3474/0.0860 = 4.0401$$

$$0.2474/0.0612 = \underline{4.0422}$$

$$16.6257$$

If the decision maker, Southcorp, were a perfectly consistent decision maker, then each of these ratios would be

exactly 4, the number of items we are comparing in this case, four criteria. Next, we average these values by summing them and dividing by 4:

$$\frac{16.6257}{4} = 4.1564$$

The consistency index, CI , is computed using the following formula:

$$CI = \frac{4.1564 - n}{n - 1}$$

where

n = the number of items being

compared

4.1564 = the average we computed previously

$$CI = \frac{4.1564 - 4}{3} \\ = 0.0521$$

If $CI = 0$, then Southcorp would be a perfectly consistent decision maker. Because Southcorp is not perfectly consistent, the next question is the degree of inconsistency that is acceptable. An acceptable level of consistency is determined by comparing the CI to a *random index*, RI , which

is the consistency index of a randomly generated pairwise comparison matrix. The *RI* has the values shown in [Table 9.5](#), depending on the number of items, n , being compared. In our example, $n = 4$ because we are comparing four criteria.

[Page 413]

Table 9.5. *RI* values for n items being compared

n	2	3	4	5	6	7	8	9	10
<i>RI</i>	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.51

The degree of consistency for the pairwise comparisons in the decision criteria matrix is determined by computing the ratio of CI to RI :

$$\frac{CI}{RI} = \frac{0.0521}{0.90} = 0.0580$$

In general, the degree of consistency is satisfactory if $CI/RI < 0.10$, and in this case, it is. If $CI/RI > 0.10$, then there are probably serious inconsistencies, and the AHP results may not be meaningful.

Remember that in this instance, we have evaluated the degree of consistency only for the pairwise comparisons in the decision criteria preference matrix. This does not mean that we have verified the consistency for the entire AHP. We would still have to evaluate the pairwise comparisons for each of the four individual criterion matrices before we could be sure the entire AHP for this problem was consistent.

AHP with Excel

Spreadsheets

The various computational steps involved in AHP can be accomplished with Excel spreadsheets. [Exhibit 9.12](#) shows a spreadsheet formatted for our Southcorp site selection example. The spreadsheet includes the pairwise comparison matrix for our customer market criterion. Notice that the pairwise comparisons have been entered as fractions. We can enter fractions in Excel by dragging the cursor over the object cells,

in this case **B5:D7**, and then clicking on "Format" at the top of the screen. When the "Format" menu comes down, we click on "Cells" and then select "Fraction" from the menu.

Exhibit 9.12.

[\[View full size image\]](#)

[illegible]

Row average
formula for F14

=B5/B8

=SUM(B5:B7)

formula =**SUM(B5:B7)** in cell B8. The sums for columns B and C can then be obtained by putting the cursor on cell B8, clicking on the right mouse button, and clicking on "Copy." Then we drag the cursor over cells **B8:D8** and press the "Enter" key. This will compute the sum for the other two columns.

Management Science

Application: Selecting a Site for a New Ice Hockey Arena by Using AHP

(This item is displayed on page 414 in the print version)

In the late 1980s the city of Åbo in Finland decided to build a new 12,000-seat ice hockey arena that could also be used for concerts and other cultural events. Åbo, an old historic city and the third largest in Finland, has two national ice hockey teams, a basketball team, and a volleyball team that could use the arena. Initially, six potential sites in and around Åbo were identified for the new arena, but debate over

which site to select and the cost of the project became hotly contested political issues that thwarted the entire selection process. Several feasibility studies by consultants did not result in a site decision, essentially because no single site alternative could gain political acceptance between the two major political parties involved and various interest groups.

In this decision environment, the authors of this study, from Åbo Akademi University, were asked by a group of city administrators to run an analytical hierarchy process (AHP) on the site selection problem. The first step in applying AHP to this problem was to teach the group involved about AHP. Five alternative sites for the arena were identified: Artukainen, Ikituuri, Kupittaa, Länsiranta, and Urheilupuisto. Some of the sites were

new, and others were sites that had been considered previously. Six criteria were identified for making the site selection decision: (1) *access* to various categories of customers and users for sports and cultural events, as well as meetings; (2) *old city* considerations that would preserve historic sites and the privacy of the local citizenry; (3) *foundation* considerations that would take into account the costly site preparation work due to different rock and clay formations and the water table level at the sites; (4) *opinions of users* (as opposed to customers) about the sites; (5) *public services* available, including water, electricity, sewers, and heating; and (6) the *architecture* of the building relative to the site, historical surroundings, use, and so on. Overall project cost was not included as a criterion because it was

similar for all the sites. The preference vector for the criteria, shown as follows, indicated that the highest-priority criterion was the preservation of Åbo's old city, historical image:

	<i>Criteria</i>
Access	0.122
Old City	0.433
Foundation	0.208
Users	0.020
Public services	0.037
Architecture	0.122

The AHP synthesis resulted in the following overall ranking of the sites:



Site	Score
Kupittaa	0.302
Urheilupuisto	0.276
Artukainen	0.177

Ikituuri	0.154
Länsiranta	0.091

The overall consistency factor for the AHP was 0.17, which indicates a high degree of inconsistency in the pairwise comparisons. This was not unexpected, given the large number of people and elements involved in the process.

This result with the Kupittaa site as the top choice was surprising in that Ikituuri had been recommended (unanimously) in a previous feasibility study. However, neither of these sites nor the AHP-recommended site at Kupittaa was selected by the city

council. The Artukainen site was selected as part of a political deal. In return for a series of political nominations, one political party allowed the other party to build the arena, provided that it was at Artukainen. There it would be far from the city's center, with easy access to customers, thus making it a "liability" to the opposing party that supported it. The arena was constructed and has been home to the Finnish national champion ice hockey team on several occasions as well as the site of numerous concerts and cultural events.

Although the AHP-recommended site was not selected, AHP did exhibit several beneficial strengths. It enabled all the elements of the decision situation to be identified and organized into a single model. It was the first time the decision makers

actually understood the scope and breadth of the problem. Through the use of pairwise comparisons, AHP forced the participants to articulate the relative importance of the decision criteria.

Source: C. Carlsson and P. Walden, "AHP in Political Group Decisions: A Study in the Art of Possibilities," *Interfaces* 25, no. 4 (July/August 1995): 1429.

The cell values for the normalized matrix at the bottom of the spreadsheet in [Exhibit 9.12](#) can be computed

as follows. First, make sure you convert the cell values to decimal numbers. To start, divide the value in cell B5 by the column sum in cell B8 by typing **=B5/B8** in cell B14. This results in the value 0.5455 in cell B14. Next, with the cursor on B14, click the right mouse button and then click on "Copy." Drag the cursor across cells **B14:D14** and press the "Enter" key. This will compute the values for cells C14 and D14. Repeat this process for each row to complete the normalized matrix.

To compute the row averages from the normalized matrix, first type the formula **=SUM(B14:D14)/3** in cell F14. This formula is displayed on the formula bar at the top of the spreadsheet. This will compute the value 0.5013 in cell F14. Next, with the cursor on cell F14, click the right mouse button and copy the formula in cell F14 to F15 and F16 by dragging the cursor across cells **F14:F16**.

Recall that these row averages represent the preference vector for the customer market

criterion. To complete this stage in AHP, we would need to compute the preference vectors for the other three criteria for income level, infrastructure, and transportation.

The next step in AHP is to develop the pairwise comparison matrix and normalized matrix for our four criteria. The spreadsheet to accomplish this step is shown in [Exhibit 9.13](#). These two matrices are constructed in the same way as the two similar matrices in [Exhibit 9.12](#).

Exhibit 9.13.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 9.13.xls

File Edit View Insert Format Tools Data Window Help

100%

Formulas

fx

014 =SUM(B5:B8)

1

2

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

18

19

A B C D E F G H I J K L M N

1 JNP for the Southcorp Development Site Selection Example

2

3

4 Criteria Market Income Infrastructure Transportation

5 Market 1 15 3 4

6 Income 5 1 9 7

7 Infrastructure 15 15 1 2

8 Transportation 14 17 12 1

9 Column sums 6 712 1 511 13 12 54

10

11 Normalized Matrix

12

13 Criteria Market Income Infrastructure Transportation

14 Market 0.1519 0.1376 0.2223 0.2857

15 Income 0.3565 0.8878 0.6667 0.5000

16 Infrastructure 0.0506 0.0764 0.0741 0.1429

17 Transportation 0.0360 0.0969 0.0370 0.0714

18

19

Row Averages

0.1553

0.8535

0.0660

0.0612

1.0000

=SUM(B5:B8)

=B5/B9

=SUM(B14:E14)/4

The final step in AHP is to develop an overall ranking. This requires Southcorp's preference matrix for each site for all four criteria ([Table 9.3](#)) and the preference vector for the criteria, both of which are included in the spreadsheet shown in [Exhibit 9.14](#).

Exhibit 9.14.

(This item is displayed on page 416 in the print version)

[\[View full size image\]](#)

Atlanta score
in cell C12

Microsoft Excel - Exhibit 9.14.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

Formulas: =SUM(B5:H5)+H5

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	AHP for the Southcorp Development Site Selection Example													
2														
3		Criteria												
4	Site	Market	Income	Infrastructure	Transportation		Market	0.1993						
5	A	0.9013	0.2819	0.1790	0.1561		Income	0.6535						
6	B	0.1185	0.0558	0.6850	0.6196		Infrastructure	0.0860						
7	C	0.3803	0.6583	0.1360	0.2243		Transportation	0.0612						
8														
9														
10														
11		Site	Score											
12		Atlanta	0.3021											
13		Birmingham	0.1505											
14		Charlotte	0.5314											
15			1.0000											
16														

Cells F14:F16 from Exhibit 9.12

Cells G14:G17 from Exhibit 9.13

All the values in the two matrices in [Exhibit 9.14](#) were entered as input from previous AHP steps. The score for

Atlanta was computed using the formula embedded in cell C12 and shown on the formula bar. Similar formulas were used for the other two site scores in cells C13 and C14.

[Exhibit 9.15](#) shows the spreadsheet for evaluating the degree of consistency for the pairwise comparisons in the preference matrix for the four criteria. The spreadsheet is broken down into steps that are self-explanatory.

Exhibit 9.15.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 9.15.xls

File Edit View Insert Format Tools Data Window Help

Formulas: =B5/G5+D5/G5+D5/G5+D5/G5

1 AHP for the Southcorp Development Site Selection Example

2

3 (1) Pairwise comparison matrix for the criteria

4

5

6

7

8

9

10

11

12

13

14

15

16

(2) Preference vector

X

0.1993

0.0535

0.0880

0.0612

(3) Product

0.8528

2.8524

0.3474

0.3474

(3/2)

4.1798

4.3048

4.0401

4.0422

16.6257

divided by 4 = 4.1564

G1 = 0.0521

CVR = 0.058

=SUM(B11:B14)

=I5/G5

=(4.1564 - 4)/3

=G12/0.90

Scoring Models

For selecting among several alternatives according to various criteria, a *scoring model* is a method similar to AHP, but it is mathematically simpler. There are several versions of scoring models. In the scoring model that we will use, the decision criteria are weighted in terms of their relative importance, and each decision alternative is graded in terms of how well it satisfies

the criteria, according to the following formula:

$$S_i = \sum g_{ij} w_j$$

where

[Page 417]

w_j = a weight between 0 and 1.00 assigned to criterion j , indicating its relative importance, where 1.00 is extremely important and 0 is not important at all. The sum of the total weights equals 1.00.

a grade between 0 and 100 indicating how well the decision alternative i

g_{ij} = satisfies criterion j , where 100 indicates extremely high satisfaction and 0 indicates virtually no satisfaction.

S_i = the total "score" for decision alternative i , where the higher the score, the better.

To demonstrate the scoring model, we will use an example. Sweats and Sweaters is a chain of stores specializing in cotton apparel. The company wants to open a new store in one of four malls around the Atlanta

metropolitan area. The company has indicated five criteria that are important in its decision about where to locate: proximity of schools and colleges, area median income, mall vehicle traffic flow and parking, quality and size (in terms of number of stores in the mall), and proximity of other malls or shopping areas. The company has weighted each of these criteria in terms of its relative importance in the decision-making process, and it has analyzed each potential mall location and graded them

according to each criterion as shown in the following table:

		Grades for Alternative (0 to 100)			
Decision Criteria	Weight (0 to 1.00)	MALL 1	MALL 2	MALL 3	MALL 4
School proximity	0.30	40	60	90	60
Median income	0.25	75	80	65	90
Vehicle	0.25	60	90	79	85

traffic

Mall quality and size	0.10	90	100	80	90
--------------------------	------	----	-----	----	----

Proximity of other shopping	0.10	80	30	50	70
-----------------------------------	------	----	----	----	----

The scores, S_i , for the decision alternative are computed as follows:

$$S_1 = (.30)(40) + (.25)(75) + (.25)(60) + (.10)(90) + (.10)$$

$$(80) = 62.75$$

$$S_2 = (.30)(60) + (.25)(80) + (.25)(90) + (.10)(100) + (.10)(30) = 73.50$$

$$S_3 = (.30)(90) + (.25)(65) + (.25)(79) + (.10)(80) + (.10)(50) = 76.00$$

$$S_4 = (.30)(60) + (.25)(90) + (.25)(85) + (.10)(90) + (.10)(70) = 77.75$$

Because mall 4 has the highest score, it would be the recommended decision, followed by mall 3, mall 2, and

finally mall 1.

Scoring Model with Excel Solution

The steps included in computing the scores of the decisions in the scoring model are relatively simple and straightforward. [Exhibit 9.16](#) shows a spreadsheet set up for the Sweats and Sweaters store location example. The formula that computes the score for mall 1 in cell D10, **=SUMPRODUCT (C5:C9,**

D5:D9), is shown on the formula bar at the top of the spreadsheet. This formula is copied and pasted using the right mouse button into cells E10, F10, and G10.

[Page 418]

Exhibit 9.16.

(This item is displayed on page 417 in the print version)

[\[View full size image\]](#)

Summary

This chapter introduced the concept of making decisions when there is more than one objective or criterion to consider. Three specific modeling techniques were presented to solve decision-making problems with multiple criteria: goal programming, the analytical hierarchy process (AHP), and scoring methods. These techniques can be applied to a wide variety of

decision-making situations when there are objectives besides just profit or cost. They are often applicable to decision-making problems in public or governmental organizations in which the levels of service or efficiency in carrying out numerous goals are more important than are profit or cost.

The presentation of goal programming completes the coverage of linear programming. One of the implicit assumptions of linear programming has been that all

parameters and values in the model were known with certainty. Upcoming chapters will consider techniques that are probabilistic and are used when a situation includes uncertainty.

References

Charnes, A., and Cooper, W. W. *Management Models and Industrial Applications of Linear Programming*. New York: John Wiley & Sons, 1961.

Dyer, J. S. "Interactive Goal Programming." *Management Science* 19, no. 1 (1972): 6270.

Ignizio, J. P. *Goal Programming and Extensions*. Lexington, MA: Lexington Books, 1976.

Ijiri, Y. *Management Goals and Accounting for Control*. Chicago: Rand McNally, 1965.

Lee, S. M. *Goal Programming for Decision Analysis*. Philadelphia: Auerbach Publishers, 1972.

Lee, S. M., and Clayton, E. R. "A Goal Programming Model for Academic Resource Allocation." *Management Science* 8, no. 8 (1972): 395408.

Saaty, T. *The Analytic Hierarchy Process*. New York: McGraw-Hill, 1988.

Zeleny, M. *Multiple Criteria Decision Making*. New York: McGraw-Hill, 1982.

Example Problems Solutions

As a prelude to the problems section, the following examples demonstrate the formulations and solutions of a goal programming problem, and an AHP problem.

Problem Statement

Rucklehouse Public Relations has contracted to do a survey

following an election primary in New Hampshire. The firm must assign interviewers to carry out the survey. The interviews are conducted by telephone and in person. One person can conduct 80 telephone interviews or 40 personal interviews in a day. It costs \$50 per day for a telephone interviewer and \$70 per day for a personal interviewer. The following three goals, which are listed in order of their priority, have been established by the firm to ensure a representative survey:

- 1.** At least 3,000 total interviews should be conducted.
- 2.** An interviewer should conduct only one type of interview each day. The firm wants to maintain its daily budget of \$2,500.
- 3.** At least 1,000 interviews should be by telephone.

Formulate a goal programming model to determine the number of interviewers to hire to satisfy these goals and then solve the model.

Solution

Model Formulation

minimize $P_1 d_1^-, P_2 d_2^+, P_3 d_3^-$

subject to

$$80x_1 + 40x_2 + d_1^- - d_1^+ = 3,000 \text{ interviews}$$

$$50x_1 + 70x_2 + d_2^- - d_2^+ = \$2,500 \text{ budget}$$

$$80x_1 + d_3^- - d_3^+ = 1,000 \text{ telephone interviews}$$

Step 1. where

x_1 = number of

telephone interviewers

x_2 = number of
personal interviewers

The QM for Windows Solution

[\[View full size image\]](#)

QM - The Quality Manager

File Edit Tools Report Data Options Help

Minimize 1.5x1 + 1.2x2

Constraint: x1 + 2x2 <= 100

For Objective Function

	x1	x2	x3	x4	x5	x6	x7
Objective	1.5	1.2	0	0	0	0	0
Constraint 1	1	2	0	0	0	0	0
Constraint 2	0	0	1	1	0	0	0

[\[View full size image\]](#)

Step 2.

Summary				
Ruckelhouse Public Relations Solution				
Item				
Decision variable analysis	Value			
X1	30.5556			
X2	13.8889			
Priority analysis	Nonachievement			
Priority 1	0			
Priority 2	0			
Priority 3	0			
Constraint Analysis	RHS	d+ (row i)	d- (row i)	
Interviews	3,000	0	.0002	
Budget (\$)	2,500	0	0	
Telephone interviews	1,000	1,444.444	0	

Problem Statement

Grace LeMans wants to purchase a new mountain bike, and she is considering three models: the Xandu Mark III, the Yellow Hawk Z9, and the Zodiac MB5. Grace has identified three criteria for selection on which she will base her decision: purchase price, gear action, and weight/durability. Grace has developed the following pairwise comparison matrices for the three criteria:

Price

Bike	<i>X</i>	<i>Y</i>	<i>Z</i>
<i>X</i>	1	3	6
<i>Y</i>	1/3	1	2
<i>Z</i>	1/6	1/2	1

[Page 420]

Gear Action

Bike	X	Y	Z
X	1	1/3	1/7
Y	3	1	1/4
Z	7	4	1

Weight/Durability

Bike	X	Y	Z
X	1	3	1

Y	1/3	1	1/2
Z	1	2	1

Grace has prioritized her decision criteria according to the following pairwise comparisons:

Criteria	Price	Gears	Weight
Price	1	3	5

Gears	$1/3$	1	2
Weight	$1/5$	$1/2$	1

Using AHP, develop an overall ranking of the three bikes Grace is considering.

Solution

**Develop Normalized I
Preference Vectors fo
Pairwise Comparison**

Criteria

		Price		
Bike	X	Y		
X	0.6667	0.6667	0	
Y	0.2222	0.2222	0	
Z	0.1111	0.1111	0	

2 0.1111 0.1111 0

Gear Action

Bike	X	Y
------	---	---

X	0.0909	0.0625	0
---	--------	--------	---

Y 0.2727 0.1875 0

Z 0.6364 0.7500 0

Step 1.

[Page 42

Weight/Durab

Bike

X

Y

X 0.4286 0.5000 0

Y 0.1429 0.1667 0

Z 0.4286 0.3333 0

The preference vectors
in the following matrix:

Criteria		
Bike	Price	Gears
X	0.6667	0.0853
Y	0.2222	0.2132
Z	0.1111	0.7014

Rank the Criteria

Criteria Price Gears Value

Price 0.6522 0.6667

Gears 0.2174 0.2222

Weight 0.1304 0.1111

Step 2.

The preference vector for

	<i>Criteria</i>
Price	$\begin{bmatrix} 0.6479 \end{bmatrix}$
Gears	$\begin{bmatrix} 0.2299 \end{bmatrix}$
Weight	$\begin{bmatrix} 0.1222 \end{bmatrix}$

Develop an Overall Ra

Step 3.

$$\begin{array}{c} \text{Price} \quad \text{Gears} \quad \text{Weight} \\ \text{X} \begin{bmatrix} 0.6667 & 0.0853 & 0.4429 \end{bmatrix} \\ \text{Y} \begin{bmatrix} 0.2222 & 0.2132 & 0.1698 \end{bmatrix} \\ \text{Z} \begin{bmatrix} 0.1111 & 0.7014 & 0.3873 \end{bmatrix} \end{array} \times \begin{array}{c} \text{Price} \\ \text{Gears} \\ \text{Weight} \end{array} \begin{bmatrix} 0.6479 \\ 0.2299 \\ 0.1222 \end{bmatrix}$$

$$\begin{aligned} \text{bike X score} &= 0.6667(0.6479) + 0.0853(0.2299) + 0.4429(0.1222) \\ &= 0.5057 \end{aligned}$$

$$\begin{aligned} \text{bike Y score} &= 0.2222(0.6479) + 0.2132(0.2299) + 0.1698(0.1222) \\ &= 0.2138 \end{aligned}$$

$$\begin{aligned} \text{bike Z score} &= 0.1111(0.6479) + 0.7014(0.2299) + 0.3873(0.1222) \\ &= 0.2806 \end{aligned}$$

[Page 422]

The ranking of the three bikes,
in order of the magnitude of

their scores, is

Bike	Score
Xandu	0.5057
Zodiak	0.2806
Yellow Hawk	0.2138
	1.0000

Problems

A manufacturing company produces three products. The resource requirements for the three products have the following resource requirements and the following profit:

Product	Labor (hr./unit)	Material (lb./unit)
1	5	4
2	2	6
3	4	3

At present the firm has a daily labor cap
daily supply of 400 pounds of material.
formulation for this problem is as follow

1. maximize $Z = 3x_1 + 5x_2 + 2x_3$
subject to

$$5x_1 + 2x_2 + 4x_3 \leq 240$$

$$4x_1 + 6x_2 + 3x_3 \leq 400$$

$$x_1, x_2, x_3 \geq 0$$

Management has developed the followi
their importance to the firm:

- 1.** Because of recent labor relations
avoid underutilization of normal p
- 2.** Management has established a sa
day.
- 3.** Overtime is to be minimized as m

4. Management wants to minimize the cost of handling and storage problems.

Formulate a goal programming model to determine the number of each product to produce to best satisfy the objectives.

In [Problem 15](#) in [Chapter 6](#), Computers & Computers distributes them from three warehouses to three universities. The supply at the three warehouses, the demand at the three universities, and the shipping costs are shown in the following table.

University			
Warehouse	Tech	A&M	State
Richmond	\$22	17	15
Atlanta	15	35	25

Washington

28

21

Demand

520

250

4

2.

[Page

Instead of its original objective of cost has indicated the following goals, arranged in order of priority:

1. A&M has been one of its better long-term customers. Computers Unlimited wants to meet all of A&M's requirements.
2. Because of recent problems with the Washington branch, at least 80 units from the Washington branch must be shipped to the customer's demand.
3. To maintain the best possible relationship with the customer, Computers Unlimited would like to keep total transportation cost at a minimum.
4. It would like to keep total transportation cost at a minimum.

of the \$22,470 total cost achieve the transportation solution method.

- 5.** Because of dissatisfaction with the to-State deliveries, it would like to shipped over this route.
- 1.** Formulate a goal programming model the number of microcomputers to goals.
- 2.** Solve this model by using the con

The Bay City Parks and Recreation Department of \$600,000 to expand its public recreation representatives have demanded four different athletic fields, tennis courts, and swimming pools. various communities in the city has been tennis courts, and 12 swimming pools. requires a certain number of acres, and amount, as follows:

Facility	Cost	Recreation
Gymnasium	\$80,000	
Athletic field	24,000	
Tennis court	15,000	
Swimming pool	40,000	

3.

The Parks and Recreation Department is planning for future construction (although more land could be acquired). The department has established the following priority:

- 1.** The department wants to spend the money that has not been spent must be returned to the state.
 - 2.** The department wants the facilities to serve at least 20,000 people each week.
 - 3.** The department wants to avoid having to acquire more than 100 acres of land already located.
 - 4.** The department would like to meet the demand for new facilities. However, this goal is secondary to the number of people expected to use the facilities.
- 1.** Formulate a goal programming model to determine the type of facility should be constructed.
 - 2.** Solve this model by using the computer software for integers.

A farmer in the Midwest has 1,000 acres of land on which to plant corn, wheat, and soybeans. Each acre of land requires 7 worker-days of preparation, requires 7 worker-days of preparation, and each acre of wheat costs \$120 to prepare, and each acre of soybeans costs \$150 to prepare.

yields \$40 profit. An acre of soybeans (10 worker-days, and yields \$20 profit. The \$80,000 for crop preparation and has 10 worker-days of labor. A midwestern grain farmer has 1000 acres of corn, 500 acres of wheat, and has established the following goals, in c

[Page

1. To maintain good relations with the community; that is, the full 6,000 worker-days must be used.
2. Preparation costs should not exceed \$80,000; additional loans will not have to be taken.
3. The farmer desires a profit of at least \$100,000 in financial condition.
4. Contracting for excess labor should be avoided.
5. The farmer would like to use as much labor as possible.

- 6.** The farmer would like to meet the goal. However, the goal should be weighed by each crop.
- 1.** Formulate a goal programming model to determine the number of acres of each crop the farmer should plant in the best possible way.
- 2.** Solve this model by using the computer.

The Growall Fertilizer Company produces three types of fertilizer: Superagro, Dynaplant, and Soilsaver. The company has a maximum of 2,000 tons of fertilizer in inventory. The cost of a ton of Superagro is \$1,200, \$1,500 for Dynaplant, and \$1,800 for Soilsaver. The production process requires 10 hours of labor for a ton of Superagro, 12 hours for a ton of Dynaplant, and 18 hours for a ton of Soilsaver. The company has 800 hours of normal production time available each week. Each week the company can expect a demand of 1,200 tons of Superagro, 900 tons of Dynaplant, and 1,100 tons of Soilsaver. The company has established the following goals, in order of priority:

- 1.** The company does not want to exceed its normal production, if possible.

5.

2. The company would like to limit o
3. The company wants to meet dem
it is twice as important to meet th
meet the demand for Dynaplant, i
the demand for Dynaplant as it is
4. It is desirable to avoid producing u
5. Because of union agreements, the
underutilization of labor.
1. Formulate a goal programming m
tons of each brand of fertilizer to
2. Solve this model by using the con

The Barrett Textile Mill was checked by
Safety and Health Administration (OSHA)
violations in four categories: hazardous
powered tools, and machine guarding.
100% compliance. Each percentage po

in each category will reduce the frequency of accidents, reduce the accident cost per worker, and constitute OSHA compliance level. However, achieving this requires money. The following table shows the frequency of accidents (accident cost per worker) and the cost of achieving compliance in each category:

Category	Accident Frequency Reduction (accidents/10^5 hr of exposure)
1. Hazardous materials	0.18
2. Fire protection	0.11
3. Hand-powered tools	0.17

6. _____

[Page

To achieve 100% compliance in all four increase compliance in hazardous materials (is now at 40% compliance), in fire protection hand-powered tools by 35 percentage 17 percentage points. However, the management dilemma, in that only \$52,000 is available expenditure could jeopardize the financial management hopes to achieve a level that is within the company's budget limit authorities enough to temporarily delay management has established four goals:

1. Do not exceed the budget constraint

- 2.** Achieve the percentage increases
100% compliance in each category
- 3.** Achieve total accident frequency rate
of exposure. (This goal denotes number
frequency of accidents even if 100%
all categories.)
- 4.** Reduce the total accident cost per
employee
- 1.** Formulate a goal programming model
points of compliance needed in each
category
- 2.** Solve this model by using the computer

Solve the following goal programming problem
computer:

7.

minimize $P_1 d_1^+, P_2 d_2^-, P_3 d_3^-$

subject to

$$4x_1 + 2x_2 + d_1^- - d_1^+ = 80$$

$$x_1 + d_2^- - d_2^+ = 30$$

$$x_2 + d_3^- - d_3^+ = 50$$

$$x_j, d_i^-, d_i^+ \geq 0$$

Solve the following goal programming problem using a computer:

minimize $P_1 d_1^- + P_1 d_1^+, P_2 d_2^-, P_3 d_3^-, 3P_4 d_2^+ + 5P_4 d_3^+$

8. subject to

$$x_1 + x_2 + d_1^- - d_1^+ = 800$$

$$5x_1 + d_2^- - d_2^+ = 2,500$$

$$3x_2 + d_3^- - d_3^+ = 1,400$$

$$x_j, d_i^-, d_i^+ \geq 0$$

Solve the following goal programming problem using a computer:

minimize $P_1 d_3^-, P_2 d_2^-, P_3 d_1^+, P_4 d_2^+$

subject to

9. $4x_1 + 6x_2 + d_1^- - d_1^+ = 48$

$$2x_1 + x_2 + d_2^- - d_2^+ = 20$$

$$d_2^+ + d_3^- - d_3^+ = 10$$

$$x_2 + d_4^- = 6$$

$$x_j, d_i^-, d_i^+ \geq 0$$

The Wearever Carpet Company manufactures shag and sculptured in 100-yard lots. It requires 4 yards of shag carpet and 6 hours to produce one lot of shag carpet. The company has the following production

[Page

1. Do not underutilize production capacity

10. 2. Achieve product demand of 40 (100-yard) lots for sculptured carpet. Meeting demand for sculptured carpet is more important than meeting demand for shag carpet.

- 3.** Limit production overtime to 20 h
- 1.** Formulate a goal programming model for the production of shag and sculptured carpet to produce the following goals.
- 2.** Solve this model by using the conventional simplex method.

The East Midvale Textile Company produces two types of carpet: shag and sculptured. The average production rate for shag is 100 yards per hour, and the normal weekly production is 80 hours. The marketing department estimates that the demand is for 60,000 yards of denim and 40,000 yards of cotton. The profit is \$3.00 per yard for denim and \$2.00 per yard for cotton. The company has established the following goals in order of importance:

- 1.** Eliminate underutilization of production capacity and employment levels.
- 2.** Limit overtime to 10 hours.
- 3.** Meet demand for denim and brushed cotton.

11.

profit for each.

- 4.** Minimize overtime as much as possible
- 1.** Formulate a goal programming model to determine the number of yards (in 1,000-yard lots) to produce of each product
- 2.** Solve this model by using the computer

The Oregon Atlantic Company produces two products: white wrapping paper (butcher paper) and newsprint. It takes 3 yards of newsprint and 8 minutes to produce a yard of wrapping paper. The company has 4,800 minutes of normal time each week. The profit is \$0.20 for a yard of wrapping paper. The weekly demand is 1,000 yards of wrapping paper. The company has the following goals in order of priority:

- 1.** Limit overtime to 480 minutes.
- 12. 2.** Achieve a profit of \$300 each week
- 3.** Fulfill the demand for the products

profits.

- 4.** Avoid underutilization of production facilities.
- 1.** Formulate a goal programming model in terms of yards of each type of paper to produce to meet the goals.
- 2.** Solve the goal programming model.

A rural clinic hires its staff from nearby hospitals. The clinic attempts to have a general practitioner or an internist on duty during at least a portion of each week with a weekly budget of \$1,200. A GP charges \$20 per hour, and an internist charges \$30 per hour. The clinic has established the following goals, in order of priority:

- 1.** A nurse should be available at least 40 hours per week.
- 2.** The weekly budget of \$1,200 should not be exceeded.
- 3.** A GP or an internist should be available at least 16 hours per week.

13.

4. An internist should be available at
1. Formulate a goal programming model to determine the number of hours to hire each staff member to
2. Solve the model by using the corner point method.

[Page

The Eaststate Manufacturing Company produces a variety of parts from fabricated sheet metal for several different customers. The manufacturing process consists of four operations: cutting, forming, finishing, and packaging. The processing times and total available hours per year to produce each part are given below.

	Part (hr./unit)		
Operation	1	2	3

Stamping	0.06	0.17	0.10
Assembly	0.18	0.20	
Finishing	0.07	0.20	0.08
Packaging	0.09	0.12	0.07

The sheet metal required for each part, the profit per part are as follows:

14.	Part	Sheet Metal (ft. ²)	Estimated Annual Demand
	1	2.6	2,600

2	1.4	1,800
3	2.5	4,100
4	3.2	1,200

The company has 15,000 square feet of space available each month. The company has the following

- 1.** Avoid overtime, which would erode profit.
- 2.** Meet parts demand.
- 3.** Achieve an annual profit of \$700,000.
- 4.** Avoid ordering more material because of a new supplier for changing the standard.

1. Formulate a goal programming model for each part to produce to achieve the goal.
2. Solve this model by using the computer.
3. How would the solution be affected if the goal is reversed?

Mac's Warehouse is a large discount store. The store needs the following number of employees each day of the week:

Day	Number of Employees	Day
Sunday	47	Thursday
Monday	22	Friday

Tuesday

28

Sat

Wednesday

35

15.

Each employee must work 5 consecutive days off. For example, any employee who has Friday and Saturday off. The store has 10 employees available to work. Mac's has prioritized goals for employee scheduling.

[Page

- 1.** The store would like to avoid hiring employees who work on Friday and Saturday.
- 2.** The most important days for the store are Friday and Sunday.
- 3.** The next most important day to hire employees is Monday.

4. The store would like to be fully staffed by the end of the week.
1. Formulate a goal programming model to determine the number of employees who should begin their shift at the beginning of the week to achieve the store's objectives.
2. Solve this model by using the computer.

Infocomp Systems Lab is a research and development organization that develops computer systems and software. The lab has proposals from its own researchers and from other departments of the proposed research projects requiring a budget. It is not possible to undertake all of them. The following table shows the developmental budget, the number of employees required, and the sales from each project if successfully completed.

Project	Developmental Budget (\$1,000,000s)	Number of Employees Required
1	10	10
2	15	15
3	20	20
4	25	25
5	30	30
6	35	35
7	40	40
8	45	45
9	50	50
10	55	55
11	60	60
12	65	65
13	70	70
14	75	75
15	80	80
16	85	85
17	90	90
18	95	95
19	100	100
20	105	105
21	110	110
22	115	115
23	120	120
24	125	125
25	130	130
26	135	135
27	140	140
28	145	145
29	150	150
30	155	155
31	160	160
32	165	165
33	170	170
34	175	175
35	180	180
36	185	185
37	190	190
38	195	195
39	200	200
40	205	205
41	210	210
42	215	215
43	220	220
44	225	225
45	230	230
46	235	235
47	240	240
48	245	245
49	250	250
50	255	255
51	260	260
52	265	265
53	270	270
54	275	275
55	280	280
56	285	285
57	290	290
58	295	295
59	300	300
60	305	305
61	310	310
62	315	315
63	320	320
64	325	325
65	330	330
66	335	335
67	340	340
68	345	345
69	350	350
70	355	355
71	360	360
72	365	365
73	370	370
74	375	375
75	380	380
76	385	385
77	390	390
78	395	395
79	400	400
80	405	405
81	410	410
82	415	415
83	420	420
84	425	425
85	430	430
86	435	435
87	440	440
88	445	445
89	450	450
90	455	455
91	460	460
92	465	465
93	470	470
94	475	475
95	480	480
96	485	485
97	490	490
98	495	495
99	500	500
100	505	505

1	\$0.675
---	---------

2	1.050
---	-------

3	0.725
---	-------

4	0.430
---	-------

5	1.240
---	-------

6	0.890
---	-------

7	1.620
---	-------

16.	8	1.200
------------	---	-------

The lab has developed the following set of projects to initiate:

- 1.** The company would like to remain within a budget of \$5,000,000.
- 2.** The number of available research scientists is limited, so the company would like to avoid obtaining extra research scientists.
- 3.** The company would like the expected number of projects implemented to be at least 10.
- 4.** Projects 1, 3, 4, and 6 are considered new product initiatives, while projects 2, 5, and 7 are considered product upgrades and thus defense projects. The company would like to select at least two projects from each category.
- 5.** Projects 2, 3, 5, 6, and 7 are considered high-priority projects, and the company would like to implement at least three of these projects.
- 6.** The lab's owner has indicated that the company should implement at least one project.

6 initiated if doing so does not interfere with the more important goals determined by the company.

1. Formulate a goal programming model. Infocomp Systems Lab should select the number of each type of computer to purchase that will minimize the total cost while satisfying the other goals.
2. Solve this model by using the computer. The solution requires 01 integer values for the number of each type of computer to purchase.

Dampier Associates is a holding company that specializes in purchasing medium-sized textile companies. The company has purchased three companies in the Carolinas and Core Textiles. The main criteria the company uses to select the companies it will purchase are current profitability, growth potential, and risk. Dampier moderately prefers growth potential over profitability in its purchase decision. Dampier's pairwise comparison matrix for the textile companies it is considering are as follows:

[Page

Profitability

Company	A	B	C
A	1	$\frac{1}{3}$	7
B	3	1	9
17. C	$\frac{1}{7}$	$\frac{1}{9}$	1

Growth Potential

Company	A	B	C
A	1	$\frac{1}{2}$	$\frac{1}{5}$

B	2	1	$1/3$
C	5	3	1

Develop an overall ranking of the three by using AHP.

- 18.** Check the pairwise comparisons for Da consistency.

Bernard Mee, the head of the department is evaluating faculty for raises at the end of the year. He is considering three faculty members for raises: John Doe and Debbie Cook. Faculty evaluations are based on three criteria: research, and service. Professor Mee's pairwise comparisons for the three faculty members for each criterion are as follows:

Faculty Member	Teaching	
	A	B
A	1	2
B	1/2	1
C	3	5
Research		

**Faculty
Member**

A

B

A

1

3

B

1/3

1

C

2

1

19.

Service

**Faculty
Member**

A

B

A

1

3

B

1/3

1

C

1/6

1/2

[Page

Criterion

Teaching Research S

Teaching

1

3

Research

1/3

1

Service

1/5

1/2

Determine an overall ranking of the three funds.

- 20.** Check the pairwise comparisons for the consistency and indicate whether the level of consistency is acceptable.

Megan Moppett is a sales representative for Temple Global Fund (TSS), and she receives a commission on every sale she makes. She has just sold a mutual fund to a client. Her earnings during the sale were \$1,000. She wants to invest in a mutual fund for retirement. She has three criteria for selection: return (based on historical trends and forecasts), risk, and the fund's performance relative to the market. She has made pairwise comparisons for the funds for each of the three criteria. The pairwise comparisons of the three criteria are as follows:

Return

Fund	Global	Blue Chip	B
Global	1	1/4	
Blue Chip	4	1	
Bond	1/2	1/6	

Risk

Fund	Global	Blue Chip	B
Global	1	2	

Blue Chip

1/2

1

21.

Bond

3

5

Load

Fund

Global

Blue Chip

B

Global

1

1

Blue Chip

1

1

Bond

3

3

Criterion	Return	Risk	L
Return	1	3	
Risk	1/3	1	
Load	1/5	1/2	

Determine the fund in which Megan should

22. In [Problem 21](#), if Megan Moppett has \$ diversify by investing in all three funds,

Alex Wall is shopping for a new four-wheel drive SUV. She has identified three models from which she will choose: the Explorer, the Trooper, and the Passport. She will make her selection based on the price, and each vehicle's appearance. For each vehicle, she has made comparisons for the vehicles for each of her preferences:

Consumer Digest Rating

Vehicle	Explorer	Trooper	Passport
Explorer	1	4	3
Trooper	1/4	1	1/2
Passport	1/3	2	1

	Price		
Vehicle	Explorer	Trooper	Passp
Explorer	1	1/4	1/6
Trooper	4	1	2
Passport	6	1/2	1

23.

Appearance

Vehicle	Explorer	Trooper	Passp
Explorer	1	4	3
Trooper	1/4	1	1/2
Passport	1/3	2	1

Criterion	<i>Consumer Digest</i> Rating	f
------------------	--	----------

*Consumer
Digest*

1

rating

Price $1/2$

Appearance $1/4$

Using AHP, determine which vehicle Alex

Station WRCH in Richmond, Virginia, is looking for a new news anchor on its 6:00 P.M. *Eyewitness News* program. Three candidates for the job are June Pawlie, Kellie Jones, and Kelli Jones. The criteria the station manager will use to evaluate the candidates are appearance, intelligence, and speaking skills. The manager's pairwise comparisons for the three criteria are as follows

Appearance

Anchor	Pawlie	Cooric	Broken
Pawlie	1	2	7
Cooric	1/2	1	5
Brokenaw	1/7	1/5	1

[Page

Intelligence

24.

Anchor	Pawlie	Cooric	Brokenaw
Pawlie	1	1/3	1/4
Cooric	3	1	1/2
Brokenaw	4	2	1

Speech

Anchor	Pawlie	Cooric	Brokenaw
Pawlie	1	1/3	2

Cooric	3	1	6
Brokenaw	$\frac{1}{2}$	$\frac{1}{6}$	1

Criterion	Appearance	Intelligence
-----------	------------	--------------

Appearance	1	8
Intelligence	$\frac{1}{8}$	1
Speech	$\frac{1}{3}$	5

Using AHP, determine which candidate the news anchor.

Carol Latta is visiting hotels in Los Angeles for a convention for a national organization that she represents. There are three hotels: the Cheraton, the Milton, and the Harriott. The criteria are ambiance, location (based on safety and restaurants), and cost. To help with her comparisons she has developed that in the following table for each criterion and her pairwise comparisons.

Ambiance			
Hotel	Cheraton	Milton	Harriott
Cheraton	1	1/2	1/5

Milton	2	1	1/3
Harriott	5	3	1

Location

Hotel	Cheraton	Milton	Harriott
Cheraton	1	5	3
Milton	1/5	1	1/4
25. Harriott	1/3	4	1

Cost			
Hotel	Cheraton	Milton	Harriott
Cheraton	1	2	5
Milton	1/2	1	2
Harriott	1/5	1/2	1

Criterion	Ambiance	Location	Cost
-----------	----------	----------	------

Ambiance	1	2	4
----------	---	---	---

Location	1/2	1	3
----------	-----	---	---

Cost	1/4	1/3	1
------	-----	-----	---

Develop an overall ranking of the three
decide where to hold the meeting.

Aaron Zeitel is a high school senior deciding where to hold his fall. He has narrowed his choices to three

Barton, and Claiborne. His criteria for selection were reputation, location (and especially proximity to the coast), tuition and room and board, and the schools available. Following are Aaron's pairwise comparisons for each of the four criteria and his pairwise comparisons for the schools.

Academic		
College	A	B
A	1	1/2
B	2	1
C	1/3	1/4

Location		
College	A	B
A	1	1
B	1	1
C	$\frac{1}{3}$	$\frac{1}{5}$
Cost		

26.	College	A	B
	A	1	$1/2$
	B	2	1
	C	4	2

Social		
College	A	B
A	1	3

	B	$\frac{1}{3}$	1
	C	$\frac{1}{6}$	$\frac{1}{3}$
Criterion	Academic	Location	Cost
Academic	1	4	$\frac{1}{2}$
Location	$\frac{1}{4}$	1	$\frac{1}{5}$
Cost	2	5	1
Social	$\frac{1}{3}$	2	$\frac{1}{3}$

Using AHP, determine which college Aar consistency of the pairwise comparison

Whitney Eggleston operates a computer Tech. She uses AHP to help match her match Chris with either Robin, Terry, or according to three criteria physical attractiveness, personality, and she had Chris do pairwise criteria, as follows:

Criterion	Attractiveness	Intelligence
Attractiveness	1	1
Intelligence	1	1

Personality

1/3

1/2

Whitney herself did the pairwise comparison based on their data sheets and a personal

Attractiveness

Client

Robin

Terry

I

Robin

1

3

Terry

1/3

1

Kelly

1/5

1/2

Intelligence			
Client	Robin	Terry	I
Robin	1	2	
Terry	1/2	1	
Kelly	2	4	

Personality

Client	Robin	Terry	I
Robin	1	2	
Terry	1/2	1	
Kelly	3	2	

Who is the best match for Chris, according to the data above?

Rockingham Systems is considering three projects, A, B, and C. Rockingham is not sure it will be able to complete all three projects. It wants to rank them in terms of preference. The company has two criteria to rank the projects: profit potential and risk. The company has ranked the projects on a scale of 1 to 3, with 1 being the highest rank and 3 being the lowest rank. The company has ranked the projects as follows:

Following are Rockingham's pairwise comparisons of the three criteria and for the criteria:

Profit		
Project	A	B
A	1	4
B	1/4	1
C	1/6	1/2

P(Success)

Project

A

B

A

1

2

B

1/2

1

C

3

6

28.

Cost

Project

A

B

A	1	$1/3$
B	3	1
C	4	2

Criteria	Profit	$P(\text{Success})$
Profit	1	2
$P(\text{Success})$	$1/2$	1
Cost	$1/6$	$1/4$

Rank the projects for Rockingham Syst

Professor Rakes is selecting a new grad
second-year MBA students. He will mal
student's grade point average (GPA) to
exam) score, and the undergraduate d
developed the following pairwise compa

Criteria	GPA	GMAT	Degr
GPA	1	5	1/3
GMAT	1/5	1	1/6
Degree	3	6	1

29.

The files of the three students, Adrian, Bonita, and Corey, are as follows:

Student	GPA	GMAT	Degree
Adrian	3.6	560	Business
Bonita	3.8	670	English
Corey	3.0	610	Engineering

Use AHP to help Professor Rakes select the best student for his research project.

The Bay City Parks and Recreation Department is considering several new facilities, including a gym, a field, and a pool. It will base its decision on which facility to build on projected usage (from surveys) and cost. The department strongly prefers usage to cost. Following are the department's pairwise comparisons of the facilities. The preferences for each facility for the two criteria are given below.

[Page 1]

Facility	Usage		
	Gym	Field	Tennis
Gym	1	1/3	3
Field	3	1	5

30.

Tennis	$\frac{1}{3}$	$\frac{1}{5}$	1
Pool	$\frac{1}{2}$	$\frac{1}{4}$	3

Cost

Facility	Gym	Field	Tennis
Gym	1	$\frac{1}{4}$	$\frac{1}{2}$
Field	4	1	3
Tennis	2	$\frac{1}{3}$	1

Pool	1/3	1/7	1/4
------	-----	-----	-----

Rank the facilities, using AHP, and check consistency.

Students at a college in Ohio are planning to visit three locations: Myrtle Beach (MB), Daytona Beach (FL). They are to base their decision on the potential fun (based on an Internet survey of other colleges). The students have developed pairwise comparisons for the locations for each

Weather

Location	MB	DB
----------	----	----

MB	1	$\frac{1}{3}$
----	---	---------------

DB	3	1
----	---	---

FL	3	1
----	---	---

Cost

Location	MB	DB
-----------------	----	----

MB	1	3
----	---	---

DB	$\frac{1}{3}$	1
----	---------------	---

FL

 $1/5$ $1/2$ **31.**

Fun**Location**

MB

DB

MB

1

 $1/2$

DB

2

1

FL

 $1/5$ $1/3$

Criteria	Weather	Cost
Weather	1	4
Cost	1/4	1
Fun	4	5

If the students use AHP to help make a select for their spring break vacation?

32. Check the pairwise comparisons in [Pro](#)

The management science and informat one of two available options within the (DSS) or operations management (OM students to determine which option the the advisers are student aptitude and ir options, and potential job availability. A develop the following pairwise comparis

Aptitude		
Option	DSS	OM
DSS	1	3
OM	1/3	1

Faculty

Option	DSS	OM
--------	-----	----

DSS	1	1/5
-----	---	-----

33.

OM	5	1
----	---	---

Jobs

Option	DSS	OM
--------	-----	----

DSS	1	4
-----	---	---

OM	1/4	1
----	-----	---

Criterion	Aptitude	Faculty
-----------	----------	---------

Aptitude	1	1/2
----------	---	-----

Faculty	2	1
---------	---	---

Jobs	4	3
------	---	---

Which option should the student select?

The Des Moines Twisters, a women's professional basketball team, has signed a new point guard and is considering signing Keisha Jones, Natasha Franklin, and Kathleen Thomas. The coach is undecided as to which player to sign on four criteria: rebounding ability, ball handling skills, accuracy, and speed. Below are the three players' college game statistics.

[Page 1 of 1]

Player Statistics				Per
Player	Points	Field Goal (%)	Free Throw (%)	
34. Keisha Jones	18.6	47.5	75.7	
Natasha Franklin	22.9	38.5	64.2	

Kathleen Taylor	16.5	62.1	88.3
-----------------	------	------	------

Although the coach can base some of his statistics, it is more difficult to assess a player, and she moderately prefers Keisha Taylor's and strongly prefers Keisha Jordan.

Assume the coach's role and develop priorities between the criteria and between the criteria. Use the criteria to select.

The town of Blacksburg needs a larger middle school in the center of town and two proposals for a new school renovation: keep it in town or build a new school on the outskirts. Groups in town have strong feelings about the school and want to retain the sense of tradition of the school where it helps engender a sense of community.

as antiquated and beyond saving and beyond repair. The school board is near bars, traffic, and college students. The school board will make the final decision. The school board will have management science professors from the school board to evaluate the proposals. The school board will have which it wants to solicit input regarding the school board. PTA, the middle school teachers, current school board and the town council. The management science professors will have the following pairwise comparison matrix.

PTA		
Proposal	Renovate	New
Renovate	1	1/3
New	3	1

Teachers

Proposal	Renovate	New
Renovate	1	1/9
New	9	1

Students

<u>35.</u>	Proposal	Renovate	New
-------------------	----------	----------	-----

Renovate	1	2
----------	---	---

New	1/2	1
-----	-----	---

Town Council

Proposal	Renovate	New
-----------------	----------	-----

Renovate	1	5
----------	---	---

New	1/5	1
-----	-----	---

The school board's pairwise comparison soliciting preferences is as follows:

Group	PTA	Teachers
PTA	1	5
Teachers	1/5	1
Students	1/2	4
Town council	4	7

1. Based on the AHP analysis conducted by the professors, which proposal should be chosen?
2. Check the school board's pairwise consistency.

Given the following pairwise comparisons according to the preference scale in [Table 1](#):

1. Steak to chicken
2. Hot dogs to hamburgers
3. Republicans to Democrats
4. Soccer to football
5. College basketball to professional basketball
6. Management science to management information systems
36. 7. Domino's to Pizza Hut
8. McDonald's to Wendy's

- 9.** Ford to Honda
- 10.** Dickens to Faulkner
- 11.** Beatles to Beethoven
- 12.** New York to Los Angeles
- 13.** Chicago to Atlanta
- 14.** American League to National League

Federated Health Care has contracted with a local health care provider for faculty and staff. There are three hospitals (within 35 miles) County, Memorial, and St. Vincent, each with an emergency room. Federated wants to select one of the three for its primary care emergency rooms for its selection are quality of medical care, as measured by the distance to the emergency room by the nearest hospital, the medical attention at the emergency room, and the pairwise comparisons of the emergency rooms. Federated wants to select one of the three for its primary care emergency rooms for its selection are quality of medical care, as measured by the distance to the emergency room by the nearest hospital, the medical attention at the emergency room, and the pairwise comparisons of the emergency rooms.

Medical Care			
Hospital	County	Memorial	Gene
County	1	1/6	1/3
Memorial	6	1	3
General	3	1/3	1

Distance			
Hospital	County	Memorial	Gene

County	1	7	4
Memorial	1/7	1	2
General	1/4	1/2	1

[Page

Speed of Attention

Hospital	County	Memorial	Gene
County	1	1/2	3

37.

Memorial	2	1	4
General	1/3	1/4	1

Cost

Hospital	County	Memorial	Gene
County	1	6	4
Memorial	1/6	1	1/2
General	1/4	2	1

Criterion	Medical Care	Distance	Speed of Attention
Medical care	1	8	6
Distance	1/8	1	1/2
Speed of attention	1/6	2	1
Cost	1/3	6	4

Using AHP, determine which hospital emergency care should designate as its primary care.

The department of management science has introduced quantitative methods in the classroom for different teachers. A group of students has been asked to rank the sections: time and day, grading, and classroom atmosphere (i.e., relaxed or strict). The teacher's sense of humor. Following are the criteria:

Criterion	Time/Day	Grading	Atmosphere
Time/day	1	2	3
Grading	1/2	1	2
Atmosphere	2	2	1

Atmosphere	$1/7$	$1/6$
------------	-------	-------

Homework	$1/3$	$1/3$
----------	-------	-------

Jokes	$1/8$	$1/9$
-------	-------	-------

Following are the students' pairwise co
criteria:

Time/Day

Section	1	2	3
----------------	---	---	---

1	1	3	5
---	---	---	---

2

 $1/3$

1

2

3

 $1/5$ $1/2$

1

4

 $1/7$ $1/5$ $1/3$

[Page

Grading

Section

1

2

3

1	1	$\frac{1}{7}$	$\frac{1}{8}$
---	---	---------------	---------------

2	7	1	2
---	---	---	---

3	8	$\frac{1}{2}$	1
---	---	---------------	---

4	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{5}$
---	---------------	---------------	---------------

38.

Atmosphere

Section

1

2

3

1

1

6

3

2	$1/6$	1	3
3	$1/3$	$1/3$	1
4	$1/3$	$1/2$	1

Homework			
Section	1	2	3
1	1	$1/4$	$1/8$
2	4	1	$1/3$

3	8	3	1
4	2	$\frac{1}{4}$	$\frac{1}{5}$

Jokes			
Section	1	2	3
1	1	$\frac{1}{5}$	3
2	5	1	6
3	$\frac{1}{3}$	$\frac{1}{6}$	1

4

1/4

1/7

1/3

1. Using AHP, rank the courses for the department.
2. Using this framework and these criteria, develop a ranking of the courses for a course with multiple sections at your school, develop a ranking of the sections.

A faculty committee in the department of business is evaluating three new textbooks for its introductory business course, which all business students are required to take. The authors of the three textbooks, by the authors, are Adams/Jones, Barron, and Smith. The committee's selection criteria are topic coverage, readability, and available supplements. Following are the pairwise comparisons for each of the criteria:

Coverage

Textbook	A	B	C
A	1	$1/5$	$1/4$
B	5	1	3
C	4	$1/3$	1

[Page

Readability

Textbook	A	B	C
----------	---	---	---

A	1	2	3
B	$\frac{1}{2}$	1	3
C	$\frac{1}{3}$	$\frac{1}{3}$	1

39.

	Cost		
Textbook	A	B	C
A	1	$\frac{1}{2}$	$\frac{1}{5}$
B	2	1	$\frac{1}{3}$

C

5

3

1

Supplements

Textbook

A

B

C

A

1

4

7

B

$\frac{1}{4}$

1

3

C

$\frac{1}{7}$

$\frac{1}{3}$

1

Criterion	Coverage	Readability	Cost
Coverage	1	1/2	1/3
Readability	2	1	1/2
Cost	4	3	1
Supplements	1/2	1/5	1/4

Using AHP, determine which textbook to purchase and check the consistency of the pairwise comparison matrix.

On the day of the professional football match, the

Player	Speed	
	A	B
A	1	$\frac{1}{5}$
B	5	1
C	$\frac{1}{3}$	$\frac{1}{6}$

40.

Position

Player

A

B

A

1

$1/2$

B

2

1

C

5

3

Size

Player

A

B

A	1	4
B	1/4	1
C	1/2	3

Criterion	Salary	Speed	Posit
Salary	1	1/5	1/3
Speed	5	1	3
Position	3	1/3	1

Size

2

1/2

1/2

Using AHP, determine which player the consistency of the pair-wise comparison

Visit one or more local car dealers and might like to purchase. Using AHP and selections according to the following criteria: reliability/maintenance, engine size, gas

- 41.** features/options. (You may need to access such as *Consumer Reports*, to facilitate ranked your car selections using AHP, do the same analysis. Then compare the results which method you prefer.

Arsenal Electronics is to construct a new

and has selected four small towns in the
important decision criteria and grades for

[Page

Decision Criterion	Weight	Abbeton Ba
--------------------	--------	------------

Work ethic	0.18	80
------------	------	----

Quality of life	0.16	75
-----------------	------	----

Labor laws/unionization	0.12	90
----------------------------	------	----

Infrastructure	0.10	60
----------------	------	----

42.

Education	0.08	75
Labor skill and education	0.07	75
Cost of living	0.06	70
Taxes	0.05	65
Incentive package	0.05	90
Government regulations	0.03	40
Environmental regulations	0.03	65

Transportation	0.03	90
Space for expansion	0.02	90
Urban proximity	0.02	60

Develop a scoring model to determine built.

The Dynaco Manufacturing Company is bearings (used in automobiles and trucks) evaluating three sites, and it has graded each as follows:

Decision Criterion	Weight	1
--------------------	--------	---

Labor pool and climate	0.30	80
------------------------	------	----

Proximity to suppliers	0.20	100
------------------------	------	-----

43.

Wage rates	0.15	60
------------	------	----

Community environment	0.15	75
-----------------------	------	----

Proximity to customers	0.10	65
------------------------	------	----

Shipping modes	0.05	85
Air service	0.05	50

Develop a scoring model to determine recommend.

Exotech Computers manufactures computer circuit boards, motherboards, keyboards and sells them around the world. It wants to choose a distribution center in Asia to serve emerging sites in Shanghai, Hong Kong, and Singapore. weighted decision criteria for each site.

44.

Decision Criterion	Weight	Score
Political stability	0.25	
Economic growth	0.18	
Port facilities	0.15	
Container support	0.10	
Land construction cost	0.08	
Transportation/distribution	0.08	

Duties and tariffs	0.07
Trade regulations	0.05
Airline service	0.02
Area roads	0.02

Recommend a site based on these dec

State University is to construct a new s
that will include a bookstore, a post off
meeting rooms, a swimming pool, and
university administration has hired a sit
best potential sites on campus for the i
identified four sites on campus and has

decision criteria for each site as follows

Decision Criterion	Weight	South
Proximity to housing	0.23	70
Student traffic	0.22	75
Parking availability	0.16	90
Plot size, terrain	0.12	80
Infrastructure	0.10	50

45.

Off-campus accessibility	0.06	90
Proximity to dining facilities	0.05	60
Visitor traffic	0.04	70
Landscape and aesthetics	0.02	50

Which site should the specialist recommend?

Following an all-star school soccer career, a student has received scholarship offers from five universities. She has made a decision and has decided to use a scoring system to evaluate the different offers. The following table includes the scores for each criterion.

developed and a grade showing how w

[Page

Decision Criterion	Weight	Tec
Playing time	0.29	61
Coach	0.20	87
Conference affiliation	0.17	42
School prestige	0.13	98
Program status	0.08	78

46.

Degree program/major	0.08	72
Dollar value of scholarship	0.05	93

Rank the universities according to their decision.

The Carter family wants to purchase a during the summer in Hilton Head, South complexes from which to choose, and five. They have graded their choices according to criteria:

Decision Criterion	Weight	Albermarle
--------------------	--------	------------

Specific summer week	.40	80
----------------------	-----	----

Cost	.20	50
------	-----	----

Proximity to beach	.15	70
--------------------	-----	----

Condominium quality	.05	90
---------------------	-----	----

Pool size	.05	40
-----------	-----	----

Cleanliness	.05	70
-------------	-----	----

47.

Crowdedness of beach	.05	30
Condominium size	.05	100

Use a scoring model to determine a recommended time-share purchase.

Robin Dillon has recently accepted a new job and has been hunting for a condominium. Her five coworkers she has compiled a list of five complexes that she might move into. The following table is that Robin intends to use in her decision-making process, indicating how well each complex satisfies the criteria.

48.	Decision Criterion	Weight	Fairfax Forrest	Du Gal
	Purchase price	.30	92	8
	Neighborhood location	.18	76	0
	Proximity to Metro train	.12	78	7
	Shopping	.10	65	8
	Cost of living	.10	75	7

Security	.10	75	(0.075)
Recreational facilities	.05	96	(0.048)
Distance to job	.05	85	(0.0425)
Condo floor plan	.05	80	(0.04)
Complex size	.05	65	(0.0325)

[Page

Use a scoring model to help Robin determine the best purchase.

The New River Rapids Under-15 Girls' Team identified which tournaments it should enter during the summer. The team manager first surveyed the parents and players, identified six possible tournaments, and then asked the parents and players to rank them. The team manager first surveyed the parents and players, determined a list of decision criteria and then asked the parents and players to rank the tournaments that they satisfied the criteria. The criteria, weighted by the team manager, are summarized as follows:

Decision Criterion	Weight	Roanoke Rapids	Greensboro
Distance	0.24	81	73
Tournament dates	0.21	91	88

Hotel cost	0.16	86	75
Level of competition	0.14	88	93
Field quality	0.10	84	91
Shopping	0.07	81	83
Attractions	0.06	73	82
Restaurants	0.02	83	88

To what three tournaments should the apply, given the preferences indicated b

Select four fast-food restaurants (e.g.,
50. Domino's) in your area and develop a set of
criteria, weights, and grades to rank them.

Case Problem

OAKDALE COUNTY SCHOOL BUSING

The Oakdale County School Board was meeting in special session. A federal judge had ordered the board to present an acceptable busing plan for racially balancing the four high schools in Oakdale County within a week. The judge had previously given the school board several opportunities to informally present a plan, but

the members had been unable to agree among themselves. Every time they met and started to develop a plan to bus students from one high school district to another, an argument would arise before they got past the first busing move, and they would adjourn the meeting. This time, however, they knew the judge had lost patience and they had to agree on something.

Of the four schools, only West High School was racially balanced, with 500 white students and 500 black

students. North High School had 1,000 white students and only 300 black students; East High School was almost as bad, with 1,050 white students and 400 black students. South High School was predominantly black, with 800 black students and 450 white students. Overall, of the 5,000 students in Oakdale County, 60% were white and 40% black.

"Look," said John Connor, a school board member from the West district, "rather than starting off by trying to shift students from one district to

another, why don't we try to establish what we want to accomplish you know, what our goals are?"

Several of the other members nodded in agreement, and Fred Harvey, the board chairman, said, "Good idea, John."

"Okay, the first goal seems pretty evident to me," John said. "Sixty percent of our students are white, and 40% are black, so that's what we need our schools to be, 60% and 40%."

"That's okay for you to say, John," Betty Philips argued, "because your district has those proportions already so you won't have any busing. But my district in the North is a long way from that ratio, and we would have to bus a lot of our students to achieve a 60%/40% ratio."

"*I'm* not saying it, Betty," said John. "That is basically what Judge Barry has been saying for 6 months."

"John's right, Betty, and we're not busing students yet; we're

just putting down our objectives," said Fred. "I think that has to be our highest-priority objective. How about the rest of you?"

[Page 448]

They all nodded their agreement, even Betty Philips, reluctantly.

"Since we know we're going to have to bus students to achieve this ratio at each school, I think we ought to try to minimize the amount of traveling the

students will have to do," suggested Mickey Gibboney, a member from the South district.

Fred Harvey noted that page 10 of their handout had a chart showing the average mileage a student in one district would have to travel on a bus to the high school in each of the other districts. The chart looked like this:

Distance (mi.)

District/School *North South East West*

North		30	12	20
South	30		18	26
East	12	18		24
West	20	26	24	

"Why don't we try to set some reasonable objectives for total busing miles, for the students' sake and for budgeting reasons?" Cassandra Watkins

asked. "I would suggest about 30,000 miles per day, based on the miles we bus students now. If we get much higher than that, we're not going to have the money to pay for it, and it means we'll be busing students all over the place."

The other members nodded and agreed.

"Okay," said Fred Harvey, "that'll be our number-two goal."

Betty Philips spoke up again. "I'll tell you another thing I

don't want to see happen, and that's any more overcrowding at North High School. We have 100 students more than capacity now."

"You think you have problems!" Bob Wilson exclaimed. "In East we have 1,450 students and capacity for 1,000. I think no overcrowding is a great idea!"

"I agree," said Mickey Gibboney. "We're 250 over our capacity at South High School."

"That's a nice idea," John Connor responded, "and I

realize that we have 200 students less than our capacity at West High School. However, let's face it, in the county we have capacity for 4,400, not 5,000, students, so there's going to be some overcrowding. I think our objective should be that all four schools share in the overcrowding proportionally."

"That sounds reasonable to me," said Fred Harvey. "How about the rest of you? Okay to say our number-three goal is to be as close to capacity at each school as possible but share

proportionally in the overcrowding?"

They all voiced their approval.

"Well," John Connor concluded, "I think we have identified the things we want to accomplish in our plan. Now if we could just use some magic trick to find a plan for busing students between the districts that would achieve all these goals."

The others nodded and frowned.

1. Formulate a goal

programming model to help the board with its dilemma.

- 2.** Solve the goal programming model by using the computer.

Case Problem

CATAWBA VALLEY HIGHWAY PATROL

Broderick Crawford is the district commander for the Catawba Valley highway patrol district in western Pennsylvania. He is attempting to assign highway patrol cars to different road segments in his district. The primary function of the highway patrol force is to patrol roads outside incorporated city and town

limits in the district to deter traffic violators and accidents. This objective is typically achieved by maintaining a visible presence letting motorists see patrol units on a regular basis and giving out warnings, citations, and so forth. Secondary activities of a patrol unit include providing assistance to motorists, answering distress calls, handling emergencies and accidents when called to the scene, and occasionally apprehending criminals.

Commander Crawford has 23

patrol cars that he wants to assign to the following six major road segment areas:

road segment 1, interstate north

road segment 2, urban area, north

road segment 3, four-lane highway, east

road segment 4, two-lane highway, west

road segment 5, interstate/four-lane highway,

south

road segment 6, two-lane
highway (heavy truck traffic),
south

Each of these road segments includes the primary arteries, as indicated earlier, plus adjoining roads. All the road segments have different levels of traffic density and accident rates, which are key factors in determining how many patrol units to assign. However, these factors do not always coincide. For example, interstate highway segments typically

have high traffic density but low accident rates, whereas some two-lane highways have low traffic density but high accident rates. Differences often occur because of variations in road conditions (such as sharp curves, visibility, and width). Other conditions, such as heavy truck traffic (as on segment 6), also contribute to high accident rates.

Each segment requires different operating costs, including maintenance and repair, fuel, and so on because of different operating

conditions. The commander's most pressing objective is to limit daily operating costs to \$450. The daily operating costs per road segment are as follows:

[Page 449]

Road Segment Operating Cost

1	\$20
2	18
3	22

4

24

5

17

6

19

The commander would like to reduce the accident rate for the district as well as increase both physical and sight contacts, which are deterrents to potential traffic violators. The commander would also like to achieve a reasonable average

response time for a patrol unit to respond to a call for each road segment. The average accident rate reduction (per million miles traveled) and physical contacts and sight contacts per car for each road segment are shown in the following table:

Patrol Unit			
Road Segment	Physical Accident Reduction (per million mi. traveled)	Sight Contacts (per day)	Contacts (per day)

1	0.27	18	1,700
2	0.21	26	900
3	0.28	10	650
4	0.19	34	230
5	0.23	25	1,600
6	0.33	17	520

The commander's second-most-important goal is to reduce the

average accident rate for the district by five accidents per million miles traveled. The commander's next goals (in order) are to achieve 350 physical contacts and 30,000 sight contacts per day in the district.

If no patrol units are assigned in the district, the average time to respond to a distress call anywhere in the district from the main district headquarters and motor pool is 28 minutes. Each car assigned to a road segment reduces the overall average response time in the

district by the following amounts:

Road Segment	Reduction in Average Response Time (min.)
1	0.32
2	0.65
3	0.43
4	0.87
5	0.55

The commander's last objective is to achieve an average response time to distress calls of 15 minutes. Because of local and political pressure, the commander has to assign at least two patrol units to each road segment. In addition, the commander believes that a maximum of five patrol units is sufficient for any particular road segment.

Formulate and solve a goal programming model to determine the number of patrol units to assign to each road segment to achieve the commander's goals.

Case Problem

KATHERINE MILLER'S JOB SELECTION

Katherine Miller is a senior in the department of information technology at Tech. For the past few months she has been involved in the job search process. She has an excellent résumé, with a high grade point average and a strong record of campus participation in clubs and activities. As a result, she has had a number of

good interviews with various companies. She now has job offers from five companies American Systems Developers, Anderssun Consulting, National Computing Software Systems (NCSS), the Gulf-South Company, and Electronic Village.

American Systems Developers and Anderssun Consulting are both large national consulting firms with offices in several major cities. If Katherine accepted the offer of either of these firms, she would primarily work on project

teams assigned to develop decision support and information systems for corporate clients around the country. If she went with American Systems Developers, her home base would be in Atlanta, and if she accepted Anderssun's offer, she would be located in Washington, DC. However, in both cases she would be traveling a great deal and could sometimes be on the road at a client location for as much as 6 to 9 months. NCSS is a software and computer systems development company

with a campus-like location in Chicago. Although her job with NCSS would involve some traveling, it would never be more than several weeks at any one time. Gulf-South is a bank holding company that operates eight different banks and its various branches in six southeastern states. If Katherine accepted Gulf-South's offer, she would be located in Tampa, where she would work in operations systems. She would be involved in developing information and support systems for bank

operations, and she would have minimal travel. Electronic Village is a national chain of discount stores specializing in electronic products, such as televisions, DVRs, CD players, MP3 players, and computers. Her job with Electronic Village would be at its corporate headquarters in Nashville, Tennessee, where she would develop and maintain computer systems to be used for inventory control at the hundreds of Electronic Village stores across the country. She would be required to travel

very little.

[Page 450]

American Systems Developers has offered Katherine a starting salary of \$38,000 annually, and Anderssun Consulting has offered her \$41,000 per year. NCSS has offered her an annual salary of \$46,000, whereas Gulf-South has offered her \$35,000 per year, and Electronic Village has offered her a salary of \$32,000 per year.

Katherine is having a difficult time making her decision. All the companies have excellent reputations, are financially healthy, and have good opportunities for advancement. All are demanding in terms of the workload they require. All five companies have given Katherine only a few weeks to make a decision regarding their offers.

Katherine has decided to use the analytical hierarchy process (AHP) to help decide which job offer she should accept. She has developed a list of criteria

that are important to her in deciding which job to take. The criteria, in no particular order, are (1) salary; (2) cost of living in the city where she would be located; (3) amount of travel associated with her job; (4) climate (weather) where she would be located; (5) entertainment and cultural opportunities, including sports, theater, museums, parks, and so on; (6) universities where she can work on an MBA degree part time and at night; (7) the crime rate in the city where she would live; (8) the

nature of the job and what she would be doing; and (9) her proximity to friends and relatives. Katherine realizes that she has very limited information on which to compare the five jobs, based on most of these criteria, so she knows she needs to go to the library and do some research, especially on the five different job locations.

Put yourself in Katherine's shoes and, using all or some of her criteria and your own preferences and knowledge, develop an overall ranking of

the jobs, using AHP.

-

Chapter 10.

Nonlinear

Programming

[Page 452]

More attention has been devoted to linear programming in this text than to any other single topic. It is a very versatile technique that can be and has been applied to a wide variety of problems. Besides chapters devoted specifically to linear programming models and

applications, we have also presented several variations of linear programming, integer and goal programming, and unique applications of linear programming for transportation and assignment problems. In all these cases, all the objective functions and constraints were linear; that is, they formed a line or plane in space. However, many realistic business problems have relationships that can be modeled only with nonlinear functions. When problems fit the general linear programming format but

include nonlinear functions, they are referred to as **nonlinear programming** problems.

Nonlinear programming problems are given a separate name because they are solved in a different manner than are linear programming problems. In fact, their solution is considerably more complex than that of linear programming problems, and it is often difficult, if not impossible, to determine an optimal solution, even for a relatively small problem. In

linear programming problems, solutions are found at the intersections of lines or planes, and though there may be a very large number of possible solution points, the number is finite, and a solution can eventually be found. However, in nonlinear programming there may be no intersection or corner points; instead, the solution space can be an undulating line or surface, which includes virtually an infinite number of points. For a realistic problem, the solution space may be like a mountain

range, with many peaks and valleys, and the maximum or minimum solution point could be at the top of any peak or at the bottom of any valley. What is difficult in nonlinear programming is determining if the point at the top of a peak is just the highest point in the immediate area (called a *local optimal*, in calculus terms) or the highest point of all (called the *global optimal*).

The solution techniques for nonlinear programming problems generally involve searching the solution surface

for peaks or valleys that is, high points or low points. The problem encountered by these methods is that they sometimes have trouble determining whether the high point they have identified is just a local optimal solution or the global optimal solution. Thus, finding a solution is often difficult and can involve very complex mathematics that are beyond the scope of this text.

In this chapter we present the basic structure of nonlinear programming problems and use Excel to solve simple models.

Nonlinear Profit Analysis

We begin our presentation of nonlinear programming models by determining the optimal value for a single nonlinear function. To demonstrate the solution procedure, we will use a profit function based on break-even analysis. In [Chapter 1](#) we used break-even analysis to begin our study of model building, so it seems appropriate that we return to this basic model to complete

our study of model building.

Recall that in break-even analysis the profit function, Z , is formulated as

$$Z = vp - c_f - v c_v$$

where

v = sales volume (i.e., demand)

p = price

c_f = fixed cost

c_v = variable cost

One important but somewhat unrealistic assumption of this break-even model is that volume, or demand, is independent of price (i.e., volume remains constant, regardless of the price of the product). It would be more realistic for the demand to vary as price increased or decreased. For our Western Clothing Company example from [Chapter 1](#), let us suppose that the dependency of demand on price is defined by the following linear function:

$$v = 1,500 - 24.6p$$

This linear relationship is illustrated in [Figure 10.1](#). The figure illustrates the fact that as price increases, demand decreases, up to a particular price level (\$60.98) that will result in no sales volume.

Figure 10.1. Linear relationship of volume to price

Volume (v)

1,500

Slope = -24.6

0

\$60.98

Price (p)

Now we will insert our new relationship for volume (v) into

our original profit equation:

$$\begin{aligned} Z &= vp - c_f - vc_v \\ &= (1,500 - 24.6p)p - c_f - (1,500 - 24.6p)c_v \\ &= 1,500p - 24.6p^2 - c_f - 1,500c_v + 24.6pc_v \end{aligned}$$

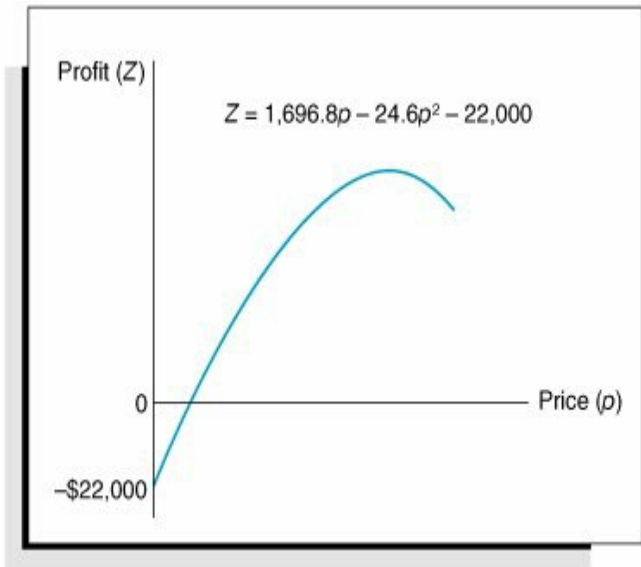
Substituting values for fixed cost ($c_f = \$10,000$) and variable cost ($c_v = \8) into this new profit function results in the following equation:

$$\begin{aligned} Z &= 1,500p - 24.6p^2 - 10,000 - 1,500(8) + 24.6p(8) \\ &= 1,696.8p - 24.6p^2 - 22,000 \end{aligned}$$

Because of the squared term, this equation for profit is now a nonlinear, or quadratic, function that relates profit to

price, as shown in [Figure 10.2](#).

Figure 10.2. The nonlinear profit function

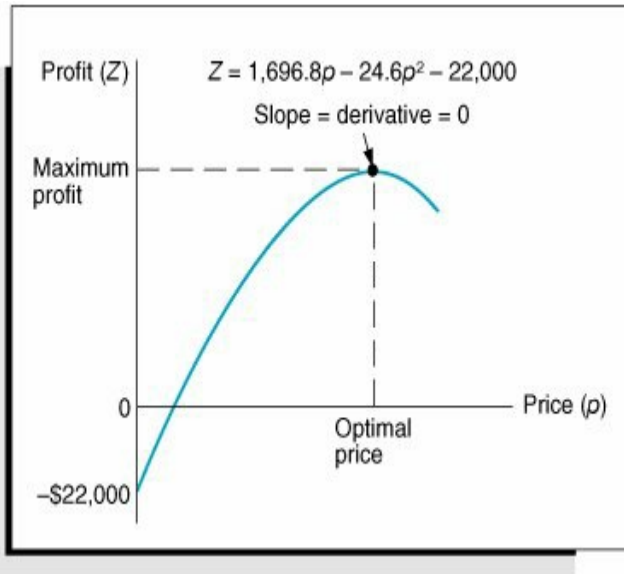


[Page 454]

In [Figure 10.2](#), the greatest

profit will occur at the point where the profit curve is at its highest. At that point the slope of the curve will equal zero, as shown in [Figure 10.3](#).

Figure 10.3.
**Maximum profit for
the profit function**



In calculus the slope of a curve at any point is equal to the

derivative of the mathematical function that defines the curve. The derivative of our profit function is determined as follows:

$$Z = 1,696.8p - 24.6p^2 - 22,000$$
$$\frac{\partial Z}{\partial p} = 1,696.8 - 49.2p$$

The slope of a curve at any point is equal to the derivative of the curve's function.

Given this derivative, the slope of the profit curve at its highest point is defined by the following relationship:

$$0 = 1,696.8 - 49.2p$$

The slope of a curve at its highest point equals zero.

Now we can solve this relationship for the optimal price, p , which will maximize

total profit:

$$0 = 1,696.8 - 49.2p$$

$$49.2p = 1,696.8$$

$$p = 1,696.8/49.2$$
$$= \$34.49$$

The optimal volume of denim jeans to produce is computed by substituting this price into our previously developed linear relationship for volume:

$$v = 1,500 - 24.6p$$

$$= 1,500 - 24.6(34.49)$$

$$= 651.6 \text{ pairs of denim jeans}$$

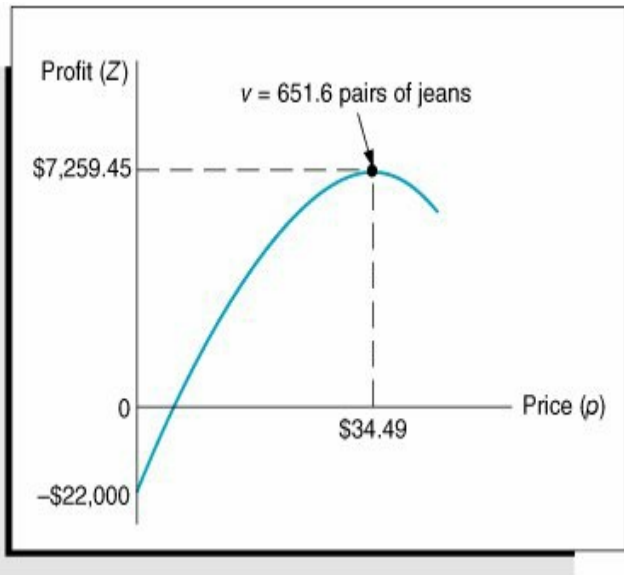
The maximum total profit is computed as follows:

$$\begin{aligned} Z &= \$1,696.8p - 24.6p^2 - 22,000 \\ &= \$1,696.8(34.49) - 24.6(34.49)^2 - 22,000 \\ &= \$7,259.45 \end{aligned}$$

The maximum profit, optimal price, and optimal volume are shown graphically in [Figure 10.4](#).

[Page 455]

Figure 10.4.
Maximum profit,
optimal price, and
optimal volume



An important concept we have yet to mention is that by

extending the break-even model this way, we have converted it into an *optimization* model. In other words, we are now able to maximize an objective function (profit) by determining the optimal value of a variable (price). This is exactly what we did in linear programming when we determined the values of decision variables that optimized an objective function. The use of calculus to find optimal values for variables is often referred to as **classical optimization**.

Classical optimization is the use of calculus to determine the optimal value of a variable.

Constrained Optimization

In the preceding section, the profit analysis model was developed as an extension of the break-even model. Recall that the total profit function was

$$Z = vp - c_f v - VC_v$$

where

v = volume

p = price

c_f = fixed cost

c_v = variable cost

and the demand function (i.e., volume as a function of price) was

$$v = 1,500 - 24.6p$$

By substituting this demand function into our total profit equation, we developed a non-linear function:

$$Z = (1,500 - 24.6p)p - c_f - 1,500c_v +$$

$$24.6pc_v$$

Then, by substituting values for c_f (\$10,000) and c_v (\$8) into this function, we obtained

$$Z = 1,696.8p - 24.6p^2 - 22,000$$

We then differentiated this function, set it equal to zero, and solved for the value of p (\$34.49), which corresponded to the maximum point on the profit curve (where the slope equaled zero).

This type of model is referred to as an **unconstrained**

optimization model. It consists of a single nonlinear objective function and *no* constraints. If we add one or more constraints to this model, it becomes a **constrained optimization model**. A constrained optimization model is more commonly referred to as a *nonlinear programming model*. The reason this type of model is designated as a form of mathematical programming is because all types of mathematical programming models are actually constrained optimization models. That is,

they all have the general form of an objective function that is *subject to one or more constraints*. Linear programming has an objective function and constraints that happen to be linear. A nonlinear programming model has the same general form as a linear programming model, except that the objective function *and/or* the constraint(s) are nonlinear.

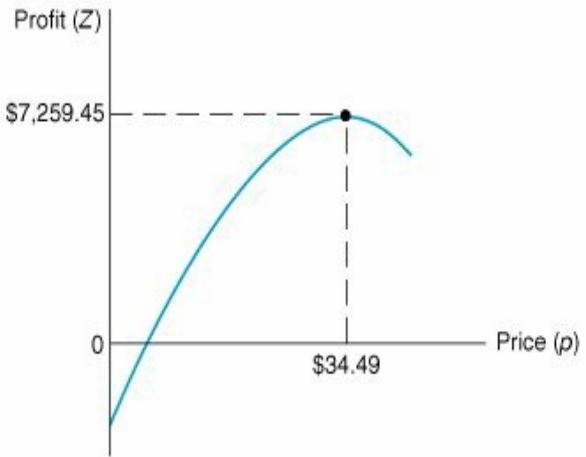
[Page 456]

Nonlinear programming differs

from linear programming, however, in one other critical aspect: The solution of nonlinear programming problems is much more complex. In linear programming a particular procedure is guaranteed to lead to a solution if the problem has been correctly formulated, whereas in nonlinear programming no guaranteed procedure exists. The reason for this complexity can be illustrated by a graph of our profit analysis model. [Figure 10.5](#) shows the nonlinear profit

curve for the example model.

Figure 10.5.
Nonlinear profit curve
for the profit analysis
model



As stated previously, the solution is found by taking the derivative of the profit

function, setting it equal to zero, and solving for p (price). This results in an optimal value for p that corresponds to the maximum profit, as shown in [Figure 10.5](#).

Now we will transform this unconstrained optimization model into a nonlinear programming model by adding the constraint

$$p \leq \$20$$

In other words, because of market conditions, we are restricting the price to a

maximum of \$20. This constraint results in a feasible solution space, as shown in [Figure 10.6](#).

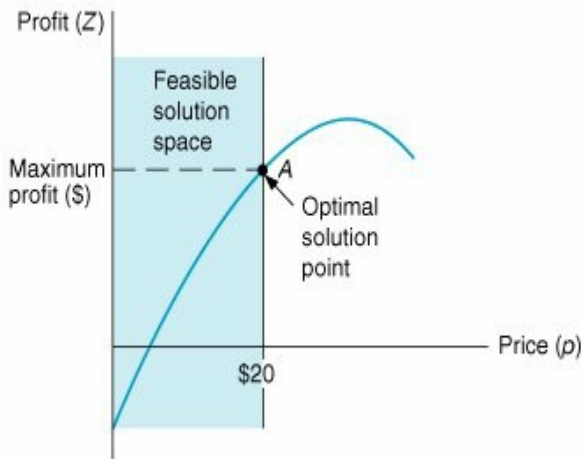
As in a linear programming problem, the solution is on the boundary of the feasible solution space formed by the constraint. Thus, in [Figure 10.6](#), point A is the optimal solution. It corresponds to the maximum value of the portion of the objective function that is still feasible. However, the difficulty with nonlinear programming is that the solution is not always on the

boundary of the feasible solution space formed by the constraint. For example, consider the addition of the following constraint to our original nonlinear objective function:

$$p \leq \$40$$

[Page 457]

**Figure 10.6. A
constrained
optimization model**



This constraint also creates a feasible solution space, as shown in [Figure 10.7](#).

**Figure 10.7. A
constrained
optimization model
with a solution point
not on the constraint
boundary**

Profit (Z)

Feasible
solution
space

Optimal
solution
point

C

B

Price (p)

\$40

But notice in [Figure 10.7](#) that the solution is no longer on the boundary of the feasible

solution space, as it would be in linear programming. Point C represents a greater profit than point B , and it is also in the feasible solution space. This means that we cannot simply look at points on the solution space boundary to find the solution; instead, we must also consider other points on the surface of the objective function. This greatly complicates the process of finding a solution to a nonlinear programming problem, and this difficulty is aggravated by an increased

number of variables and constraints and by nonlinear functions of a higher order. You can imagine the difficulties of solution if you contemplate a model in space that is made up of intersecting cones, ellipses, and undulating surfaces, as well as planes, and a solution that is not even on the boundary of the solution space.

A number of different solution approaches to nonlinear programming problems are available. As we have already indicated, they typically represent a convergence of the

principles of calculus and mathematical programming. However, as noted earlier, the solution techniques can be very complex. Thus, we will confine our discussion of nonlinear programming solution methods to several different model applications and the Excel solution.

Solution of Nonlinear Programming Problems with Excel

Excel can solve nonlinear programming problems by using the "Solver" option from the "Tools" menu that we used previously in this text to solve linear programming problems. [Exhibit 10.1](#) shows an Excel spreadsheet set up to solve our initial Western Clothing Company example. The

demand function contained in cell C4 is **=1500-24.6*C5**. The formula for profit is contained in cell C3 and is shown on the formula bar at the top of the spreadsheet.

Exhibit 10.1.

[\[View full size image\]](#)

Formula for profit

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Western Clothing Company														
2															
3		Profit =	-22000												
4		Demand =	1500												
5		Price =	0.00												
6		Fixed cost =	10000												
7		Variable cost =	8												
8															

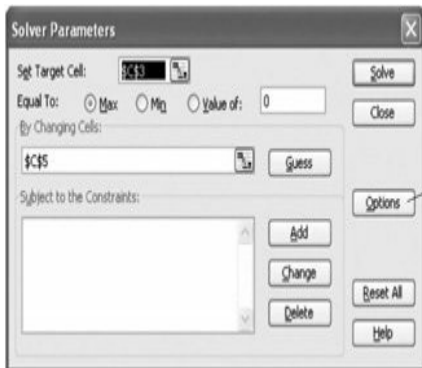
$$=1500-24.6 \times C5$$

The Solver Parameters window for this problem is shown in [Exhibit 10.2](#). It is important to check the "Options" settings from Solver and make sure that the "Assume Linear Models" option has not been

selected. Solver will automatically solve the problem as a nonlinear model if the "Assume Linear Model" option has not been selected. [Exhibit 10.3](#) shows the Excel solution for the Western Clothing Company example.

Exhibit 10.2.

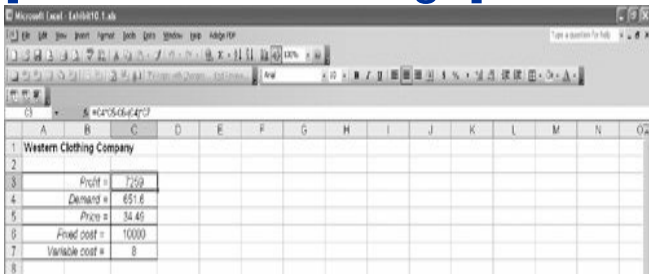
[\[View full size image\]](#)



[Page 459]

Exhibit 10.3.

[\[View full size image\]](#)



The screenshot shows a Microsoft Excel spreadsheet titled "Exhibit10.1.xls". The spreadsheet is set up for a nonlinear optimization problem for the Western Clothing Company. The data is organized as follows:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Western Clothing Company														
2															
3		Profit =	7259												
4		Demand =	651.6												
5		Price =	34.49												
6		Fixed cost =	10000												
7		Variable cost =	8												
8															

[Exhibit 10.4](#) shows an Excel spreadsheet set up to solve a nonlinear version of the Beaver Creek Pottery Company example from [Chapter 2](#) that is formulated as

$$\text{maximize } Z = \$ (4 - 0.1x_1)x_1 + (5 - 0.2x_2)x_2$$

subject to

$$x_1 + 2x_2 = 40 \text{ hours (labor)}$$

where

x_1 = number of bowls
produced

x_2 = number of mugs produced

$4 - 0.1x_1$ = profit (\$) per bowl

$5 - 0.2x_2 = \text{profit (\$) per mug}$

Exhibit 10.4.

[\[View full size image\]](#)

profit contained in cell C11 is computed using the formula **=SUMPRODUCT(C5:C6,D5:D6)** and is shown on the formula bar at the top of the spreadsheet. The formula for labor, **=C5+2*C6**, is contained in cell C9. The Solver Parameters window for this problem is shown in [Exhibit 10.5](#), and the final solution is shown in [Exhibit 10.6](#).

Exhibit 10.5.

(This item is displayed on page 460 in the print version)

Solver Parameters ✕

Set Target Cell: 

Equal To: ☒ Max ☐ Min ☐ Value of:

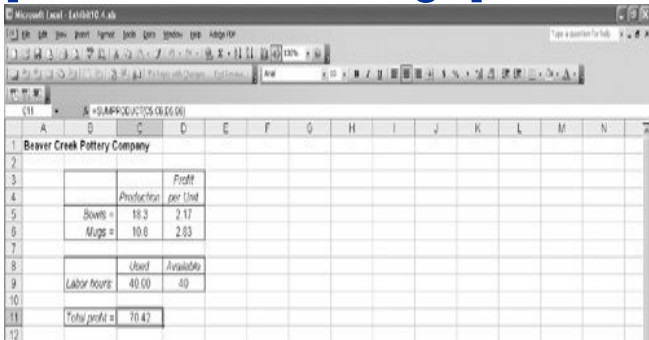
By Changing Cells: 

Subject to the Constraints:  

Exhibit 10.6.

(This item is displayed on page 460 in the print version)

[\[View full size image\]](#)



The screenshot shows a Microsoft Excel spreadsheet titled "Excel - Exhibit10.4.xls". The spreadsheet contains a linear programming problem for the Beaver Creek Pottery Company. The data is organized as follows:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Beaver Creek Pottery Company													
2														
3				Profit										
4			Production	per Unit										
5		Bowls =	18.3	2.17										
6		Mugs =	10.6	2.83										
7														
8			Used	Available										
9		Labor hours	40.00	40										
10														
11		Total profit =	70.47											
12														

Excel will also provide the value of the *Lagrange*

multiplier, which provides the dual value of the labor resource. To derive the Lagrange multiplier, after you click on "Solve" in Solver, the Solver Results screen shown in [Exhibit 10.7](#) will appear. On this screen, under "Reports," select "Sensitivity." This will generate the sensitivity report shown in [Exhibit 10.8](#). Note that in addition to the problem solution, the value of the Lagrange multiplier is also provided for the labor constraint.

Exhibit 10.7.

[\[View full size image\]](#)

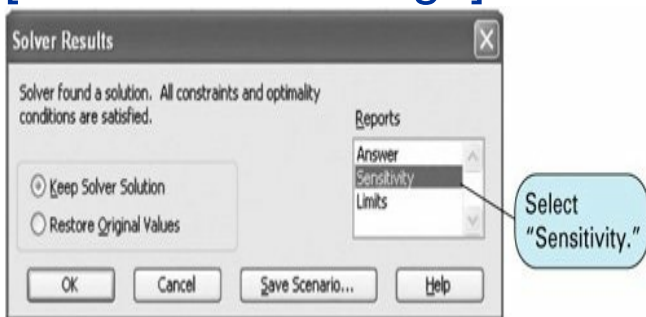


Exhibit 10.8.

Changing Cells

Cell	Name	Final Value	Reduced Gradient
\$C\$5	Bowls = Production	18.3	0.0
\$C\$6	Mugs = Production	10.8	0.0

Constraints

Cell	Name	Final Value	Lagrange Multiplier
\$C\$9	Labor hours: Used	40.00	0.33

Lagrange multiplier
for labor

The Lagrange multiplier value of .33 is analogous to the dual value in a linear programming

problem. It reflects the approximate change in the objective function resulting from a unit change in the quantity (right-hand-side) value of the constraint equation. For this example, if the quantity of labor hours is increased from 40 to 41 hours, the value of Z will increase by \$0.33 from \$70.42 to \$70.75.

A Nonlinear Programming Model with Multiple Constraints

Now that we have shown how Excel can be used to solve nonlinear problems, we can look at more complex problems for example, a problem with more than one constraint. Consider the Western Clothing Company example presented earlier, except now the company

produces two kinds of jeans, designer and straight-leg, and production is subject to resource constraints for denim cloth, cutting time, and sewing time. The company sells its jeans to several upscale clothing store chains, and sales demand is dependent on the price at which the company sells the jeans. The demand for designer jeans (x_1) and the demand for straight-leg jeans (x_2) are defined by the following relationships:

$$x_1 = 1,500 - 24.6p_1$$

$$x_2 = 2,700 - 63.8p_2$$

where

p_1 = price of designer jeans

p_2 = price of straight-leg jeans

The cost of producing designer jeans is \$12 per pair, and the cost of producing straight-leg jeans is \$9 per pair; thus, the objective function for profit is

$$\text{maximize } Z = (p_1 - 12)x_1 + (p_2 - 9)x_2$$

The production of jeans is

subject to the following
resource constraints for
available cloth, cutting time,
and sewing time:

$$\text{cloth: } 2x_1 + 2.7x_2 \leq 6,000 \text{ yards}$$

$$\text{cutting: } 3.6x_1 + 2.9x_2 \leq 8,500 \text{ minutes}$$

$$\text{sewing: } 7.2x_1 + 8.5x_2 \leq 15,000 \text{ minutes}$$

The complete nonlinear
programming model
formulation is summarized as
follows:

$$\text{maximize } Z = (p_1 - 12)x_1 + (p_2 - 9)x_2$$

subject to

$$2x_1 + 2.7x_2 \leq 6,000$$

$$3.6x_1 + 2.9x_2 \leq 8,500$$

$$7.2x_1 + 8.5x_2 \leq 15,000$$

where

$$x_1 = 1,500 - 24.6p_1$$

$$x_2 = 2,700 - 63.8p_2$$

p_1 = price of designer jeans

p_2 = price of straight-leg jeans

Notice that the decision variables for this problem are p_1 and p_2 , not x_1 and x_2 . The demand variables, x_1 and x_2 ,

are functions of price and, thus, are dependent variables. Also notice that we did not substitute the functional relationships for x_1 and x_2 into the objective function, creating a quadratic function. The model is still nonlinear, but we can solve it directly with Excel in its present form.

[Exhibit 10.9](#) shows an Excel spreadsheet set up to solve our Western Clothing Company example with constraints. The decision variables for price are contained in cells **D5:D6**. The

formula for demand for designer jeans is contained in cell C5 and is shown on the formula bar at the top of the screen. The formula for total profit, **=SUMPRODUCT(C5:C6,E5:E6)**, is contained in cell E7. The formulas for the resource constraints are contained in cells C11, C12, and C13, and the resource availabilities for each constraint are in cells **D11:D13**. The Solver Parameters window and solution are shown in [Exhibit 10.10](#) and [Exhibit 10.11](#), respectively.

Exhibit 10.9.

[\[View full size image\]](#)

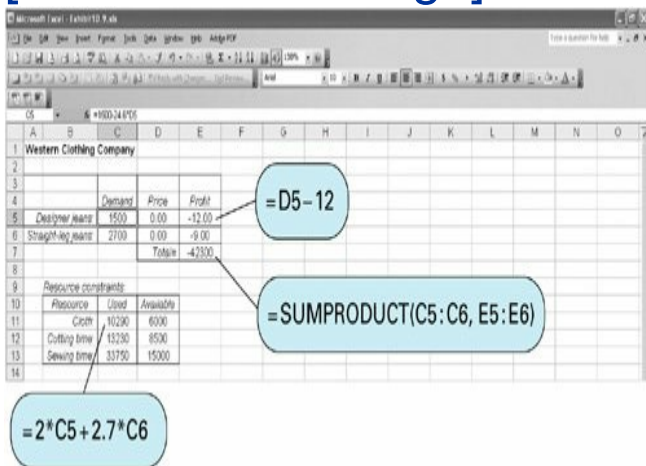


Exhibit 10.10.

Solver Parameters

Set Target Cell:

Equal To:

☒ Max

☐ Min

☐ Value of:

0

Solve

Close

By Changing Cells:

Guess

Subject to the Constraints:

Add

Change

Delete

Options

Reset All

Help

Exhibit 10.11.

[\[View full size image\]](#)

[illegible]

Nonlinear Model Examples

In our previous profit analysis examples, the models followed the traditional mathematical (i.e., linear) programming formulation, with an objective function, decision variables, and constraints. However, many types of problems employ well-known nonlinear functions but are not typically thought of in the traditional math programming model

framework. In this section we consider several additional examples that include familiar nonlinear functions: the facility location model and the investment portfolio selection model.

Facility Location

In the facility location problem, the objective, in general, is to locate a centralized facility that serves several customers or other facilities to minimize distance or miles traveled

between the facility and its customers. This problem uses as a measure of distance the formula for the straight-line distance between two points on a set of x, y coordinates, which is also the hypotenuse of a right triangle:

$$d = \sqrt{(x_i - x)^2 + (y_i - y)^2}$$

where

(x, y) = coordinates of proposed facility

(x_i, y_i) = coordinates of

customer or location facility, i

Consider, for example, the Clayton County Rescue Squad and Ambulance Service, which serves five rural towns, Abbeville, Benton, Clayton, Dunning, and Eden. The rescue squad wants to construct a centralized facility and garage to minimize its total annual travel mileage to the towns. The locations of the five towns in terms of their graphical x, y coordinates, measured in miles relative to the point $x = 0, y = 0$, and the expected number of annual trips the squad will

have to make to each town are as follows:

Town	Coordinates		Annual Trips
	x	y	
Abbeville	20	20	75
Benton	10	35	105
Clayton	25	9	135
Dunning	32	15	60

Eden

10

8

90

The objective of the problem is to determine a set of coordinates (x, y) for the rescue squad facility that minimizes the total miles traveled to the town, according to the following function:

$$\sum_i d_i t_i$$

where

d_i = distance to town i

t_i = annual trips to town i

[Page 464]

The Excel solution to this problem is shown in [Exhibit 10.12](#). The coordinates (x , y) for the new rescue squad facility are in cells **C14:C15**. The formulas for the distances between the rescue squad facility and each of the towns (d_i) are in cells **E6:E10**. For example, the formula for the distance, d_A , between the

rescue squad facility and Abbeville in cell E6 is

=SQRT((B6C14)^2+(C6C15)^2). The formula for the total annual distance in cell D18 is

=SUMPRODUCT(E6:E10,D6:D10) which is also shown on the formula bar at the top of the spreadsheet. In the Solver window for this problem, D18 is the target cell that is minimized, and the changing cells (i.e., decision variables) are **C14:C15**. There are no constraints.

Exhibit 10.12.

[\[View full size image\]](#)



Microsoft Excel - Exhibit10.12.xls

File Home Insert Page Layout Formulas Data Window Help

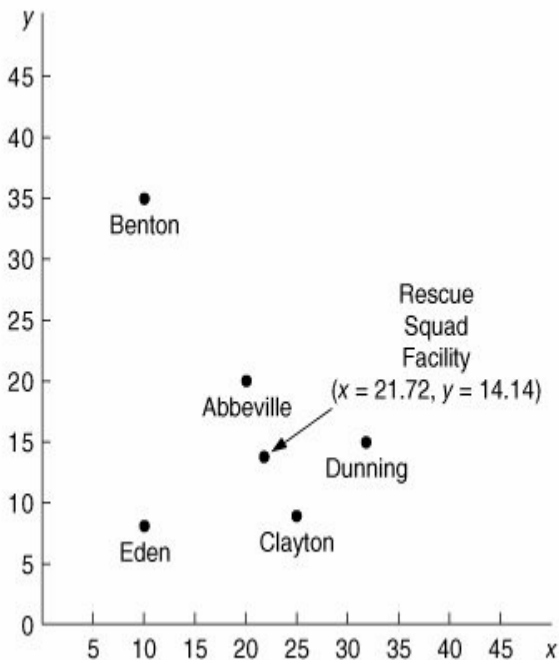
Formulas: =SUMPRODUCT(B6:E10,D6:D10)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Clayton County Rescue Squad														
2															
3															
4		Coordinates		Annual											
5	Town	x	y	Trips	Distance										
6	Abbeville	20	20	75	5.11										
7	Benton	10	35	105	23.93										
8	Clayton	25	9	135	5.09										
9	Dunning	32	15	60	1.32										
10	Eden	10	8	90	5.22										
11															
12															
13	Rescue Squad Facility Location														
14		x =	21.72												
15		y =	14.14												
16															
17															
18	Total Annual Distance =			4432.53											
19															

Formula: $=SQRT((B6-C14)^2+(C6-C15)^2)$

The location of the rescue squad facility is shown graphically in [Figure 10.8](#).

Figure 10.8. Rescue squad facility location



Investment Portfolio Selection

A classic example of nonlinear programming is the investment portfolio selection model developed by Harry Markowitz in 1959. This model is based on the assumption that most investors are concerned with two factors: return on investment and risk. Thus, the

objective of the portfolio selection model is to minimize some measure of portfolio risk while achieving a minimum return on the total portfolio investment. Risk is reflected by the variability in the value of the investment, and in this model, *variance* in the return on investment is the measure of risk.

Covariance, which is a measure of correlation, is also used to reflect risk. Individual investment returns within a portfolio typically exhibit statistical dependence. Over

time, the returns of any two stocks may exhibit positive or negative correlation; that is, two stocks of the same general type will go up or down together. To adjust for this possible correlation, investors often attempt to diversify their portfolios. To reflect the risk associated with *not* diversifying, the model includes covariance.

The minimization of portfolio risk, as measured by the portfolio variance, is the model objective. The variance, S , on the annual return from the

portfolio, is determined by the following formula:

$$S = x_1^2 s_1^2 + x_2^2 s_2^2 + \dots x_n^2 s_n^2 + \sum_{i \neq j} x_i x_j r_{ij} s_i s_j$$

where

x_i, x_j = the proportion of money invested in investments i and j

s_i^2 = the variance for investment i

r_{ij} = the correlation between returns on investments i and j

s_i, s_j = the standard deviations

of returns for investments i and j

In the preceding formula for S , the first part is a measure of variance, and the second part is a measure of the covariance. In many cases, the parameters in this equation may be estimates or sample values (the sample variance, sample covariance, etc.).

The general formulation of the portfolio selection model is as follows. The investor desires to achieve a minimum expected annual return from the

portfolio, which is formulated as a model constraint as follows:

$$r_1x_1 + r_2x_2 + \dots + r_nx_n \geq r_m$$

where

r_i = expected annual return on investment i

x_i = proportion (or fraction) of money invested in investment i

r_m = minimum desired annual return from the portfolio

A second constraint specifies

that all the money is invested:

$$x_1 + x_2 + \dots + x_n = 1.0$$

The following example and Excel solution will demonstrate how this model is applied.

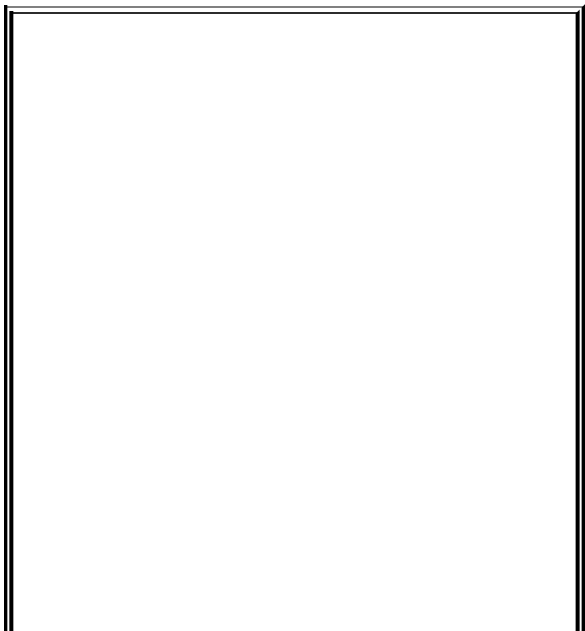
Jessica Todd has identified four stocks she wants to include in her investment portfolio. She wants a total annual return of at least .11. From historical data, she has estimated the average returns and variances for the four investments as follows:

Stock (x_i)	Annual Return (r_i)	Variance (s_i)
1. Altacam	.08	.009
2. Bestco	.09	.015
3. Com.com	.16	.040
4. Delphi	.12	.023

She has also estimated the

covariances between the stocks, as follows:

Stock Combination (<i>i, j</i>)	Covariances (<i>r_{ij}</i>)
A,B (1,2)	.4
A,C (1,3)	.3
A,D (1,4)	.6
B,C (2,3)	.2
B,D (2,4)	.7



Management Science

Application: Gas Production in Australia

Santos, Ltd., is a publicly owned mineral exploration and production company in Australia. The company has developed a nonlinear mathematical programming model to determine how to manage its Cooper Basin oil and gas reservoirs, which are located in the eastern half of Australia. The basic questions the company needs to answer are as follows:

How many oil and gas wells should be drilled, and when should they be drilled?

How much compressor capacity

should be installed and when?

At what rate should each reservoir be produced to meet demand?

The nonlinear mathematical programming model determines the production schedule that meets physical and logistical constraints while maximizing net present value (subject to a requirement that the incremental present value ratio of each reservoir investment exceed a minimum value). It is estimated that the model saves the company approximately \$43.6 million per year.



Source: E. Dougherty, D. Dare, P. Hutchison, and E. Lombardino. "Optimizing SANTOS's Gas Production and Processing Operations in Central Australia Using the Decomposition Method,"

[Page 467]

The nonlinear programming model formulation for this problem is as follows:

$$\begin{aligned} \text{minimize } Z = S = & x_1^2(.009) + x_2^2(.015) + x_3^2(.040) + x_4^2(.230) \\ & + x_1x_2(.4)(.009)^{1/2}(.015)^{1/2} + x_1x_3(.3)(.009)^{1/2}(.040)^{1/2} \\ & + x_1x_4(.6)(.009)^{1/2}(.023)^{1/2} + x_2x_3(.2)(.015)^{1/2}(.040)^{1/2} \\ & + x_2x_4(.7)(.015)^{1/2}(.023)^{1/2} + x_3x_4(.4)(.040)^{1/2}(.023)^{1/2} \\ & + x_2x_1(.4)(.015)^{1/2}(.009)^{1/2} + x_3x_1(.3)(.040)^{1/2}(.009)^{1/2} \\ & + x_4x_1(.6)(.023)^{1/2}(.009)^{1/2} + x_3x_2(.2)(.040)^{1/2}(.015)^{1/2} \\ & + x_4x_2(.7)(.023)^{1/2}(.015)^{1/2} + x_4x_3(.4)(.023)^{1/2}(.040)^{1/2} \end{aligned}$$

subject to

$$.08x_1 + .09x_2 + .16x_3 + .12x_4 \geq 0.11$$

$$x_1 + x_2 + x_3 + x_4 = 1.00$$

$$x_i \geq 0$$

The Excel spreadsheet for Jessica Todd's portfolio is shown in [Exhibit 10.13](#). The decision variables representing the proportion of each stock in the portfolio (x_i) are in cells **E6:E9**. The formula for total return, **=SUMPRODUCT(B6:B9, E6:E9)**, is in cell B19. The objective is to minimize the total portfolio variance formula, which is shown on the formula

bar at the top of the spreadsheet. This is a complicated formula and difficult to type in. Notice that it has been shortened slightly by doubling the covariance terms in the last part of the equation to reflect all the different investment pairs that are included in **B14:E17** (i.e., 2 times AB, AC, AD, BC, BD, and CD will include BA, CA, DA, CB, DB, and DC). In other words, all the covariance terms to the right of the diagonal in the matrix **B14:E17** are doubled to include the same values to the

left of the diagonal.

Exhibit 10.13.

[\[View full size image\]](#)

Doubling covariances will include all investment pairs.

include all investment pairs.

	Average Return	Estimated Variance	Variables	Investment Proportion
Stock Altacorn	0.08	0.009	X1 =	0.000
Bestco	0.09	0.015	X2 =	0.000
Com.com	0.16	0.040	X3 =	0.000
Delphi	0.12	0.023	X4 =	0.000
Total desired return =	0.11		Sum =	1.000

Correlations				
Stock	Altacorn	Bestco	Com.com	Delphi
Altacorn	1	0.40	0.30	0.60
Bestco	0.40	1	0.20	0.70
Com.com	0.30	0.20	1	0.40
Delphi	0.60	0.70	0.40	1

Total portfolio return = 0

Total variance = 0.000

Total proportion invested = 0.00

=SUMPRODUCT(B6:B9,E6:E9)

=SUM(E6:E9)

=SUMPRODUCT(B6:B9,E6:E9)

The Solver Parameters window is shown in [Exhibit 10.14](#).

Notice that nonnegativity constraints for the variables are included **E6:E9** ≥ 0 . The model solution is shown in [Exhibit 10.15](#). It indicates that Jessica should invest 36.0% of her funds in Altacam (x_1), 27.2% in Bestco (x_2), 31.5% in Com.com (x_3), and 5.3% in Delphi (x_4). This will result in her desired minimum portfolio annual return of .11 and a variance of .0104. What does this variance mean? A variance of .0104 equals a standard

deviation of .102. If returns are normally distributed (which is likely), then there is a .95 probability that the annual portfolio return will be between .089 $\{.11 - (1.96)(.102)\}$ and .310 $\{.11 + (1.96)(.102)\}$.

[Page 468]

Exhibit 10.14.

[\[View full size image\]](#)



Exhibit 10.15.

[\[View full size image\]](#)

Summary

In this chapter we have presented an overview of nonlinear programming problems and solutions. We have used relatively simple examples because of the complexity involved in formulating and solving larger problems. A more in-depth understanding of this topic requires significant knowledge of calculus and higher mathematics. Nevertheless,

this chapter has provided a general understanding of the types of problems that require nonlinear solutions and the general format for modeling nonlinear problems. It has also demonstrated the capability of Excel for solving nonlinear programming problems.

References

Gottfried, B. S., and Weisman, J. *Introduction to Optimization Theory*. Upper Saddle River, NJ: Prentice Hall, 1973.

Hillier, F. S., and Lieberman, G. J. *Operations Research*, 4th ed. San Francisco: Holden-Day, 1986.

Kuhn, H. W., and Tucker, A. W. "Non-Linear Programming." In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, ed. Jerzy Neyman, pp. 481-92. Berkeley: University of California Press,

1951.

Luenberger, D. G. *Introduction to Linear and Non-Linear Programming*. Reading, MA: Addison-Wesley, 1973.

Markowitz, H. M. *Portfolio Selection: Efficient Diversification of Investments*. Cowles Foundation Monograph, No. 16. New York: John Wiley & Sons, 1959.

McMillan, C., Jr. *Mathematical Programming*, 2nd ed. New York: John Wiley & Sons, 1975.

Example Problem Solution

The following example illustrates the use of the substitution method and the method of Lagrange multipliers for solving a nonlinear programming problem.

Problem Statement

The Hickory Cabinet and Furniture Company makes

chairs and tables. The company has developed the following nonlinear programming model to determine the optimal number of chairs (x_1) and tables (x_2) to produce each day to maximize profit, given a constraint for available mahogany wood:

$$\text{maximize } Z = \$280x_1 - 6x_1^2 + 160x_2 - 3x_2^2$$

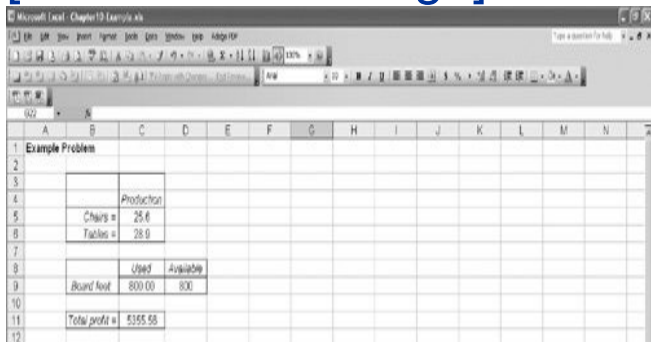
subject to

$$20x_1 + 10x_2 = 800 \text{ bd. ft.}$$

Determine the Excel solution to this model.

Solution

[\[View full size image\]](#)



The screenshot shows a Microsoft Excel spreadsheet titled "Microsoft Excel - Chapter19 Example.xls". The spreadsheet contains a linear programming problem solution. The data is organized as follows:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Example Problem													
2														
3														
4			Production											
5		Chairs =	25.6											
6		Tables =	28.9											
7														
8			Used	Available										
9		Board feet	800.00	800										
10														
11		Total profit =	5355.58											
12														

Problems

The Hickory Cabinet and Furniture Company makes chairs. The fixed cost per month making chairs is \$7,500, and the variable cost per chair is \$40. Price is related to demand, according to the following line equation:

1. $v = 400 - 1.2p$

[Page 470]

Develop the nonlinear profit function for the company and determine the price that maximizes profit, the optimal volume, and the maximum profit per month.

- Graphically illustrate the profit curve developed in [Problem 1](#). Indicate the optimal price and the maximum profit per month.

The Rainwater Brewery produces beer. The annual fixed cost is \$150,000, and the variable cost per barrel is \$16. Price is related to demand, according to the following linear equation:

$$3. \quad v = 75,000 - 1,153.8p$$

Develop the nonlinear profit function for the brewery and determine the price that will maximize profit, the optimal volume, and the maximum profit per year.

The Rolling Creek Textile Mill makes denim. The monthly fixed cost is \$8,000, and the variable cost per yard of denim is \$0.30. Price is related to demand, according to the following linear equation:

4. $v = 17,000 - 5,666p$

Develop the nonlinear profit function for the textile mill and determine the optimal price, the optimal volume, and the maximum profit per month.

The Grady Tire Company recaps tires. The weekly fixed cost is \$2,500, and the variable cost per tire is \$9. Price is related to demand, according to the following linear equation:

5. $v = 200 - 4.75p$

Develop the nonlinear profit function for the tire company and determine the optimal price, the optimal volume, and the maximum profit per week.

Andy Mendoza makes handcrafted dolls which he sells at craft fairs. He is consid

- mass-producing the dolls to sell in stores. He estimates that the initial investment for the factory and equipment will be \$25,000, while labor, materials, packaging, and shipping will be about \$10 per doll. He has determined that sales volume is related to price, according to the following linear equation:

$$v = 4,000 - 80p$$

Develop the nonlinear profit function for Andy and determine the price that will maximize profit, the optimal volume, and the maximum profit per month.

The Rainwater Brewery produces beer, which it sells to distributors in barrels. The brewery incurs a monthly fixed cost of \$12,000 and the variable cost per barrel is \$17. The brewery has developed the following profit function and demand constraint:

7. maximize $Z = vp - \$12,000 - 17v$

subject to

$$v = 1,800 - 15p$$

Solve this nonlinear programming model to determine the optimal price (p).

The Beaver Creek Pottery Company has developed the following nonlinear programming model to determine the optimal number of bowls (x_1) and mugs (x_2) to produce each day:

8. maximize $Z = \$7x_1 - 0.3x_1^2 + 8x_2 - 0.4x_2^2$

subject to

$$4x_1 + 5x_2 = 100 \text{ hr.}$$

Determine the optimal solution to this nonlinear programming model.

The Evergreen Fertilizer Company produces two types of fertilizers, Fastgro and Super Two. The company has developed the following nonlinear programming model to determine the optimal number of bags of Fastgro (x_1) and Super Two (x_2) that it should produce each day to maximize profit, given a constraint for available potassium:

9.

$$\text{maximize } Z = \$30x_1 - 2x_1^2 + 25x_2 - 0.5x_2^2$$

subject to

$$3x_1 + 6x_2 = 300 \text{ lb.}$$

Determine the optimal solution to this nonlinear programming model.

The Rolling Creek Textile Mill produces combed and brushed cotton cloth. The company

developed the following nonlinear programming model to determine the optimal number of yards of denim (x_1) brushed cotton (x_2) to produce each d order to maximize profit, subject to a l constraint:

10.

$$\text{maximize } Z = \$10x_1 - 0.02x_1^2 + 12x_2 - 0.03x_2^2$$

subject to

$$0.2x_1 + 0.1x_2 = 40 \text{ hr.}$$

Determine the optimal solution to this nonlinear programming model.

In the investment portfolio selection m this chapter, what if Jessica Todd decide that she wanted to maximize her retur while incurring a portfolio variance of no more than 0.020? Analyze this scenari

11.

using the Excel spreadsheet in [Exhibit 1](#) to determine what Jessica's portfolio a return would be.

The Riverwood Paneling Company makes two kinds of wood paneling: Colonial and Western. The company has developed the following nonlinear programming model to determine the optimal number of sheets of Colonial paneling (x_1) and Western paneling (x_2) to produce to maximize profit, subject to a labor constraint:

12.

$$\text{maximize } Z = \$25x_1 - 0.8x_1^2 + 30x_2$$

subject to

$$x_1 + 2x_2 = 40 \text{ hr.}$$

Determine the optimal solution to this nonlinear programming model.

13. Interpret the meaning of the Lagrange multiplier in [Problem 12](#).

Metro Telcom Systems develops, sells, installs computer systems. The company divided its customer base into five regions and it has 15 representatives who sell and install the company's systems. The company wants to allocate salespeople to regions that they maximize daily sales revenue. However, whereas the sales increase as the number of salespeople allocated to a region increases, they do so at a declining rate according to the following nonlinear formula:

$$\text{total sales} = a (b/x)$$

Following are the a and b parameters for daily sales in each region.

Region

	1	2	3	4
--	---	---	---	---

14. a \$15,000 \$24,000 \$8,100 \$12,000 \$2

b 9,000 15,000 5,300 7,600 12

[Page 472]

Because some of the regions are in urban areas and some are not, the representatives' daily expenses will differ between regions. The company has a daily expense budget of \$6,500, and the daily expenses (including travel costs) per representative for each region average for region 1, \$540 for region 2, \$290 for region 3, \$275 for region 4, and \$490

region 5. Formulate and solve a nonlinear programming model for this problem to determine the number of representatives to allocate to each region to maximize sales.

Blue Ridge Power Company has a maximum capacity of 2.5 million kwh of electric power available on a daily basis. The demand (in millions of kwh) for power from its customers for high-demand (peak) hours and low-demand (off-peak) hours is determined by the following formulas:

high demand: $5.8 - 0.06p_h + 0.005p_l$

15. low demand: $3.0 - 0.11p_l + 0.008p_h$

The variable p_l equals the price per kwh during low-demand hours, and p_h is the price per kwh during high-demand (peak) hours.

Formulate and solve a nonlinear programming model to determine the optimal structure (per kwh) that will maximize revenue.

The Metro Police Department has partitioned the city into four quadrants. The department has 20 police patrol cars available for each shift per day to assign to the quadrants. The department wants to assign patrol cars to the quadrants during a shift so that the crime rate is minimized while providing average response times to calls of less than 10 minutes. As patrol cars are assigned to each quadrant, the crime rate and response time decrease, but at a decreasing rate according to the following formula:

$$y = a + (b/x)$$

The a and b parameters for daily crime rate (expressed as crimes per 1,000 population) and response time (in minutes) are provided in the following table:

in the following table:

16.

Quadrant	Crime Rate		Response Time (min)
	a	b	a
1	0.24	0.15	4
2	0.37	0.21	8
3	0.21	0.12	6
4	0.48	0.30	3

Formulate and solve a nonlinear programming model to determine the number of police patrol cars to assign to each quadrant that will result in the minimum overall crime rate.

The Burger Doodle restaurant chain purchases ingredients from four different food suppliers. The company wants to construct a new central distribution center to process and package the ingredients into its menu items before shipping them to their various restaurants. The suppliers transport the food items in 40-foot trailer trucks. The coordinates of the food suppliers and the annual number of truckloads that will be transported to the distribution center are as follows:

Coordinates

Supplier	x	y	Annual Truckload
----------	-----	-----	---------------------

17.

A	200	200	65
B	100	500	120
C	250	600	90
D	500	300	75

[Page 473]

Determine the set of coordinates for the new distribution center that will minimize total miles traveled from the suppliers.

Home-Base, a home improvement and building supply chain, is going to build a warehouse facility to serve its stores in North Carolina cities Charlotte, Winston-Salem, Greensboro, Durham, Raleigh, and Wilmington. The coordinates of these cities (in miles), using Columbia, South Carolina as the graphical origin (0, 0) and the annual truckload trips that supply each store are as follows:

	Coordinates		
	<i>x</i>	<i>y</i>	Truckloads
Charlotte	15	85	16
Winston-Salem	42	145	9

Greensboro	88	145	10
Durham	125	140	3
Raleigh	135	125	6
Wilmington	180	18	7

Determine the set of coordinates for the new warehouse that will minimize the total miles traveled to the stores and identify the closest town to these coordinates.

An investment adviser is helping a couple

plan a retirement portfolio. The adviser recommended three stocks: Allied Electronics, Bank United, and Consolidated Computers. Following are the annual return and variance for each stock and the covariance between the stocks:

Stock	Annual Return	Covariance
Allied Electronics	.14	.10
Bank United	.10	.04
Consolidated Computers	.12	.08

Stock Combination (i, j)	Covariance
A,B	.4
A,C	.7
B,C	.3

The couple wants a total portfolio return at least .11. Determine the proportion each stock to include in the portfolio to minimize the overall risk.

Mark Decker has identified four stocks portfolio, and he wants to determine the percentage of his total available funds he should invest in each stock. The alternative stocks include an Internet company, a computer software company, a computer manufacturer, and an entertainment conglomerate. He wants a total annual return of .12. From historical data, he has determined the average annual return and variance for each of the funds, as follows:

Stock	Annual Return	Variance
1. Internet	.18	.11
2. Software	.12	.06

3. Computer	.10	.04
4. Entertainment	.15	.08

20.

[Page 474]

He has also estimated the covariances between stocks, as follows:

Stock Combination (i, j)	Covariance
1,2	.9
1,3	.7

1,4	.3
2,3	.8
2,4	.4
3,4	.2

Determine the percentage of Mark's total funds that he should invest in each stock to minimize his overall risk.

The Accentuate Consulting Firm has eight projects for which it has contracted with clients to develop computer systems and

software. The firm has 35 team members that can assign to the projects. The firm has developed the following profit functions for each project, where x_i equals the number of team members assigned to a project:

Project	Profit
1	$\$250,000x_1^{0.4}$
2	$370,000x_2^{0.3}$
3	$140,000x_3^{0.7}$
4	$500,000x_4^{0.2}$

5	$230,000x_5^{0.3}$
6	$170,000x_6^{0.6}$
7	$280,000x_7^{0.4}$
8	$315,000x_8^{0.4}$

These functions take into account the project completion time, cost, and probability of successful completion based on the number of team members assigned. These functions reflect the fact that profit increases but at a decreasing rate as team members are assigned. Thus, as additional team members are assigned to a project, the

marginal effect decreases. Determine the optimal number of team members to assign to each project in order to maximize profit.

Case Problem

ADMISSIONS AT STATE UNIVERSITY

State University has increased its tuition for in-state and out-of-state students in each of the past 5 years to offset cuts in its budget allocation from the state legislature. The university administration always thought that the number of applications received was independent of tuition; however, drops in applications and enrollments

the past 2 years have proved this theory to be wrong.

University admissions officials have developed the following relationships between the number of applicants who accept admission and enter State (x_1) and the cost of tuition per semester (t_1):

$$x_1 = 21,000 - 12t_1 \text{ (in-state)}$$

$$x_2 = 35,000 - 6t_2 \text{ (out-of-state)}$$

The university would like to develop a planning model that will indicate the in-state and

out-of-state tuitions, as well as the number of students that could be expected to enroll in the freshman class. The university doesn't have enough classroom space for more than 1,400 freshmen, and it needs at least 700 freshmen to meet all its class-size objectives. The university knows from historical data that approximately 55% of all in-state freshmen will want to live in the school dormitories, 72% of all out-of-state students will want to live in the dormitories, and there will be at most 800

dormitory rooms available for freshmen.

[Page 475]

The university also wants to make sure that it maintains high academic standards in its admissions decisions. It knows from historical data that the average SAT score for an in-state student is 960 and the average SAT score for an out-of-state student is 1,150. The university wants the entering freshman class to have an average SAT score of at least

1,000.

State University is a state-supported institution, so the state legislature wants to make sure that the university doesn't enroll just out-of-state students because they pay more tuition and they have better SAT scores. Thus, the government has instituted a policy that no more than 55% of the entering freshman class can be out-of-state students.

Develop and solve a nonlinear programming model for State University to indicate the

tuition the university should charge, the total tuition, and the number of in-state and out-of-state students it can expect with these tuition rates.

Case Problem

SELECTING A EUROPEAN DISTRIBUTION
CENTER SITE FOR AMERICAN
INTERNATIONAL AUTOMOTIVE
INDUSTRIES

American International
Automotive Industries (AIAI)
manufactures engine,
transmission, and chassis parts
for manufacturers and repair
companies in the United States,
South America, Canada,
Mexico, Asia, and Europe. The

company transports parts to its foreign markets by container ships. To serve its customers in South America and Asia, AIAI has large warehouse and distribution centers. In Europe, it ships into Hamburg and Gdansk, where it has contracted with independent distribution companies to deliver its products to customers throughout Europe. However, AIAI has been displeased with a recent history of late deliveries and rough handling of its products. For a time AIAI was not overly

concerned because its European market wasn't too big, and its European customers didn't complain. Plus, it had more pressing problems elsewhere. However, in the past few years, trade barriers have fallen in Europe, and Eastern European markets have opened up. AIAI's European business has expanded, as has new competition, and its customers have become more demanding and quality conscious. As a result, AIAI has initiated the selection process for the site of

a new European warehouse and distribution center. Although it provides parts to a number of smaller truck and auto maintenance and service centers in Europe, it has seven major customersauto and truck manufacturers in Vienna, Leipzig, Budapest, Prague, Krakow, Munich, and Frankfurt. Its customers have adopted manufacturing processes requiring continuous replenishment of parts and materials.

AIAl's European headquarters is in Hamburg. The vice

president for construction and development in Dayton, Ohio, has asked the Hamburg office to do a preliminary site search based on proximity to customers and mileage. Each container is converted to a single tractor-trailer truckload. The number of containers shipped annually to each customer is as follows: Vienna, 160; Leipzig, 100; Budapest, 180; Prague, 210; Krakow, 90; Munich, 120; and Frankfurt, 50. When the vice president of construction in Dayton received this information, he pulled out

his map of Europe and began to look for possible sites.

Assist AIAI with its site selection process in Europe. Recommend a city site that minimizes the total annual delivery mileage.

Chapter 11.

Probability and Statistics

[Page 477]

The techniques presented in [Chapters 2](#) through [10](#) are typically thought of as **[deterministic](#)**; that is, they are not subject to uncertainty or variation. With deterministic techniques, the assumption is that conditions of complete certainty and perfect

knowledge of the future exist. In the linear programming models presented in previous chapters, the various parameters of the models and the model results were assumed to be known with certainty. In the model constraints, we did not say that a bowl would require 4 pounds of clay "70% of the time." We specifically stated that each bowl would require exactly 4 pounds of clay (i.e., there was no uncertainty in our problem statement). Similarly, the solutions we derived for the

linear programming models contained no variation or uncertainty. It was assumed that the results of the model would occur in the future, without any degree of doubt or chance.

Deterministic
techniques assume that no uncertainty exists in model parameters.

In contrast, many of the techniques in management science do reflect *uncertain* information and result in *uncertain* solutions. These techniques are said to be **probabilistic**. This means that there can be more than one outcome or result to a model and that there is some doubt about which outcome will occur. The solutions generated by these techniques have a probability of occurrence. They may be in the form of *averages*; the actual values that occur will vary over time.

Probabilistic
techniques include
uncertainty and
assume that there
can be more than
one model solution.

Many of the upcoming chapters in this text present probabilistic techniques. The presentation of these techniques requires that the reader have a fundamental understanding of probability. Thus, the purpose of this

chapter is to provide an overview of the fundamentals, properties, and terminology of probability and statistics.

Types of Probability

Two basic types of probability can be defined: *objective probability* and *subjective probability*. First, we will consider what constitutes objective probability.

Objective Probability

Consider a referee's flipping a coin before a football game to determine which team will kick

off and which team will receive. Before the referee tosses the coin, both team captains know that they have a .50 (or 50%) chance (or probability) of winning the toss. None of the onlookers in the stands or anywhere else would argue that the probability of a head or a tail was not .50. In this example, the probability of .50 that either a head or a tail will occur when a coin is tossed is called an **objective probability**. More specifically, it is referred to as a **classical** or **a priori** (prior to the

occurrence), probability, one of the two types of objective probabilities.

*Objective probabilities that can be stated prior to the occurrence of an event are **classical**, or **a priori**, probabilities.*

A classical, or a priori, probability can be defined as

follows: Given a set of outcomes for an activity (such as a head or a tail when a coin is tossed), the probability of a specific (desired) outcome (such as a head) is the ratio of the number of specific outcomes to the total number of outcomes. For example, in our coin-tossing example, the probability of a head is the ratio of the number of specific outcomes (a head) to the total number of outcomes (a head and a tail), or $1/2$. Similarly, the probability of drawing an ace from a deck of 52 cards

would be found by dividing 4 (the number of aces) by 52 (the total number of cards in a deck) to get $1/13$. If we spin a roulette wheel with 50 red numbers and 50 black numbers, the probability of the wheel landing on a red number is 50 divided by 100, or $1/2$.

These examples are referred to as a priori probabilities because we can state the probabilities *prior to* the actual occurrence of the activity (i.e., ahead of time). This is because we know (or assume we know) the number of specific outcomes

and total outcomes prior to the occurrence of the activity. For example, we know that a deck of cards consists of 4 aces and 52 total cards before we draw a card from the deck and that a coin contains one head and one tail before we toss it. These probabilities are also known as classical probabilities because some of the earliest references in history to probabilities were related to games of chance, to which (as the preceding examples show) these probabilities readily apply.

The second type of objective probability is referred to as **relative frequency probability**. This type of objective probability indicates the relative frequency with which a specific outcome has been observed to occur in the long run. It is based on the observation of past occurrences. For example, suppose that over the past 4 years, 3,000 business students have taken the introductory management science course at State University, and 300 of

them have made an A in the course. The relative frequency probability of making an A in management science would be $300/3,000$ or $.10$. Whereas in the case of a classical probability we indicate a probability before an activity (such as tossing a coin) takes place, in the case of a relative frequency we determine the probability after observing, for example, what 3,000 students have done in the past.

*Objective
probabilities that are*

*stated after the
outcomes of an event
have been observed
are **relative
frequency
probabilities.***

The relative frequency definition of probability is more general and widely accepted than the classical definition. Actually, the relative frequency definition can encompass the classical case. For example, if

we flip a coin many times, in the long run the relative frequency of a head's occurring will be .50. If, however, you tossed a coin 10 times, it is conceivable that you would get 10 consecutive heads. Thus, the relative frequency (probability) of a head would be 1.0. This illustrates one of the key characteristics of a relative frequency probability: The relative frequency probability becomes more accurate as the total number of observations of the activity increases. If a coin were tossed about 350 times,

the relative frequency would approach $1/2$ (assuming a fair coin).

Relative frequency is the more widely used definition of objective probability.

Subjective Probability

When relative frequencies are not available, a probability is

often determined anyway. In these cases a person must rely on personal belief, experience, and knowledge of the situation to develop a probability estimate. A probability estimate that is not based on prior or past evidence is a **subjective probability**. For example, when a meteorologist forecasts a "60% chance of rain tomorrow," the .60 probability is usually based on the meteorologist's experience and expert analysis of the weather conditions. In other words, the meteorologist is not saying that

these exact weather conditions have occurred 1,000 times in the past and on 600 occasions it has rained, thus there is a 60% probability of rain.

Likewise, when a sportswriter says that a football team has an 80% chance of winning, it is usually not because the team has won 8 of its 10 previous games. The prediction is judgmental, based on the sportswriter's knowledge of the teams involved, the playing conditions, and so forth. If the sportswriter had based the probability estimate solely on

the team's relative frequency of winning, then it would have been an objective probability. However, once the relative frequency probability becomes colored by personal belief, it is subjective.

Subjective probability is an estimate based on personal belief, experience, or knowledge of a situation.

Subjective probability estimates are frequently used in making business decisions. For example, suppose the manager of Beaver Creek Pottery Company (referred to in [chapters 2](#) and [3](#)) is thinking about producing plates in addition to the bowls and mugs it already produces. In making this decision, the manager will determine the chances of the new product's being successful and returning a profit. Although the manager can use

personal knowledge of the market and judgment to determine a *probability of success*, direct relative frequency evidence is not generally available. The manager cannot observe the frequency with which the introduction of this new product was successful in the past. Thus, the manager must make a subjective probability estimate.

This type of subjective probability analysis is common in the business world. Decision makers frequently must

consider their chances for success or failure, the probability of achieving a certain market share or profit, the probability of a level of demand, and the like without the benefit of relative frequency probabilities based on past observations. Although there may not be a consensus as to the accuracy of a subjective estimate, as there is with an objective probability (e.g., everyone is sure there is a .50 probability of getting a head when a coin is tossed), subjective probability analysis

is often the only means available for making probabilistic estimates, and it is frequently used.

[Page 479]

A brief note of caution must be made regarding the use of subjective probabilities. Different people will often arrive at different subjective probabilities, whereas everyone should arrive at the same objective probability, given the same numbers and correct calculations. Therefore, when a

probabilistic analysis is to be made of a situation, the use of objective probability will provide more consistent results. In the material on probability in the remainder of this chapter, we will use objective probabilities unless the text indicates otherwise.

Fundamentals of Probability

Let us return to our example of a referee's tossing a coin prior to a football game. In the terminology of probability, the coin toss is referred to as an **experiment**. An experiment is an activity (such as tossing a coin) that results in one of the several possible outcomes. Our coin-tossing experiment can result in either one of two outcomes, which are referred

to as events: a head or a tail.
The probabilities associated
with each event in our
experiment follow:

<i>Event</i>	<i>Probability</i>
Head	.50
Tail	.50
	1.00

*An **experiment** is an activity that results in one of several possible outcomes.*

This simple example highlights two of the fundamental characteristics of probability. First, *the probability of an event is always greater than or equal to zero and less than or equal to one [i.e., $0 \leq P(\text{event}) \leq 1.0$]*. In our coin-tossing example, each event has a

probability of .50, which is in the range of 0 to 1.0. Second, *the probability of all the events included in an experiment must sum to one*. Notice in our example that the probability of each of the two events is .50, and the sum of these two probabilities is 1.0.

Two fundamentals of probability: $0 \leq P$ (events) ≤ 1.0 , and the probabilities of all events in an experiment sum to one.

The specific example of tossing a coin also exhibits a third characteristic: The events in a set of events are **mutually exclusive**. The events in an experiment are mutually exclusive if only one of them can occur at a time. In the context of our experiment, the term *mutually exclusive* means that any time the coin is tossed, *only one* of the two events can take place either a head or a tail can occur, but not both. Consider a customer who

enters a store to shop for shoes. The store manager estimates that there is a .60 probability that the customer will buy a pair of shoes and a .40 probability that the customer will not buy a pair of shoes. These two events are mutually exclusive because it is impossible to buy shoes and not buy shoes at the same time. In general, events are mutually exclusive if only one of the events can occur, but not both.

The events in an

experiment are
mutually exclusive
if only one can occur
at a time.

Because the events in our example of obtaining a head or tail are mutually exclusive, we can infer that the probabilities of mutually exclusive events *sum to 1.0*. Also, the probabilities of mutually exclusive events can be added. The following example will

demonstrate these fundamental characteristics of probability.

The probabilities of mutually exclusive events sum to one.

The staff of the dean of the business school at State University has analyzed the records of the 3,000 students who received a grade in management science during the past 4 years. The dean

wants to know the number of students who made each grade (A, B, C, D, or F) in the course. The dean's staff has developed the following table of information:

[Page 480]

Event Grade	Number of Students	Relative Frequency	Probability
A	300	$300/3,000$	.10
B	600	$600/3,000$	.20

C	1,500	$1,500/3,000$	.50
D	450	$450/3,000$	.15
F	150	$150/3,000$	.05
	3,000		1.00

This example demonstrates several of the characteristics of probability. First, the *data* (numerical information) in the second column show how the students are distributed across

the different grades (events). The third column shows the relative frequency with which each event occurred for the 3,000 observations. In other words, the relative frequency of a student's making a C is $1,500/3,000$, which also means that the probability of selecting a student who had obtained a C at random from those students who took management science in the past 4 years is .50.

This information, organized according to the events in the experiment, is called a **frequency distribution**. The

list of the corresponding probabilities for each event in the last column is referred to as a **probability distribution**.

*A **frequency distribution** is an organization of numeric data about the events in an experiment.*

All the events in this example are mutually exclusive; it is not

possible for two or more of these events to occur at the same time. A student can make only one grade in the course, not two or more grades. As indicated previously, mutually exclusive probabilities of an experiment can be summed to equal one. There are five mutually exclusive events in this experiment, the probabilities of which (.10, .20, .50, .15, and .05) sum to one.

This example exhibits another characteristic of probability: Because the five events in the example are all that can occur

(i.e., no other grade in the course is possible), the experiment is said to be **collectively exhaustive**.

Likewise, the coin-tossing experiment is collectively exhaustive because the only two events that can occur are a head and a tail. In general, when a *set of events* includes all the events that can possibly occur, the set is said to be collectively exhaustive.

*A set of events is
**collectively
exhaustive** when it*

includes all the events that can occur in an experiment.

The probability of a single event occurring, such as a student receiving an A in a course, is represented symbolically as $P(A)$. This probability is called the **marginal probability** in the terminology of probability. For our example, the marginal probability of a student's

getting an A in management science is

*A **marginal probability** is the probability of a single event occurring.*

$$P(A) = .10$$

For mutually exclusive events, it is possible to determine the probability that one or the other of several events will

occur. This is done by summing the individual marginal probabilities of the events. For example, the probability of a student receiving an A or a B is determined as follows:

$$P(A \text{ or } B) = P(A) + P(B) = .10 + .20 = .30$$

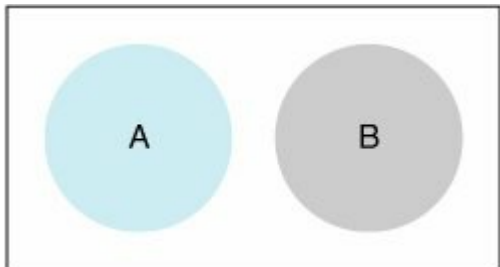
In other words, 300 students received an A and 600 students received a B; thus, the number of students who received an A or a B is 900. Dividing 900 students who received an A or a B by the total number of students, 3,000, yields the

probability of a student's receiving an A or a B [i.e., $P(A \text{ or } B) = .30$].

Mutually exclusive events can be shown pictorially with a **Venn diagram**. [Figure 11.1](#) shows a Venn diagram for the mutually exclusive events A and B in our example.

Figure 11.1. Venn diagram for mutually exclusive events

(This item is displayed on page 481 in the print version)



A Venn diagram
visually displays
mutually exclusive
and non-mutually
exclusive events.

Now let us consider a case in which two events are *not* mutually exclusive. In this case the probability that A or B or *both* will occur is expressed as

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

where the term $P(AB)$, referred to as the **joint probability** of A and B, is the probability that *both* A and B will occur. For mutually exclusive events, this term would have to equal zero because both events cannot occur together. Thus, for

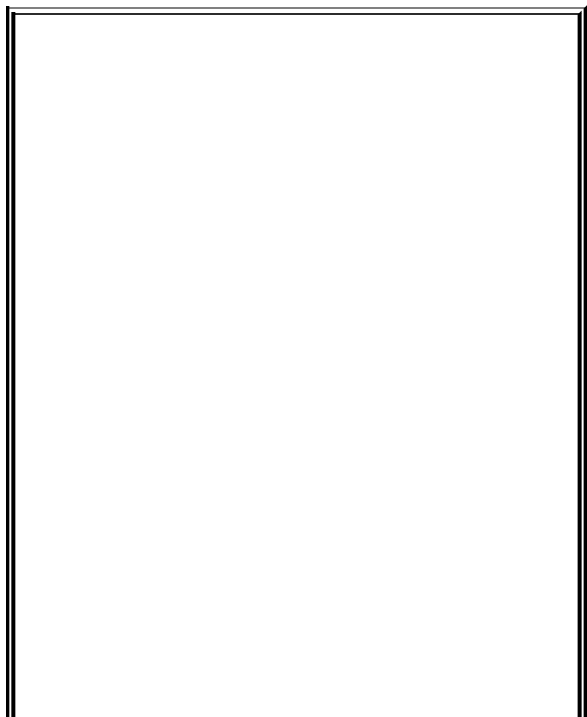
mutually exclusive events our formula would become

A joint probability
is the probability that two or more events that are not mutually exclusive can occur simultaneously.

$$\begin{aligned}P(\text{A or B}) &= P(\text{A}) + P(\text{B}) - P(\text{AB}) \\&= P(\text{A}) + P(\text{B}) - 0 \\&= P(\text{A}) + P(\text{B})\end{aligned}$$

which is the same formula we

developed previously for
mutually exclusive events.



Management Science

Application: Treasure

Hunting with Probability and Statistics

In 1857 the SS *Central America*, a wooden-hulled steamship, sank during a hurricane approximately 20 miles off the coast of Charleston, South Carolina, in the Atlantic Ocean. A total of 425 passengers and crew members lost their lives, and approximately \$400 million in gold bars and coins sank to the ocean bottom approximately 8,000 feet (i.e., 1.5 miles) below the surface. This shipwreck was the most famous of its time, causing a financial panic in the United States and a recommendation from insurance

investigators that new ships be built with iron hulls and watertight compartments.

In 1985 the Columbus-America Discovery Group was formed to locate and recover the remains of the *SS Central America*. A search plan was developed, using statistical techniques and Monte Carlo simulation. From the available historical information about the wreck, three scenarios were constructed, describing different search areas based on different accounts from nearby ships and witnesses. Using historical statistical data on weather, winds and currents in the area, and estimates of uncertainties in celestial navigation, probability distributions were developed for different possible sites where the ship might have sunk. Using Monte Carlo simulation, a map

was configured for each scenario, consisting of cells representing locations. Each cell was assigned a probability that the ship sank in that location. The probability maps for the three scenarios were combined into a composite probability map, which provided a guide for searching the ocean bottom using sonar. The search consisted of long, straight paths that concentrated on the high-probability areas. Based on the results of the sonar search of 1989, the discovery group recovered 1 ton of gold bars and coins from the wreck. In 1993 the total gold recovered from the wreck was valued at \$21 million. However, because of the historical significance of all recovered items, they were expected to bring a much higher return when sold.



Source: L. D. Stone, "Search for the SS *Central America*: Mathematical Treasure Hunting," *Interfaces* 22, no. 1 (January/February 1992): 3254.

[Page 482]

The following example will illustrate the case in which two

events are *not* mutually exclusive. Suppose it has been determined that 40% of all students in the school of business are at present taking management, and 30% of all the students are taking finance. Also, it has been determined that 10% take both subjects. Thus, our probabilities are

$$P(M) = .40$$

$$P(F) = .30$$

$$P(MF) = .10$$

The probability of a student's taking one or the other or both of the courses is determined as

follows:

$$\begin{aligned} P(M \text{ or } F) &= P(M) + P(F) - P(MF) \\ &= .40 + .30 - .10 = .60 \end{aligned}$$

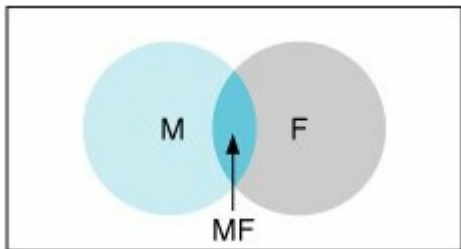
Observing this formulation closely, we can see why the joint probability, $P(MF)$, was subtracted out. The 40% of the students who were taking management also included those students taking both courses. Likewise, the 30% of the students taking finance also included those students taking both courses. Thus, if we add the two marginal probabilities, we are *double-counting* the

percentage of students taking both courses. By subtracting out one of these probabilities (that we added in twice), we derive the correct probability.

[Figure 11.2](#) contains a Venn diagram that shows the two events, M and F , that are not mutually exclusive, and the joint event, MF .

Figure 11.2. Venn diagram for non-mutually exclusive events and the joint

event



An alternative way to construct a probability distribution is to add the probability of an event to the sum of all previously listed probabilities in a probability distribution. Such a

list is referred to as a **cumulative probability distribution**. The cumulative probability distribution for our management science grade example is as follows:

Event Grade	Probability	Cumulative Probability
A	.10	.10
B	.20	.30
C	.50	.80
D	.15	.95

F

.05

1.00

1.00

The value of a cumulative probability distribution is that it organizes the event probabilities in a way that makes it easier to answer certain questions about the probabilities. For example, if we want to know the probability that a student will get a grade

of C or higher, we can add the probabilities of the mutually exclusive events A, B, and C:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) = .10 + .20 + .50 = .80$$

Or we can look directly at the cumulative probability distribution and see that the probability of a C and the events preceding it in the distribution (A and B) equals .80. Alternatively, if we want to know the probability of a grade lower than C, we can subtract the cumulative probability of a C from 1.00 (i.e., $1.00 - .80 =$

.20).

-

Statistical Independence and Dependence

Statistically, events are either independent or dependent. If the occurrence of one event does not affect the probability of the occurrence of another event, the events are *independent*. Conversely, if the occurrence of one event affects the probability of the occurrence of another event, the events are *dependent*. We will first turn our attention to a

discussion of independent events.

Independent Events

When we toss a coin, the two events getting a head and getting a tail are independent. If we get a head on the first toss, this result has absolutely no effect on the probability of getting a head or a tail on the next toss. The probability of getting either a head or a tail will still be .50, regardless of the outcomes of previous

tosses. In other words, the two events are **independent events**.

*A succession of events that do not affect each other are **independent events**.*

When events are independent, it is possible to determine the probability of both events occurring in succession by

multiplying the probabilities of each event. For example, what is the probability of getting a head on the first toss and a tail on the second toss? The answer is

$$P(HT) = P(H) \bullet P(T)$$

where

$P(H)$ = probability of a head

$P(T)$ = probability of a tail

$P(HT)$ = joint probability of a head and a tail

Therefore,

$$P(HT) = P(H) \bullet P(T) = (.5)(.5) \\ = .25$$

The probability of independent events occurring in succession is computed by multiplying the probabilities of each event.

As we indicated previously, the probability of both events occurring, $P(HT)$, is referred to as the *joint probability*.

Another property of independent events relates to **conditional probabilities**. A conditional probability is the probability that event A will occur given that event B has already occurred. This relationship is expressed symbolically as

A conditional probability is the probability that an event will occur, given that another event has already occurred.

$$P(A|B)$$

The term in parentheses, "A slash B," means "A, given the occurrence of B." Thus, the entire term $P(A|B)$ is

interpreted as the probability that A will occur, given that B has already occurred. If A and B are independent events, then

$$P(A|B) = P(A)$$

In words, this result says that if A and B are independent, then the probability of A, given the occurrence of event B, is simply equal to the probability of A. Because the events are independent of each other, the occurrence of event B will have no effect on the occurrence of A. Therefore, the probability of A is in no way dependent on

the occurrence of B.

In summary, if events A and B are independent, the following two properties hold:

1. $P(AB) = P(A) \bullet P(B)$

2. $P(A|B) = P(A)$

[Page 484]

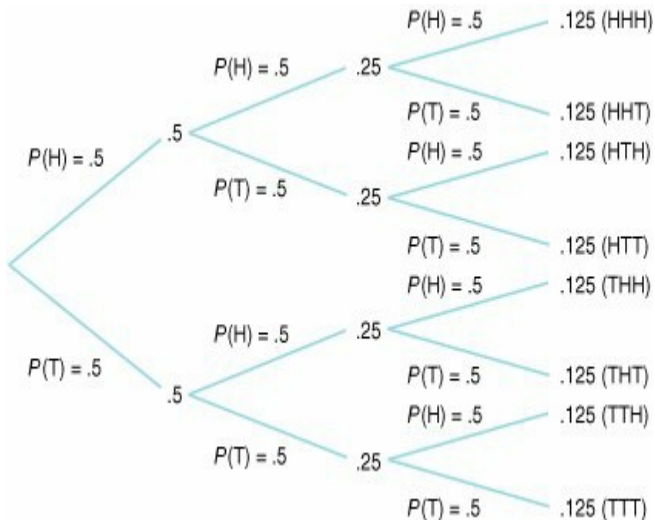
Probability Trees

Consider an example in which a coin is tossed three consecutive

times. The possible outcomes of this example can be illustrated by using a **probability tree**, as shown in [Figure 11.3](#).

Figure 11.3.

Probability tree for coin-tossing example



The probability tree in [Figure 11.3](#) demonstrates the probabilities of the various occurrences, given three tosses

of a coin. Notice that at each toss, the probability of either event's occurring remains the same: $P(H) = P(T) = .5$. Thus, the events are independent. Next, the joint probabilities of events occurring in succession are computed by multiplying the probabilities of all the events. For example, the probability of getting a head on the first toss, a tail on the second, and a tail on the third is .125:

$$P(HTT) = P(H) \cdot P(T) \cdot P(T) = (.5)(.5)(.5) = .125$$

However, do not confuse the results in the probability tree with conditional probabilities. The probability of a head and then two tails occurring on three consecutive tosses is computed prior to any tosses taking place. If the first two tosses have already occurred, the probability of getting a tail on the third toss is still .5:

$$P(T|HT) = P(T) = .5$$

The Binomial Distribution

Some additional information can be drawn from the probability tree of our example. For instance, what is the probability of achieving exactly two tails on three tosses? The answer can be found by observing the instances in which two tails occurred. It can be seen that two tails in three tosses occurred three times, each time with a probability of .125. Thus, the probability of getting exactly two tails in three tosses is the sum of these three probabilities, or .375. The use of a probability tree can

become very cumbersome, especially if we are considering an example with 20 tosses. However, the example of tossing a coin exhibits certain properties that enable us to define it as a **Bernoulli process**. The properties of a Bernoulli process follow:

- 1.** There are two possible outcomes for each trial (i.e., each toss of a coin). Outcomes could be success or failure, yes or no, heads or tails, good or bad, and so on.

*Properties of a **Bernoulli process.***

2. The probability of the outcomes remains constant over time. In other words, the probability of getting a head on a coin toss remains the same, regardless of the number of tosses.

[Page 485]

3. The outcomes of the trials are independent. The fact

that we get a head on the first toss does not affect the probabilities on subsequent tosses.

4. The number of trials is *discrete* and an integer. The term *discrete* indicates values that are *countable* and, thus, usually an integer for example, 1 car or 2 people rather than 1.34 cars or 2.51 people. There are 1, 2, 3, 4, 5, ... tosses of the coin, not 3.36 tosses.

Given the properties of a

Bernoulli process, a **binomial distribution** function can be used to determine the probability of a number of successes in n trials. The binomial distribution is an example of a **discrete distribution** because the value of the distribution (the number of successes) is discrete, as is the number of trials. The formula for the binomial distribution is

$$P(r) = \frac{n!}{r!(n-r)!} p^r q^{n-r}$$

where

p = probability of a success

$q = 1 - p$ = probability of a failure

n = number of trials

r = number of successes in n trials

A binomial distribution indicates the probability of r successes in n trials.

The terms $n!$, $(n\ r)!$, and $r!$ are called factorials. Factorials are computed using the formula

$$m! = m(m-1)(m-2)(m-3) \dots (2)(1)$$

The factorial $0!$ always equals one.

The binomial distribution formula may look complicated, but using it is not difficult. For example, suppose we want to determine the probability of getting exactly two tails in

three tosses of a coin. For this example, getting a tail is a success because it is the object of the analysis. The probability of a tail, p , equals .5; therefore, $q = 1 - .5 = .5$. The number of tosses, n , is 3, and the number of tails, r , is 2. Substituting these values into the binomial formula will result in the probability of two tails in three coin tosses:

$$\begin{aligned}
 P(2 \text{ tails}) &= P(r = 2) = \frac{3!}{2!(3-2)!} (.5)^2 (.5)^{3-2} \\
 &= \frac{(3 \cdot 2 \cdot 1)}{(2 \cdot 1)(1)} (.25)(.5) \\
 &= \frac{6}{2} (.125) \\
 P(r = 2) &= .375
 \end{aligned}$$

Notice that this is the same result achieved by using a probability tree in the previous section.

Now let us consider an example of more practical interest. An electrical manufacturer produces microchips. The microchips are inspected at the end of the production process

at a quality control station. Out of every batch of microchips, four are randomly selected and tested for defects. Given that 20% of all transistors are defective, what is the probability that each batch of microchips will contain exactly two defective microchips?

The two possible outcomes in this example are a good microchip and a defective microchip. Because defective microchips are the object of our analysis, a defective item is a success. The probability of a success is the probability of a

defective microchip, or $p = .2$. The number of trials, n , equals 4. Now let us substitute these values into the binomial formula:

[Page 486]

$$\begin{aligned}P(r = \text{ defectives}) &= \frac{4!}{2!(4-2)!} (.2)^2 (.8)^2 \\&= \frac{(4 \cdot 3 \cdot 2 \cdot 1)}{(2 \cdot 1)(2 \cdot 1)} (.04)(.64) \\&= \frac{24}{4} (.0256) \\&= .1536\end{aligned}$$

Thus, the probability of getting exactly two defective items out

of four microchips is .1536.

Now, let us alter this problem to make it even more realistic. The manager has determined that four microchips from every large batch should be tested for quality. If *two or more* defective microchips are found, the whole batch will be rejected. The manager wants to know the probability of rejecting an entire batch of microchips, if, in fact, the batch has 20% defective items.

From our previous use of the binomial distribution, we know

that it gives us the probability of an *exact* number of *integer* successes. Thus, if we want the probability of two or more defective items, it is necessary to compute the probability of two, three, and four defective items:

$$P(r \geq 2) = P(r = 2) + P(r = 3) + P(r = 4)$$

Substituting the values $p = .2$, $n = 4$, $q = .8$, and $r = 2, 3$, and 4 into the binomial distribution results in the probability of two or more defective items:

$$\begin{aligned}
 P(r \geq 2) &= \frac{4!}{2!(4-2)!} (.2)^2 (.8)^2 + \frac{4!}{3!(4-3)!} (.2)^3 (.8)^1 + \frac{4!}{4!(4-4)!} (.2)^4 (.8)^0 \\
 &= .1536 + .0256 + .0016 \\
 &= .1808
 \end{aligned}$$

Thus, the probability that a batch of microchips will be rejected due to poor quality is .1808.

Notice that the *collectively exhaustive* set of events for this example is 0, 1, 2, 3, and 4 defective transistors. Because the sum of the probabilities of a collectively exhaustive set of events equals 1.0,

$$P(r = 0, 1, 2, 3, 4) = P(r = 0)$$

$$+ P(r = 1) + P(r = 2) + P(r = 3) + P(r = 4) = 1.0$$

Recall that the results of the immediately preceding example show that

$$P(r = 2) + P(r = 3) + P(r = 4) = .1808$$

Given this result, we can compute the probability of "less than two defectives" as follows:

$$\begin{aligned} P(r < 2) &= P(r = 0) + P(r = 1) \\ &= 1.0 - [P(r = 2) + P(r = 3) + P(r = 4)] \\ &= 1.0 - .1808 \\ &= .8192 \end{aligned}$$

Although our examples included very small values for n and r that enabled us to work out the examples by hand, problems containing larger values for n and r can be solved easily by using an electronic calculator.

[Page 487]

Dependent Events

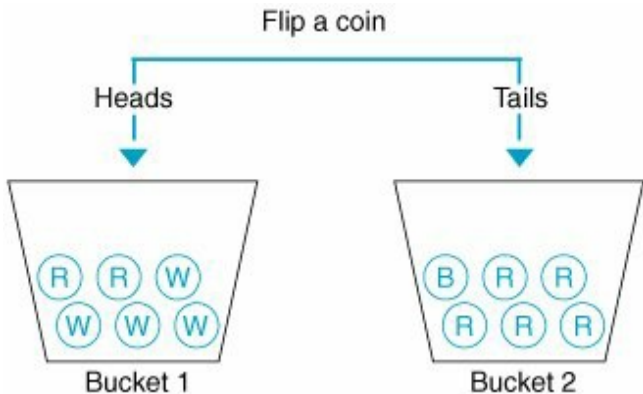
As stated earlier, if the occurrence of one event affects the probability of the

occurrence of another event, the events are **dependent events**. The following example illustrates dependent events.

Two buckets each contain a number of colored balls. Bucket 1 contains two red balls and four white balls, and bucket 2 contains one blue ball and five red balls. A coin is tossed. If a head results, a ball is drawn out of bucket 1. If a tail results, a ball is drawn from bucket 2. These events are illustrated in **Figure 11.4**.

Figure 11.4.

Dependent events



In this example the probability of drawing a *blue* ball is clearly

dependent on whether a head or a tail occurs on the coin toss. If a tail occurs, there is a $1/6$ chance of drawing a blue ball from bucket 2. However, if a head results, there is no possibility of drawing a blue ball from bucket 1. In other words, the probability of the event "drawing a blue ball" is dependent on the event "flipping a coin."

Like statistically independent events, dependent events exhibit certain defining properties. In order to describe these properties, we will alter

our previous example slightly, so that bucket 2 contains one white ball and five red balls. Our new example is shown in [Figure 11.5](#). The outcomes that can result from the events illustrated in [Figure 11.5](#) are shown in [Figure 11.6](#). When the coin is flipped, one of two outcomes is possible, a head or a tail. The probability of getting a head is .50, and the probability of getting a tail is .50:

$$P(H) = .50$$

$$P(T) = .50$$

Figure 11.5. Another set of dependent events

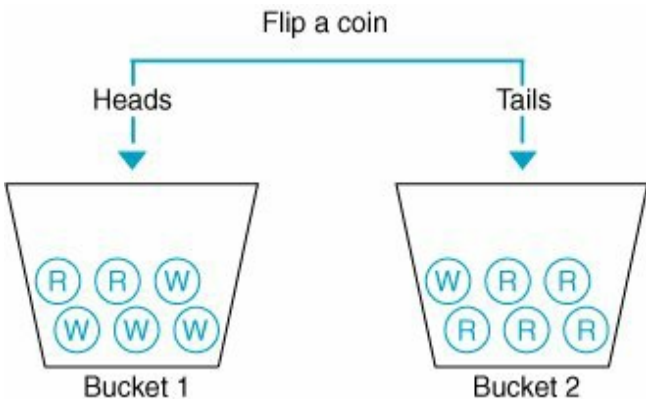
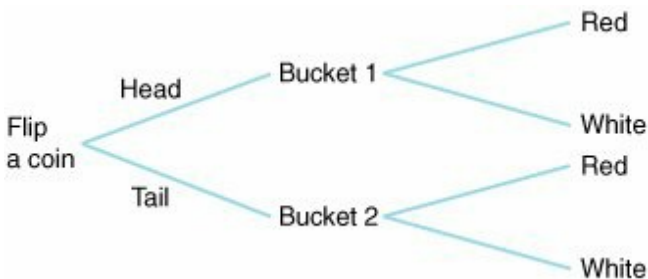


Figure 11.6.

Probability tree for dependent events



[Page 488]

As indicated previously, these

probabilities are referred to as *marginal* probabilities. They are also *unconditional* probabilities because they are the probabilities of the occurrence of a single event and are not conditional on the occurrence of any other event(s). They are the same as the probabilities of independent events defined earlier, and like those of independent events, the marginal probabilities of a collectively exhaustive set of events *sum to one*.

Once the coin has been tossed and a head or tail has resulted,

a ball is drawn from one of the buckets. If a head results, a ball is drawn from bucket 1; there is a $2/6$, or $.33$, probability of drawing a red ball and a $4/6$, or $.67$, probability of drawing a white ball. If a tail resulted, a ball is drawn from bucket 2; there is a $5/6$, or $.83$, probability of drawing a red ball and a $1/6$, or $.17$, probability of drawing a white ball. These probabilities of drawing red or white balls are called *conditional* probabilities because they are conditional on the outcome of the event of

tossing a coin. Symbolically, these conditional probabilities are expressed as follows:

$$P(R|H) = .33$$

$$P(W|H) = .67$$

$$P(R|T) = .83$$

$$P(W|T) = .17$$

The first term, which can be expressed verbally as "the probability of drawing a red ball, given that a head results from the coin toss," equals 33. The other conditional probabilities are expressed similarly.

Conditional probabilities can

also be defined by the following mathematical relationship.

Given two dependent events A and B,

$$P(A|B) = \frac{P(AB)}{P(B)}$$

the term $P(AB)$ is the joint probability of the two events, as noted previously. This relationship can be manipulated by multiplying both sides by $P(B)$, to yield

$$P(A|B) \cdot P(B) = P(AB)$$

Thus, the joint probability can

be determined by multiplying the conditional probability of A by the marginal probability of B.

Recall from our previous discussion of independent events that

$$P(AB) = P(A) \bullet P(B)$$

Substituting this result into the relationship for a conditional probability yields

$$\begin{aligned} P(A|B) &= \frac{P(A) \bullet P(B)}{P(B)} \\ &= P(A) \end{aligned}$$

which is consistent with the property for independent events.

Returning to our example, the joint events are the occurrence of a head and a red ball, a head and a white ball, a tail and a red ball, and a tail and a white ball. The probabilities of these joint events are as follows:

$$\begin{aligned}P(RH) &= P(R|H) \cdot P(H) = (.33)(.5) = .165 \\P(WH) &= P(W|H) \cdot P(H) = (.67)(.5) = .335 \\P(RT) &= P(R|T) \cdot P(T) = (.83)(.5) = .415 \\P(WT) &= P(W|T) \cdot P(T) = (.17)(.5) = .085\end{aligned}$$

The marginal, conditional, and joint probabilities for this example are summarized in [Figure 11.7](#). [Table 11.1](#) is a *joint probability table*, which summarizes the joint probabilities for the example.

Figure 11.7.
**Probability tree with
marginal, conditional,
and joint probabilities**

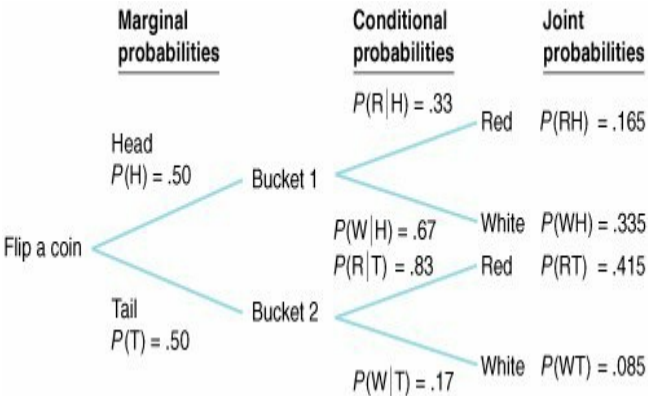


Table 11.1. Joint Probability Table

Draw a Ball

Flip a Coin	Draw a Ball		Marginal Probabilities
	RED	WHITE	

Head	$P(RH)$ $= .165$	$P(WH)$ $= .335$	$P(H) = .50$
Tail	$P(RT)$ $= .415$	$P(WT)$ $= .085$	$P(T) = .50$
Marginal probabilities	$P(R) = .580$	$P(W) = .420$	1.00

Bayesian Analysis

The concept of conditional probability given statistical dependence forms the necessary foundation for an area of probability known as *Bayesian analysis*. The technique is named after Thomas Bayes, an eighteenth-century clergyman who pioneered this area of analysis.

In Bayesian analysis, additional information is used to alter the marginal probability of the occurrence of an event.

The basic principle of Bayesian analysis is that additional information (if available) can sometimes enable one to alter (improve) the marginal probabilities of the occurrence of an event. The altered probabilities are referred to as *revised*, or **posterior**, probabilities.

*A **posterior** probability is the altered marginal probability of an*

event, based on additional information.

The concept of posterior probabilities will be illustrated using an example. A production manager for a manufacturing firm is supervising the machine setup for the production of a product. The machine operator sets up the machine. If the machine is set up correctly, there is a 10% chance that an

item produced on the machine will be defective; if the machine is set up incorrectly, there is a 40% chance that an item will be defective. The production manager knows from past experience that there is a .50 probability that a machine will be set up correctly or incorrectly by an operator. In order to reduce the chance that an item produced on the machine will be defective, the manager has decided that the operator should produce a sample item. The manager wants to know the probability

that the machine has been set up incorrectly if the sample item turns out to be defective.

The probabilities given in this problem statement can be summarized as follows:

$$\begin{array}{ll} P(C) = .50 & P(D|C) = .10 \\ P(IC) = .50 & P(D|IC) = .40 \end{array}$$

where

C = correct

IC = incorrect

D = defective

[Page 490]

The posterior probability for our example is the conditional probability that the machine has been set up incorrectly, given that the sample item proves to be defective, or $P(IC|D)$. In Bayesian analysis, once we are given the initial marginal and conditional probabilities, we can compute the posterior probability by

using **Bayes's rule**, as follows:

$$\begin{aligned}P(\text{IC} \mid \text{D}) &= \frac{P(\text{D} \mid \text{IC})P(\text{IC})}{P(\text{D} \mid \text{IC})P(\text{IC}) + P(\text{D} \mid \text{C})P(\text{C})} \\&= \frac{(.40)(.50)}{(.40)(.50) + (.10)(.50)} \\&= .80\end{aligned}$$

Previously, the manager knew that there was a 50% chance that the machine was set up incorrectly. Now, after producing and testing a sample item, the manager knows that if it is defective, there is an .80 probability that the machine was set up incorrectly. Thus, by gathering some additional

information, the manager can revise the estimate of the probability that the machine was set up correctly. This will obviously improve decision making by allowing the manager to make a more informed decision about whether to have the machine set up again.

In general, given two events, A and B, and a third event, C, that is conditionally dependent on A and B, Bayes's rule can be written as

$$P(A|C) = \frac{P(C|A)P(A)}{P(C|A)P(A) + P(C|B)P(B)}$$

Expected Value

It is often possible to assign numeric values to the various outcomes that can result from an experiment. When the values of the variables occur in no particular order or sequence, the variables are referred to as **random variables**. Every possible value of a variable has a probability of occurrence associated with it. For example, if a coin is tossed three times, the number

of heads obtained is a random variable. The possible values of the random variable are 0, 1, 2, and 3 heads. The values of the variable are random because there is no way of predicting which value (0, 1, 2, or 3) will result when the coin is tossed three times. If three tosses are made several times, the values (i.e., numbers of heads) that will result will have no sequence or pattern; they will be random.

Like the variables defined in previous chapters in this text, random variables are typically

represented symbolically by a letter, such as x , y , or z .

Consider a vendor who sells hot dogs outside a building every day. If the number of hot dogs the vendor sells is defined as the random variable x , then x will equal 0, 1, 2, 3, 4, . . . hot dogs sold daily.

Although the exact values of the random variables in the foregoing examples are not known prior to the event, it is possible to assign a probability to the occurrence of the possible values that can result. Consider a production

operation in which a machine breaks down periodically. From experience it has been determined that the machine will break down 0, 1, 2, 3, or 4 times per month. Although managers do not know the exact number of breakdowns (x) that will occur each month, they can determine the relative frequency probability of each number of breakdowns $P(x)$. These probabilities are as follows:

x	$P(x)$
0	.10
1	.20
2	.30
3	.25
4	.15
	1.00

These probability values taken together form a *probability distribution*. That is, the probabilities are distributed over the range of possible values of the random variable X .

The **expected value** of the random variable (the number of breakdowns in any given month) is computed by multiplying each value of the random variable by its probability of occurrence and summing these products.

The **expected value** of a random variable is computed by multiplying each possible value of the variable by its probability and summing these products.

For our example, the expected number of breakdowns per month is computed as follows:

$$\begin{aligned} E(x) &= (0)(.10) + (1)(.20) + (2)(.30) + (3)(.25) + (4)(.15) \\ &= 0 + .20 + .60 + .75 + .60 \\ &= 2.15 \text{ breakdowns} \end{aligned}$$

This means that, on the average, management can expect 2.15 breakdowns every month.

The expected value is often referred to as the weighted average, or *mean*, of the probability distribution and is a measure of central tendency of the distribution. In addition to knowing the mean, it is often desirable to know how the values are dispersed (or

scattered) around the mean. A measure of dispersion is the **variance**, which is computed as follows:

1. Square the difference between each value and the expected value.

2. Multiply these resulting amounts by the probability of each value.

3. Sum the values compiled in step 2.

The expected value is the mean of the probability distribution of the random variable.

Variance *is a measure of the dispersion of random variable values about the expected value, or mean.*

The general formula for computing the variance, which we will designate as σ^2 , is

$$\sigma^2 = \sum_{i=1}^n [x_i - E(x_i)]^2 P(x_i)$$

The variance (σ^2) for the machine breakdown example is computed as follows:

x_i	$P(x_i)$	$x_i E(x)$	$[x_i - E(x)]^2$	$[x_i - E(x)]^2 \bullet P(x_i)$
-------	----------	------------	------------------	---------------------------------

0	.10	2.15	4.62	.462
---	-----	------	------	------

1	.20	1.15	1.32	.264
2	.30	0.15	.02	.006
3	.25	0.85	.72	.180
4	.15	1.85	3.42	.513
	1.00			1.425

$\sigma^2 = 1.425$ breakdowns per month

The **standard deviation** is another widely recognized measure of dispersion. It is designated symbolically as σ and is computed by taking the square root of the variance, as follows:

$$\begin{aligned}\sigma &= \sqrt{1.425} \\ &= 1.19 \text{ breakdowns per month}\end{aligned}$$

*The **standard deviation** is computed by taking the square root of the variance.*

[Page 492]

Management Science

Application: A Probability

Model for Analyzing Coast Guard Patrol Effectiveness

A primary responsibility of the U.S. Coast Guard is to monitor and protect the maritime borders of the United States. One measure of Coast Guard effectiveness in meeting this responsibility is determined by a probability model that indicates the probability that the Coast Guard will intercept a randomly selected target vessel attempting to penetrate a Coast Guard patrol perimeter. The probability $P(I)$, the probability of intercepting the target vessel, is computed according to the formula

$$P(I) = P(I|D)P(D|A)P(A|O)P(O)$$

$P(I|D)$ is the probability that a target vessel will be intercepted (I), given that it is detected (D). It is a constant value estimated from Coast Guard historical data. $P(D|A)$ is the probability that a target vessel is detected, given that a Coast Guard vessel is available (A). This value is computed using a submodel based on the patrol area and speed of the Coast Guard vessel and the speed of the target vessel. $P(A|O)$ is the probability that the Coast Guard vessel is available, given that it is on the scene (O) in the patrol area. $P(O)$ is the probability that a Coast Guard vessel is on the scene in the patrol area, and currently it is assigned a constant value of one. The Coast Guard used this model to perform sensitivity analysis and analyze the effect of different parameter changes

on effectiveness. An example might be testing the impact of a change in the patrol area or vessel speed.



Source: S. O. Kimbrough, J. R. Oliver, and C. W. Pritchett, "On Post-Evaluation Analysis: Candle-Lighting and Surrogate Models," *Interfaces* 23, no. 3 (May/June 1993): 1728.

A small standard deviation or variance, relative to the expected value, or mean, indicates that most of the values of the random variable distribution are bunched close to the expected value.

Conversely, a large relative value for the measures of dispersion indicates that the values of the random variable are widely dispersed from the expected value.

The Normal Distribution

Previously, a *discrete* value was defined as a value that is countable (and usually an integer). A random variable is discrete if the values it can equal are finite and countable. The probability distributions we have encountered thus far have been discrete distributions. In other words, the values of the random variables that made up these discrete distributions were

always finite (for example, in the preceding example, there were five possible values of the random variable breakdowns per month). Because every value of the random variable had a unique probability of occurrence, the discrete probability distribution consisted of all the (finite) values of a random variable and their associated probabilities.

***A continuous
random variable
can take on an***

*infinite number of
values within an
interval, or range.*

In contrast, a **continuous random variable** can take on an infinite number of values within some interval. This is because continuous random variables have values that are not specifically countable and are often fractional. The distinction between discrete and continuous random

variables is sometimes made by saying that *discrete* relates to things that can be counted and *continuous* relates to things that are measured. For example, a load of oil being transported by tanker may consist of not just 1 million or 2 million barrels but 1.35 million barrels. If the range of the random variable is between 1 million and 2 million barrels, then there is an infinite number of possible (fractional) values between 1 million and 2 million barrels, even though the value 1.35 million

corresponds to a discrete value of 1,350,000 barrels of oil. No matter how small an interval exists between two values in the distribution, there is always at least one value and, in fact, an infinite number of values between the two values.

[Page 493]

Because a continuous random variable can take on an extremely large or infinite number of values, assigning a unique probability to every value of the random variable

would require an infinite (or very large) number of probabilities, each of which would be infinitely small.

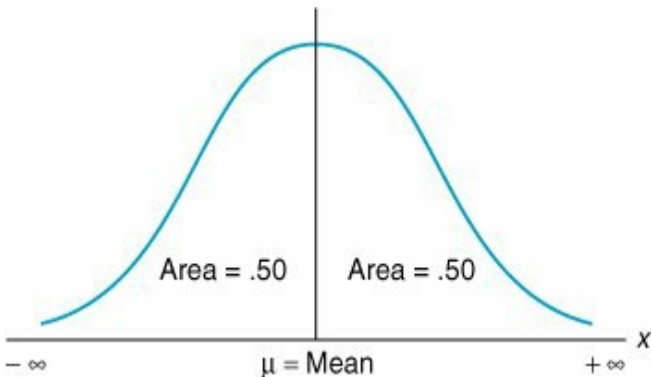
Therefore, we cannot assign a unique probability to each value of the continuous random variable, as we did in a discrete probability distribution. In a **continuous distribution**, we can refer only to the probability that a value of the random variable is within some *range*. For example, we can determine the probability that between 1.35 million and 1.40 million barrels of oil are transported,

or the probability that fewer than or more than 1.35 million barrels are shipped, but we cannot determine the probability that exactly 1.35 million barrels of oil are transported.

One of the most frequently used continuous probability distributions is the **normal distribution**, which is a continuous curve in the shape of a bell (i.e., it is symmetrical). The normal distribution is a popular continuous distribution because it has certain mathematical

properties that make it easy to work with, and it is a reasonable approximation of the continuous probability distributions of a number of natural phenomena. [Figure 11.8](#) is an illustration of the normal distribution.

Figure 11.8. The normal curve



*The **normal distribution** is a continuous probability distribution that is symmetrical on both*

sides of the mean that is, it is shaped like a bell.

The fact that the normal distribution is a continuous curve reflects the fact that it consists of an infinite or extremely large number of points (on the curve). The bell-shaped curve can be flatter or taller, depending on the degree to which the values of the random variable are dispersed

from the center of the distribution. The center of the normal distribution is referred to as the *mean* (μ), and it is analogous to the average of the distribution.

The center of a normal distribution is its mean, μ .

Notice that the two ends (or tails) of the distribution in [Figure 11.8](#) extend from $-\infty$ to

+ ∞ . In reality, random variables do not often take on values over an infinite range. Therefore, when the normal distribution is applied, it actually approximates the distribution of a random variable with finite limits. The *area* under the normal curve represents *probability*. The entire area under the curve equals 1.0 because the sum of the probabilities of all values of a random variable in a probability distribution must equal 1.0. Fifty percent of the curve lies to the right of the

mean, and 50% lies to the left. Thus, the probability that a random variable x will have a value greater (or less) than the mean is .50.

The area under the normal curve represents probability, and the total area under the curve sums to one.

As an example of the

application of the normal distribution, consider the Armor Carpet Store, which sells Super Shag carpet. From several years of sales records, store management has determined that the mean number of yards of Super Shag demanded by customers during a week is 4,200 yards, and the standard deviation is 1,400 yards. It is necessary to know both the mean and the standard deviation to perform a probabilistic analysis using the normal distribution. The store management assumes that the

continuous random variable, yards of carpet demanded per week, is normally distributed (i.e., the values of the random variable have approximately the shape of the normal curve). The mean of the normal distribution is represented by the symbol μ , and the standard deviation is represented by the symbol σ :

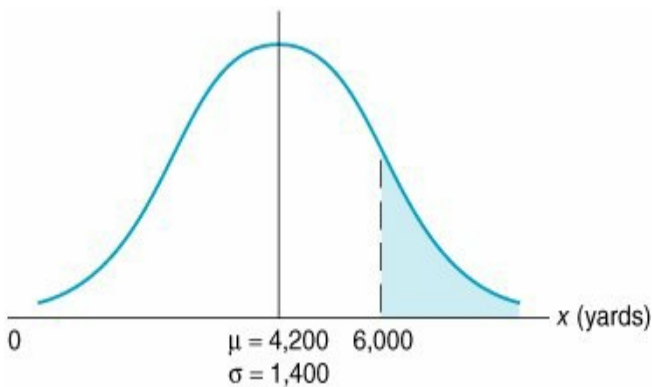
[Page 494]

$$\mu = 4,200\text{yd.}$$

$$\sigma = 1,400\text{yd.}$$

The store manager wants to know the probability that the demand for Super Shag in the upcoming week will exceed 6,000 yards. The normal curve for this example is shown in [Figure 11.9](#). The probability that x (the number of yards of carpet) will be equal to or greater than 6,000 is expressed as

Figure 11.9. The normal distribution for carpet demand



$$P(x \geq 6,000)$$

which corresponds to the area under the normal curve to the right of the value 6,000 because the area under the curve (in [Figure 11.9](#)) represents probability. In a normal distribution, area or probability is measured by determining the *number of standard deviations the value of the random variable x is from the mean*. The number of standard deviations a value is from the mean is represented by Z and is computed using the formula

$$Z = \frac{x - \mu}{\sigma}$$

The area under a normal curve is measured by determining the number of standard deviations the value of a random variable is from the mean.

The number of standard deviations a value is from the mean gives us a consistent

standard of measure for all normal distributions. In our example, the units of measure are yards; in other problems, the units of measure may be pounds, hours, feet, or tons. By converting these various units of measure into a common measure (number of standard deviations), we create a standard that is the same for all normal distributions.

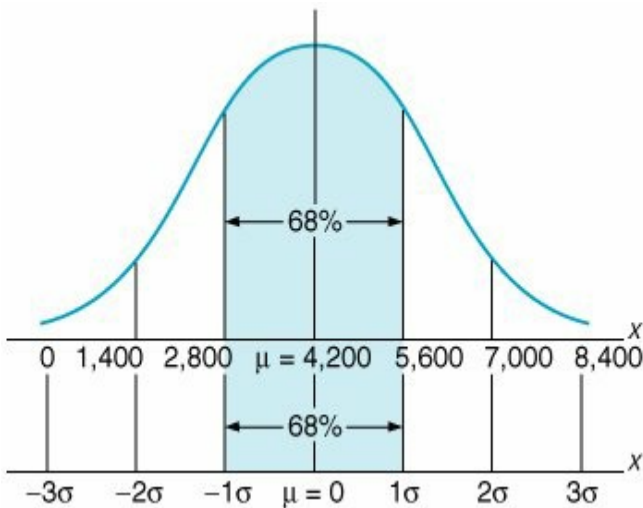
Actually, the standard form of the normal distribution has a mean of zero ($\mu = 0$) and a standard deviation of one ($\sigma = 1$). The value Z enables us to

convert this scale of measure into whatever scale our problem requires.

[Figure 11.10](#) shows the *standard normal distribution*, with our example distribution of carpet demand above it. This illustrates the conversion of the scale of measure along the horizontal axis from yards to number of standard deviations.

Figure 11.10. The standard normal distribution

(This item is displayed on page 495 in the print version)



The horizontal axis along the bottom of [Figure 11.10](#) corresponds to the standard normal distribution. Notice that the area under the normal curve between 1σ and 1σ represents 68% of the total area under the normal curve, or a probability of .68. Now look at the horizontal axis corresponding to yards in our example. Given that the standard deviation is 1,400 yards, the area between 1σ (2,800 yards) and 1σ (5,600 yards) is also 68% of the total area under the curve. Thus, if

we measure distance along the horizontal axis in terms of the number of standard deviations, we will determine the same probability, no matter what the units of measure are. The formula for Z makes this conversion for us.

[Page 495]

Returning to our example, recall that the manager of the carpet store wants to know the probability that the demand for Super Shag in the upcoming week will be 6,000 yards or

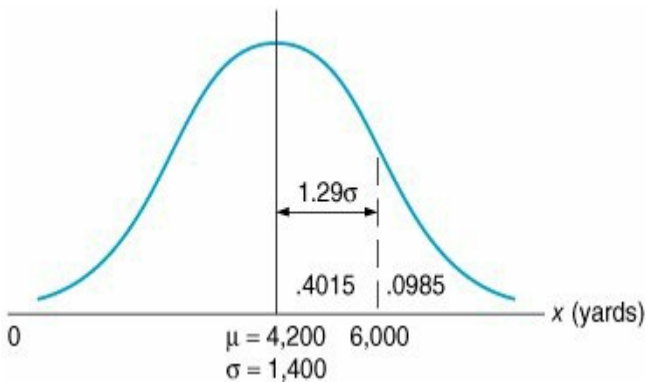
more. Substituting the values $x = 6,000$, $\mu = 4,200$, and $\sigma = 1,400$ yards into our formula for Z , we can determine the number of standard deviations the value 6,000 is from the mean:

$$\begin{aligned} Z &= \frac{x - \mu}{\sigma} \\ &= \frac{6,000 - 4,200}{1,400} \\ &= 1.29 \text{ standard deviations} \end{aligned}$$

The value $x = 6,000$ is 1.29 standard deviations from the mean, as shown in [Figure 11.11](#).

Figure 11.11.

Determination of the Z value



The area under the standard

normal curve for values of Z has been computed and is displayed in easily accessible *normal tables*. [Table A.1](#) in [Appendix A](#) is such a table. It shows that $Z = 1.29$ standard deviations corresponds to an area, or probability, of .4015. However, this is the area between $\mu = 4,200$ and $x = 6,000$ because what was measured was the area within 1.29 standard deviations of the mean. Recall, though, that 50% of the area under the curve lies to the right of the mean. Thus, we can subtract

.4015 from .5000 to get the area to the right of $x = 6,000$:

$$\begin{aligned}P(x \geq 6,000) &= .5000 - .4015 \\&= .0985\end{aligned}$$

This means that there is a .0985 (or 9.85%) probability that the demand for carpet next week will be 6,000 yards or more.

[Page 496]

Now suppose that the carpet store manager wishes to consider two additional questions: (1) What is the

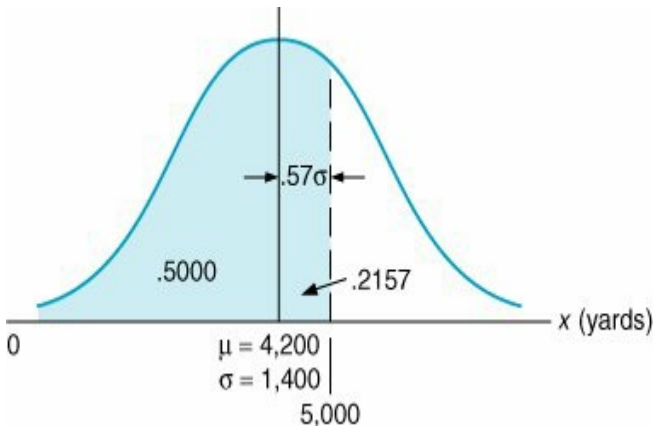
probability that demand for carpet will be 5,000 yards or less? (2) What is the probability that the demand for carpet will be between 3,000 yards and 5,000 yards? We will consider each of these questions separately.

First, we want to determine $P(x \leq 5,000)$. The area representing this probability is shown in [Figure 11.12](#). The area to the left of the mean in [Figure 11.12](#) equals .50. That leaves only the area between $\mu = 4,200$ and $x = 5,000$ to be determined. The number of

standard deviations $x = 5,000$
is from the mean is

$$\begin{aligned} Z &= \frac{x - \mu}{\sigma} \\ &= \frac{5,000 - 4,200}{1,400} \\ &= \frac{800}{1,400} \\ &= .57 \text{ standard deviations} \end{aligned}$$

Figure 11.12. Normal distribution for $P(x \leq 5,000 \text{ yards})$



The value $Z = .57$ corresponds to a probability of $.2157$ in [Table A.1](#) in [Appendix A](#). Thus, the area between $4,200$ and $5,000$ in [Figure 11.12](#) is $.2157$. To find our desired probability,

we simply add this amount to .5000:

$$P(x \leq 5,000) = .5000 + .2157 \\ = .7157$$

Next, we want to determine $P(3,000 \leq x \leq 5,000)$. The area representing this probability is shown in [Figure 11.13](#). The shaded area in [Figure 11.13](#) is computed by finding two areas: the area between $x = 3,000$ and $\mu = 4,200$ and the area between $\mu = 4,200$ and $x = 5,000$ and summing them. We already computed the area between 4,200 and 5,000 in

the previous example and found it to be .2157. The area between $x = 3,000$ and $\mu = 4,200$ is found by determining how many standard deviations $x = 3,000$ is from the mean:

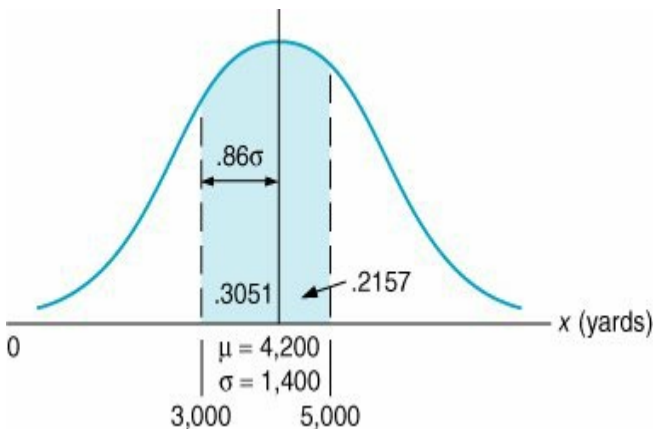
[Page 497]

$$\begin{aligned} Z &= \frac{3,000 - 4,200}{1,400} \\ &= \frac{-1,200}{1,400} \\ &= -.86 \end{aligned}$$

Figure 11.13. Normal distribution with

$$P(3,000 \text{ yards} \leq x \leq 5,000 \text{ yards})$$

(This item is displayed on page 496 in the print version)



The minus sign is ignored when we find the area corresponding to the Z value of .86 in [Table A.1](#). This value is .3051. We now find our probability by summing .2157 and .3051:

$$\begin{aligned}P(3,000 \leq x \leq 5,000) &= .2157 + .3051 \\ &= .5208\end{aligned}$$

The normal distribution, although applied frequently in probability analysis, is just one of a number of continuous probability distributions. In subsequent chapters, other continuous distributions will be identified. Being acquainted

with the normal distribution will make the use of these other distributions much easier.

Sample Mean and Variance

The mean and variance we described in the previous section are actually more correctly referred to as the **population mean** (μ) and variance (σ). The population mean is the mean of the entire set of possible measurements or set of data being analyzed,

and the population variance is defined similarly. Although we provided numeric values of the population mean and variance (or standard deviation) in our examples in the previous section, it is more likely that a **sample mean**, $\bar{x}$, would be used to estimate the population mean, and a **sample variance**, s^2 , would be used to estimate the population variance. The computation of a true population mean and variance is usually too time-consuming and costly, or the entire population of data may not be

available. Instead, a **sample**, which is a smaller subset of the population, is used. The mean and variance of this sample can then be used as an estimate of the population mean and variance if the population is assumed to be normally distributed. In other words, a smaller set of sample data is used to make generalizations or estimates, referred to as *inferences*, about the whole (population) set of data.

*The **population mean** and variance*

are for the entire set of data being analyzed.

*A **sample mean** and **variance** are derived from a subset of the population data and are used to make inferences about the population.*

For example, in the previous section we indicated that the mean weekly demand for Super Shag carpet was 4,200 yards and the standard deviation was 1,400; thus $\mu = 4,200$ and $\sigma = 1,400$. However, we noted that the management of the Armor Carpet Store determined the mean and standard deviation from "several years of sales records." Thus, the mean and standard deviation were actually a sample mean and standard deviation, based on

several years of sample data from a normal distribution. In other words, we used a sample mean and sample standard deviation to estimate the population mean and standard deviation. The population mean and standard deviation would have actually been computed from all weekly demand since the company started, for some extended period of time.

[Page 498]

The sample mean, $\bar{x}$, is computed using the following

formula:

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n}$$

The sample variance, s^2 , is computed as follows:

$$s^2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}$$

or, in shortcut form:

$$s^2 = \frac{\sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n}}{n-1}$$

The sample standard deviation is simply the square root of the variance:

$$s = \sqrt{s^2}$$

Let us consider our example for Armor Carpet Store again, except now we will use a sample of 10 weeks' demand data for Super Shag carpet, as follows:

Week i	Demand x_i
1	2,900
2	5,400
3	3,100
4	4,700
5	3,800
6	4,300
7	6,800

8

2,900

9

3,600

10

4,500

 $\Sigma = 42,000$

The sample mean is computed
as

$$\begin{aligned}\bar{x} &= \frac{\sum_{i=1}^n x_i}{n} \\ &= \frac{42,000}{10} = 4,200 \text{ yd.}\end{aligned}$$

[Page 499]

The sample variance is computed as

$$\begin{aligned}
 s^2 &= \frac{\sum_{i=1}^n x_i^2 - \frac{\left(\sum_{i=1}^n x_i\right)^2}{n}}{n-1} \\
 &= \frac{(190,060,000) - \frac{(1,764,000,000)}{10}}{9} \\
 s^2 &= 1,517,777
 \end{aligned}$$

The sample standard deviation, s , is

$$\begin{aligned}
 s &= \sqrt{s^2} \\
 &= \sqrt{1,517,777} \\
 &= 1,232 \text{ yd.}
 \end{aligned}$$

These values are very close to the mean and standard

deviation we originally estimated in this example in the previous section (i.e., $\mu = 4,200$ yd., $\sigma = 1,400$ yd.). In general, the accuracy of the sample depends on two factors: the sample size (n) and the variation in the data. The larger the sample, the more accurate the sample statistics will be in estimating the population statistics. Also, the more variable the data are, the less accurate the sample statistics will be as estimates of the population statistics.

The Chi-Square Test for Normality

Several of the quantitative techniques that are presented in the remainder of this text include probabilistic data and parameters and statistical analysis. In many cases the problem data are assumed (or stated) to be normally distributed, with a mean and standard deviation, which enable statistical analysis to be performed based on the normal distribution. However, in reality

it can never be simply assumed that data are normally distributed or in fact reflect any probability distribution.

Frequently, a statistical test must be performed to determine the exact distribution (if any) to which the data conform.

The chi-square (χ^2) test is one such statistical test to see if observed data fit a particular probability distribution, including the normal distribution. The chi-square test compares the actual

frequency distribution for a set of data with a theoretical frequency distribution that would be expected to occur for a specific distribution. This is also referred to as testing the *goodness-of-fit* of a set of data to a specific probability distribution.

To perform a chi-square goodness-of-fit test, the actual number of frequencies in each class or the range of a frequency distribution is compared to the theoretical frequencies that should occur in each class if the data

followed a particular distribution. These numeric differences between the actual and theoretical values in each class are used in a formula to compute a χ^2 test statistic, or number, which is then compared to a number from a chi-square table called a *critical value*. If the computed χ^2 test statistic is greater than the tabular critical value, then the data do not follow the distribution being tested; if the critical value is greater than the computed test statistic, the distribution does exist. In

statistical terminology this is referred to as testing the hypothesis that the data are, for example, normally distributed. If the χ^2 test statistic is greater than the tabular critical value, the hypothesis (H_0) that the data fits the hypothesized distribution is rejected, and the distribution does not exist; otherwise, it is accepted.

[Page 500]

To demonstrate how to apply the chi-square test for a normal

distribution, we will use an expanded version of our Armor Carpet Store example used in the previous section. To test the distribution, we need a lot more data than the 10 weeks of demand we used previously. Instead, we will now assume that we have collected a sample of 200 weeks of demand for Super Shag carpet and that these data have been grouped according to the following frequency distribution:

Range,

Weekly Demand (yd.)	Frequency (weeks)
0 < 1,000	2
1,000 < 2,000	5
2,000 < 3,000	22
3,000 < 4,000	50
4,000 < 5,000	62
5,000 <	40

5,000 < 6,000	20
6,000 < 7,000	15
7,000 < 8,000	3
8,000+	1
	200

Because we haven't provided the actual data, we are also

going to assume that the sample mean ($\bar{x}$) equals 4,200 yards and the sample standard deviation (s) equals 1,232 yards, although normally these values would be computed directly from the data as in the previous section.

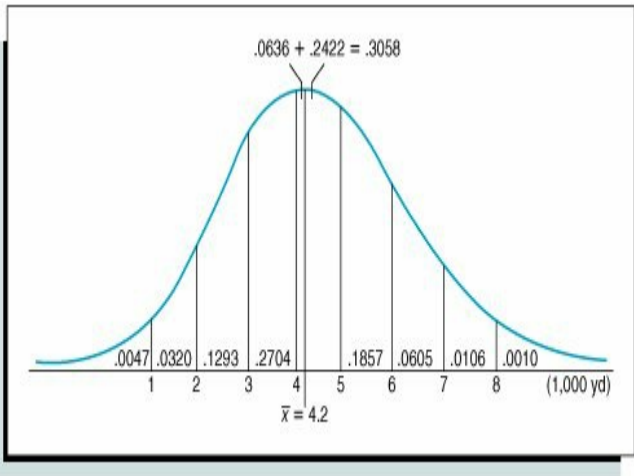
The first step in performing the chi-square test is to determine the number of observations that *should be* in each frequency range, if the distribution is normal. We start by determining the area (or probability) that should be in each class, using the sample

mean and standard deviation. [Figure 11.14](#) shows the theoretical normal distribution, with the area in each range. The area of probability for each range is computed using the normal probability table ([Table A.1](#) in [Appendix A](#)), as demonstrated earlier in this chapter. For example, the area less than 1,000 yards is determined by computing the Z statistic for $x = 1,000$:

$$\begin{aligned} Z &= \frac{x - \bar{x}}{s} \\ &= \frac{1,000 - 4,200}{1,232} \\ &= -2.60 \end{aligned}$$

Figure 11.14. The theoretical normal distribution

(This item is displayed on page 500 in the print version)



This corresponds to a normal table value of .4953. This is the area from 1,000 to the sample mean (4,200). Subtracting this

value from .5000 results in the area less than 1,000, or $.5000 - .4953 = .0047$. The area in the range of 1,000 to 2,000 yards is computed by subtracting the area from 2,000 to the mean (.4633) from the area from 1,000 to the mean (.4953), or $.4953 - .4633 = .0320$. The Z values for these ranges and all the range areas in [Figure 11.14](#) are shown in [Table 11.2](#).

Table 11.2. The determination of the theoretical range

frequencies

[\[View full size image\]](#)

Range	Z	Area: $x \rightarrow \bar{x}$	Range Area	Normal Frequency ($n = 200$)
0 < 1,000	—	.5000	.0047	0.94
1,000 < 2,000	-2.60	.4953	.0320	6.40
2,000 < 3,000	-1.79	.4633	.1293	25.86
3,000 < 4,000	-.97	.3340	.2704	54.08
4,000 < 5,000	$\left\{ \begin{array}{l} -.16 \\ .65 \end{array} \right\}$	$\left\{ \begin{array}{l} .0636 \\ .2422 \end{array} \right\}$	.3058	61.16
5,000 < 6,000	1.46	.4279	.1857	37.14
6,000 < 7,000	2.27	.4884	.0605	12.10
7,000 < 8,000	3.08	.4990	.0106	2.12
8,000+	—	.5000	.0010	0.20

Notice that the area for the range that includes the mean

between 4,000 and 5,000 yards is determined by adding the two areas to the immediate right and left of the mean, $.0636 + .2422 = .3058$.

The next step is to compute the theoretical frequency for each range by multiplying the area in each range by $n = 200$. For example, the frequency in the range from 0 to 1,000 is $(.0047)(200) = 0.94$, and the frequency for the range from 1,000 to 2,000 is $(.0320)(200) = 6.40$. These and the remaining theoretical frequencies are shown in the

last column in [Table 11.2](#).

Next we must compare these theoretical frequencies with the actual frequencies in each range, using the following chi-square test statistic:

$$\chi^2_{k-p-1} = \sum_k \frac{(f_o - f_t)^2}{f_t}$$

where

f_o = observed frequency

f_t = theoretical frequency

k = the number of classes or ranges

p = the number of estimated parameters

$k - p - 1$ = degrees of freedom

[Page 502]

However, before we can apply this formula, we must make an important adjustment. Before

we can apply the chi-square test, each range must include at least five theoretical observations. Thus, we need to combine some of the ranges so that they will contain at least five theoretical observations. For our distribution we can accomplish this by combining the two lower class ranges (01,000 and 1,0002,000) and the three higher ranges (6,0007,000; 7,0008,000; and 8,000+). This results in a revised frequency distribution with six frequency classes, as shown in [Table 11.3](#).

Table 11.3. Computation of χ^2 test statistic

Range, Weekly Demand	Observed Frequency f_o	Theoretical Frequency f_t	$(f_o - f_t)^2$	$(f_t)^2$
0 < 2,000	7	7.34	.12	.01
2,000 < 3,000	22	25.86	14.90	.57
3,000 < 4,000	50	54.08	16.64	.30
4,000 < 5,000	62	61.16	.71	.01

5,000 < 6,000	40	37.14	8.18	.22
6,000+	19	14.42	21.00	1.4
				2.5

The completed chi-square test statistic is shown in the last column in [Table 11.3](#) and is computed as follows:

$$\chi^2 = \sum_6 \frac{(f_o - f_t)^2}{f_t}$$

$$= 2.588$$

Next we must compare this test statistic with a critical value obtained from the chi-square table ([Table A.2](#)) in [Appendix A](#). The degrees of freedom for the critical value are $= k - p - 1$, where k is the number of frequency classes, or 6; and p is the number of parameters that were estimated for the distribution, which in this case is 2, the sample mean and the sample standard deviation.

Thus,

$$k - p - 1 = 6 - 2 - 1$$

= 3 degrees of freedom

Using a level of significance (degree of confidence) of .05 (i.e., $\alpha = .05$), from [Table A.2](#):

$$\chi^2_{.05,3} = 7.815$$

Because $7.815 > 2.588$, we accept the hypothesis that the

distribution is normal. If the χ^2 value of 7.815 were less than the computed χ^2 test statistic, the distribution would not be considered normal.

Statistical Analysis with Excel

QM for Windows does not have statistical program modules. Therefore, to perform statistical analysis, and specifically to compute the mean and standard deviation from sample data, we must rely on Excel.

Exhibit 11.1 shows the Excel spreadsheet for our Armor Carpet Store example. The average demand for the sample data (4,200) is computed in cell C16, using the formula **=AVERAGE(C4:C13)**, which is also shown on the formula bar at the top of the spreadsheet. Cell C17 contains the sample standard deviation (1,231.98), computed by using the formula **=STDEV(C4:C13)**.

Exhibit 11.1.

[\[View full size image\]](#)

Click on "Tools" and then
access "Data Analysis."

Microsoft Excel - Exhibit11.1.xls

File Edit Format Tools Data Window Help

Tools > Data Analysis

Formula bar: =AVERAGE(C4:C13)

Week	Demand
1	2900
2	5400
3	3100
4	4700
5	3900
6	4500
7	6800
8	2900
9	3600
10	4500

	Demand
Mean	4200.00
Standard Error	389.50
Median	4050.00
Mode	2900.00
Standard Deviation	1231.98
Sample Variance	1517777.78
Kurtosis	0.89
Skewness	1.00
Range	3900.00
Minimum	2900.00
Maximum	6800.00
Sum	42000.00
Count	10.00
Confidence Level(95.0%)	861.31

Average = 4200

Standard deviation = 1231.98

Generated from
"Data Analysis."

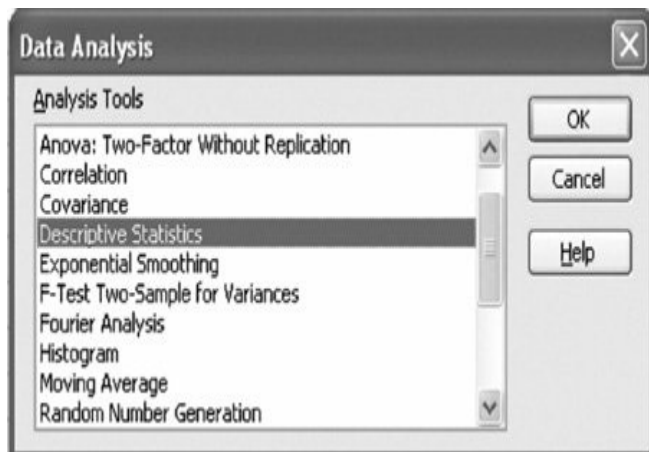
=STDEV(C4:C13)

=AVERAGE(C4:C13)

A statistical analysis of the sample data can also be obtained by using the "Data Analysis" option from the "Tools" menu at the top of the spreadsheet. (If this option is not available on your "Tools" menu, select the "Add-Ins" option from the "Tools" menu and then select the "Analysis ToolPak" option. This will insert the "Data Analysis" option on your "Tools" menu when you return to it.) Selecting the "Data Analysis" option from the

"Tools" menu will result in the Data Analysis window shown in [Exhibit 11.2](#).

Exhibit 11.2.



Select "Descriptive Statistics" from this window. This will result in the dialog window titled Descriptive Statistics shown in [Exhibit 11.3](#). This window, completed as shown, results in summary statistics for the demand data in our carpet store example. Notice that the input range we entered, **C3:C13**, includes the "Demand" heading on the spreadsheet in cell C3, which we acknowledge by checking the "Labels in first row" box.

This will result in our statistical summary being labeled "Demand" on the spreadsheet, as shown in cell E3 in [Exhibit 11.1](#). Notice that we indicated where we wanted to locate the summary statistics on our spreadsheet by typing **E3** in the "Output Range" window. We obtained the summary statistics by checking the "Summary Statistics" box at the bottom of the screen.

Exhibit 11.3.

[\[View full size image\]](#)

Cells with data

Indicates that the first row of data (in C3) is a label.

Specifies location of statistical summary on spreadsheet.

The image shows the 'Descriptive Statistics' dialog box in Microsoft Excel. The dialog box has a title bar with a close button (X). It contains several sections: 'Input', 'Output options', and 'Summary statistics'. The 'Input' section has 'Input Range' set to '\$C\$3:\$C\$13' and 'Grouped By' set to 'Columns'. The 'Labels in first row' checkbox is checked. The 'Output options' section has 'Output Range' set to '\$E\$3', 'New Worksheet Ply' and 'New Workbook' are unselected, and 'Summary statistics' is checked. The 'Confidence Level for Mean' is set to 95%. There are also checkboxes for 'Kth Largest' and 'Kth Smallest', both set to 1. Callouts from the surrounding text point to specific fields: 'Cells with data' points to the 'Input Range' field; 'Indicates that the first row of data (in C3) is a label.' points to the 'Labels in first row' checkbox; and 'Specifies location of statistical summary on spreadsheet.' points to the 'Output Range' field.

Descriptive Statistics

Input

Input Range:

Grouped By: ☒ Columns ☐ Rows

☒ Labels in first row

Output options

☒ Output Range:

☐ New Worksheet Ply:

☐ New Workbook

☒ Summary statistics

☒ Confidence Level for Mean: %

☐ Kth Largest:

☐ Kth Smallest:

OK Cancel Help

Summary

The field of probability and statistics is quite large and complex, and it contains much more than has been presented in this chapter. This chapter presented the basic principles and fundamentals of probability. The primary purpose of this brief overview was to prepare the reader for other material in the book. The topics of decision analysis ([Chapter 12](#)), simulation

([Chapter 14](#)), probabilistic inventory models ([Chapter 16](#)), and, to a certain extent, project management ([Chapter 8](#)) are probabilistic in nature and require an understanding of the fundamentals of probability.

References

Chou, Ya-lun. *Statistical Analysis*, 2nd ed. New York: Holt, Rinehart & Winston, 1975.

Cramer, H. *The Elements of Probability Theory and Some of Its Applications*, 2nd ed. New York: John Wiley & Sons, 1973.

Dixon, W. J., and Massey, F. J. *Introduction to Statistical Analysis*, 4th ed. New York: McGraw-Hill, 1983.

Hays, W. L., and Winkler, R. L. *Statistics: Probability, Inference, and Decision*, 2nd

ed. New York: Holt, Rinehart & Winston, 1975.

Mendenhall, W., Reinmuth, J. E., Beaver, R., and Duhan, D. *Statistics for Management and Economics*, 5th ed. North Scituate, MA: Duxbury Press, 1986.

Neter, J., Wasserman, W., and Whitmore, G. A. *Applied Statistics*, 3rd ed. Boston: Allyn & Bacon, 1988.

Sasaki, K. *Statistics for Modern Business Decision Making*. Belmont, CA: Wadsworth,

1969.

Spurr, W. A., and Bonini, C. P.
*Statistical Analysis for Business
Decisions*. Homewood, IL:
Richard D. Irwin, 1973.

Example Problem Solution

The following example will illustrate the solution of a problem involving a normal probability.

Problem Statement

The Radcliffe Chemical Company and Arsenal produces explosives for the U.S. Army. Because of the nature of its

products, the company devotes strict attention to safety, which is also scrutinized by the federal government. Historical records show that the annual number of property damage and personal injury accidents is normally distributed, with a mean of 8.3 accidents and a standard deviation of 1.8 accidents.

[Page 505]

- 1.** What is the probability that the company will have fewer than 5 accidents next

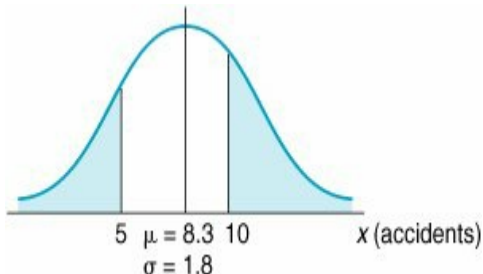
year? more than 10?

- 2.** The government will fine the company \$200,000 if the number of accidents exceeds 12 in a 1-year period. What average annual fine can the company expect?

Solution

Set Up the Normal Distribution

Step 1.



Solve Part A

$$P(x \leq 5 \text{ accidents})$$

$$\begin{aligned} Z &= \frac{x - \mu}{\sigma} \\ &= \frac{5 - 8.3}{1.8} \\ &= -1.83 \end{aligned}$$

From [Table A.1](#) in [Appendix A](#) we see that

$Z = 1.83$ corresponds to a probability of .4664; thus,

$$\begin{aligned} P(x \leq 5) &= .5000 - .4664 \\ &= .0336 \end{aligned}$$

$$P(x \geq 10 \text{ accidents})$$

$$\begin{aligned} Z &= \frac{x - \mu}{\sigma} \\ &= \frac{10 - 8.3}{1.8} \\ &= .94 \end{aligned}$$

Step 2.

From [Table A.1](#) in [Appendix A](#) we see that $Z = .94$ corresponds to a probability of .3264; thus,

$$P(X \geq 10) = .5000 - .3264$$

$$= .1736$$

Solve Part B

$P(x \geq 12 \text{ accidents})$

$$\begin{aligned} Z &= \frac{x - \mu}{\sigma} \\ &= \frac{12 - 8.3}{1.8} \\ &= 2.06 \end{aligned}$$

[Page 506]

Step 3. From [Table A.1](#) in [Appendix A](#) we see that $Z = 2.06$ corresponds to a probability of .4803; thus,

$$\begin{aligned} P(x \geq 12) &= .5000 - .4803 \\ &= .0197 \end{aligned}$$

Therefore, the

company's expected
annual fine is

$$\begin{aligned} \$200,000 \cdot P(x \geq 12) &= (\$200,000)(.0197) \\ &= \$3,940 \end{aligned}$$

Problems

Indicate which of the following probabilities are objective and which are subjective. (Note that in some cases, the probabilities may not be entirely on or the other.)

- 1.** The probability of snow tomorrow
- 2.** The probability of catching a fish
- 3.** The probability of the prime interest rate rising in the coming year
- 1.** **4.** The probability that the Cincinnati Reds will win the World Series

- 5.** The probability that demand for a product will be a specific amount next month
- 6.** The probability that a political candidate will win an election
- 7.** The probability that a machine will break down
- 8.** The probability of being dealt four aces in a poker hand

A gambler in Las Vegas is cutting a deck of cards for \$1,000. What is the probability that the card for the gamble will be the following?

- 2.**
 - 1.** A face card
 - 2.** A queen
 - 3.** A spade

4. A jack of spades

Downhill Ski Resort in Colorado has accumulated information from records of the past 30 winters regarding the measurable snowfall. This information is as follows:

<i>Snowfall (in.)</i>	<i>Frequency</i>
-----------------------	------------------

019	2
-----	---

2029	7
------	---

3039	8
------	---

4049	8
------	---

50+

5

30

- 1.** Determine the probability of each event in this frequency distribution
- 2.** Are all the events in this distribution mutually exclusive? Explain.

Employees in the textile industry can be segmented as follows:

Employees

Number

Female and union 12,000

Female and
nonunion 25,000

Male and union 21,000

4. Male and nonunion 42,000

[Page 507]

- 1.** Determine the probability of each event in this distribution.
- 2.** Are the events in this distribution mutually exclusive? Explain.

3. What is the probability that an employee is male?
4. Is this experiment collectively exhaustive? Explain.

The quality control process at a manufacturing plant requires that each lot of finished units be sampled for defective items. Twenty units from each lot are inspected. If five or more defective units are found, the lot is rejected. If a lot is known to contain 10% defective items, what is the probability that the lot will be rejected? accepted?

5.

A manufacturing company has 10 machines in continuous operation during a workday. The probability that an individual machine will break down

6. during the day is .10. Determine the probability that during any given day 3 machines will break down.

A polling firm is taking a survey regarding a proposed new law. Of the voters polled, 30% are in favor of the law. If 10 people are surveyed, what is the probability that 4 will indicate that they are opposed to the passage of the new law?

An automobile manufacturer has discovered that 20% of all the transmissions it installed in a particular style of truck one year are defective. It has contacted the owners of these vehicles and asked them to return their trucks to the dealer to check the transmission. The Friendly Auto Mart

sold seven of these trucks and has two of the new transmissions in stock. What is the probability that the auto dealer will need to order more new transmissions?

A new county hospital is attempting to determine whether it needs to add a particular specialist to its staff. Five percent of the general hospital

9. population in the county contracts the illness the specialist would treat. If 12 patients check into the hospital in a day, what is the probability that 4 or more will have the illness?

A large research hospital has accumulated statistical data on its patients for an extended period. Researchers have determined that patients who are smokers have an

18% chance of contracting a serious illness such as heart disease, cancer, or emphysema, whereas there is only a .06 probability that a nonsmoker will contract a serious illness. From hospital records, the researchers know that 23% of all hospital patients are smokers, while 77% are nonsmokers. For planning purposes, the hospital physician staff would like to know the probability that a given patient is a smoker if the patient has a serious illness.

Two law firms in a community handle all the cases dealing with consumer suits against companies in the area. The Abercrombie firm takes 40% of all suits and the Olson firm handles the other 60%. The Abercrombie firm wins 70% of its cases, and the Olson firm wins 60% of its cases.

- 11.**
- 1.** Develop a probability tree showing all marginal, conditional, and joint probabilities.
 - 2.** Develop a joint probability table.
 - 3.** Using Bayes's rule, determine the probability that the Olson firm handled a particular case, given that the case was won.

The Senate consists of 100 senators, of whom 34 are Republicans and 66 are Democrats. A bill to increase defense appropriations is before the Senate.

- 12.** Thirty-five percent of the Democrats and 70% of the Republicans favor the bill. The bill needs a simple majority to pass. Using a probability tree, determine the probability that the bill will pass.

A retail outlet receives radios from three electrical appliance companies. The outlet receives 20% of its radios from A, 40% from B, and 40% from C. The probability of receiving a defective radio from A is .01; from B, .02; and from C, .08.

13.

1. Develop a probability tree showing all marginal, conditional, and joint probabilities.
2. Develop a joint probability table.

[Page 508]

3. What is the probability that a defective radio returned to the retail store came from company B?

A metropolitan school system consists

of three districts north, south, and central. The north district contains 25% of all students, the south district contains 40%, and the central district contains 35%. A minimum-competency test was given to all students; 10% of the north district students failed, 15% of the south district students failed, and 5% of the central district students failed.

14.

- 1.** Develop a probability tree showing all marginal, conditional, and joint probabilities.
- 2.** Develop a joint probability table.
- 3.** What is the probability that a student selected at random failed the test?

A service station owner sells Goodroad tires, which are ordered from a local tire

distributor. The distributor receives tires from two plants, A and B. When the owner of the service station receives an order from the distributor, there is a .50 probability that the order consists of tires from plant A or plant B. However, the distributor will not tell the owner which plant the tires come from. The owner knows that 20% of all tires produced at plant A are defective, whereas only 10% of the tires produced at plant B are defective. When an order arrives at the station, the owner is allowed to inspect it briefly. The owner takes this opportunity to inspect one tire to see if it is defective. If the owner believes the tire came from plant A, the order will be sent back. Using Bayes's rule, determine the posterior probability that a tire is from plant A, given that the owner finds that it is defective.

15.

16. A metropolitan school system consists of two districts, east and west. The east district contains 35% of all students, and the west district contains the other 65%. A vocational aptitude test was given to all students; 10% of the east district students failed, and 25% of the west district students failed. Given that a student failed the test, what is the posterior probability that the student came from the east district?

The Ramshead Pub sells a large quantity of beer every Saturday. From past sales records, the pub has determined the following probabilities for sales:

Barrels

Probability

6

.10

7	.20
---	-----

17.

8	.40
---	-----

9	.25
---	-----

10	.05
----	-----

1.00

Compute the expected number of barrels that will be sold on Saturday.

The following probabilities for grades in management science have been determined based on past records:

18.

<i>Grade</i>	<i>Probability</i>
A	.10
B	.30
C	.40
D	.10
F	.10

The grades are assigned on a 4.0 scale where an A is a 4.0, a B a 3.0, and so on. Determine the expected grade and variance for the course.

A market in Boston orders oranges from Florida. The oranges are shipped to Boston from Florida by either railroad, truck, or airplane; an order can take 1, 2, 3, or 4 days to arrive in Boston once it is placed. The following probabilities have been assigned to the number of days it takes to receive an order once it is placed (referred to as *lead time*):

*Lead Time**Probability***19.**

1

.20

2

.50

3

.20

4

.10

1.00

Compute the expected number of days it takes to receive an order and the standard deviation.

An investment firm is considering two alternative investments, A and B, under two possible future sets of economic conditions, good and poor. There is a .60 probability of good economic conditions occurring and a .40 probability of poor economic conditions occurring. The expected gains and losses under each economic type of conditions are shown in the following table:

		Economic Conditions	
20.	Investment	Good	Poor

A	\$900,000	\$800,000
B	120,000	70,000

Using the expected value of each investment alternative, determine which should be selected.

An investor is considering two investments, an office building and bonds. The possible returns from each investment and their probabilities are as follows:

Office Building

Bonds

Return	Probability	Return	Probabilit
--------	-------------	--------	------------

\$50,000	.30	\$30,000	.60
----------	-----	----------	-----

60,000	.20	40,000	.40
--------	-----	--------	-----

80,000	.10		1.00
--------	-----	--	------

10,000	.30		
--------	-----	--	--

0	.10		
---	-----	--	--

	1.00		
--	------	--	--

21.

Using expected value and standard deviation as a basis for comparison, discuss which of the two investments should be selected.

22. The Jefferson High School Band Boosters Club has organized a raffle. The prize is a \$6,000 car. Two thousand tickets to the raffle are to be sold at \$1 apiece. If a person purchases four tickets, what will be the expected value of the tickets?

The time interval between machine breakdowns in a manufacturing firm is defined according to the following probability distribution:

Time Interval (hr.)	Probability
--------------------------------	--------------------

23.

1	.15
2	.20
3	.40
4	.25
	1.00

[Page 510]

Determine the cumulative probability distribution and compute the expected time between machine breakdowns.

The life of an electronic transistor is normally distributed, with a mean of 500 hours and a standard deviation of 80 hours. Determine the probability that a transistor will last for more than 400 hours.

The grade point average of students at a university is normally distributed, with a mean of 2.6 and a standard deviation of 0.6. A recruiter for a company is interviewing students for summer employment. What percentage of the students will have a grade point average of 3.5 or greater?

The weight of bags of fertilizer is normally distributed, with a mean of 50 pounds and a standard deviation of 6 pounds. What is the probability that a

bag of fertilizer will weigh between 45 and 55 pounds?

The monthly demand for a product is normally distributed, with a mean of 700 units and a standard deviation of 200 units. What is the probability that demand will be greater than 900 units in a given month?

27.

The Polo Development Firm is building a shopping center. It has informed renters that their rental spaces will be ready for occupancy in 19 months. If the expected time until the shopping center

28. is completed is estimated to be 14 months, with a standard deviation of 4 months, what is the probability that the renters will not be able to occupy in 19 months?

29. A warehouse distributor of carpet keep 6,000 yards of deluxe shag carpet in stock during a month. The average demand for carpet from the stores that purchase from the distributor is 4,500 yards per month, with a standard deviation of 900 yards. What is the probability that a customer's order will not be met during a month? (This situation is referred to as a stockout.)

The manager of the local National Video Store sells videocassette recorders at discount prices. If the store does not have a video recorder in stock when a customer wants to buy one, it will lose the sale because the customer will purchase a recorder from one of the many local competitors. The problem is that the cost of renting warehouse space to keep enough recorders in inventory to meet all demand is

30. excessively high. The manager has determined that if 90% of customer demand for recorders can be met, then the combined cost of lost sales and inventory will be minimized. The manager has estimated that monthly demand for recorders is normally distributed, with a mean of 180 recorders and a standard deviation of 60. Determine the number of recorders the manager should order each month to meet 90% of customer demand.

The owner of Western Clothing Company has determined that the company must sell 670 pairs of denim jeans each month to break even (i.e., to reach the point where total revenue equals total cost). The company's marketing department has estimated that monthly demand is normally distributed, with a mean of 805 pairs o

31.

jeans and a standard deviation of 207 pairs. What is the probability that the company will make a profit each month?

- 32.** Lauren Moore, a professor in management science, is computing her final grades for her introductory management science class. The average final grade is a 63, with a standard deviation of 10. Professor Moore wants to curve the final grades according to a normal distribution so that 10% of the grades are Fs, 20% are Ds, 40% are Cs, 20% are Bs, and 10% are As. Determine the numeric grades that conform to the curve Professor Moore wants to establish.

The SAT scores of all freshmen

accepted at State University are normally distributed, with a mean of 1,050 and a standard deviation of 120. The College of Business at State

- 33.** University has accepted 620 of these freshmen into the college. All students in the college who score over 1,200 are eligible for merit scholarships. How many students can the college administration expect to be eligible for merit scholarships?

Erin Richards is a junior at Central High School, and she has talked to her guidance counselor about her chances of being admitted to Tech after her graduation. The guidance counselor has told her that Tech generally accepts only those applicants who graduate in the top 10% of their high school class. The average grade point average of the last four senior classes has been 2.67, with

34.

a standard deviation of 0.58. What GPA will Erin have to achieve to be in the top 10% of her class?

[Page 511]

The associate dean in the college of business at Tech is going to purchase a new copying machine for the college. One model he is considering is the Xerox X10. The sales representative has told him that this model will make an average of 125,000 copies, with a standard deviation of 40,000 copies, before breaking down. What is the probability that the copier will make 200,000 copies before breaking down?

35.

The Palace Hotel believes its customers may be waiting too long for room service. The hotel operations manager knows that the time for room service

orders is normally distributed, and he sampled 10 room service orders during a 3-day period and timed each (in minutes), as follows:

23

23

15

12

36. 26

16

19

18

30

25

The operations manager believes that only 10% of the room service orders should take longer than 25 minutes if the hotel has good customer service. Does the hotel room service meet this goal?

Agnes Hammer is a senior majoring in management science. She has been interviewing with several companies for a job when she graduates, and she is curious about what starting salary offers she might receive. There are 140 seniors in the graduating class for her major, and more than half have received job offers. She asked 12 of her classmates at random what their annual starting salary offers were, and she received the following responses:

\$28,500

\$35,500

32,600

36,000

37. 34,000

25,700

27,500

29,000

24,600

31,500

34,500

26,800

Assume that starting salaries are normally distributed. Compute the mean and standard deviation for these data and determine the probability that Agnes will receive a salary offer of less

than \$27,000.

The owner of Gilley's Ice Cream Parlor has noticed that she sells more ice cream on hotter days during the summer, especially on days when the temperature is 85° or higher. To plan how much ice cream to stock, she would like to know the average daily high temperature for the summer months of July and August. Assuming that daily temperatures are normally distributed, she has gathered the following data for the high temperature for 20 days from a local almanac:

86°

92°

85

94

78

83

91

81

38.

90

84

92

76

83

78

80

78

69

85

74

90

Compute the mean and standard deviation for these data and determine the expected number of days in July and August that the high temperature will be 85° or greater.

The state of Virginia has implemented a Standard of Learning (SOL) test that all public school students must pass before they can graduate from high school. A passing grade is 75. Montgomery County High School administrators want to gauge how well their students might do on the SOL test, but they don't want to take the time to test the whole student population. Instead, they selected 20 students at random and

gave them the test. The results are as follows:

83	79	56	93
----	----	----	----

48	92	37	45
----	----	----	----

39.

72	71	92	71
----	----	----	----

66	83	81	80
----	----	----	----

58	95	67	78
----	----	----	----

Assume that SOL test scores are normally distributed. Compute the

mean and standard deviation for these data and determine the probability that a student at the high school will pass the test.

The department of management science at Tech has sampled 250 of its majors and compiled the following frequency distribution of grade point averages (on a 4.0 scale) for the previous semester:

GPA	Frequency
$0 < 0.5$	1
$0.5 < 1.0$	4

40.

$1.0 < 1.5$ 20

$1.5 < 2.0$ 35

$2.0 < 2.5$ 67

$2.5 < 3.0$ 58

$3.0 < 3.5$ 47

$3.5 < 4.0$ 18

250

The sample mean () for this

distribution is 2.5, and the sample standard deviation (s) is 0.72.

Determine whether the student GPAs are normally distributed, using a .05 level of significance (i.e., $\alpha = .05$).

Geo-net, a cellular phone company, has collected the following frequency distribution for the length of calls outside its normal customer roaming area:

Length (min.)	Frequency
0 < 5	26
5 < 10	75
10 < 15	139

41.

15 < 20

105

20 < 25

37

25+

18

400

The sample mean () for this distribution is 14.3 minutes, and the sample standard deviation is 3.7 minutes. Determine whether these data are normally distributed ($\alpha = .05$).

Case Problem

VALLEY SWIM CLUB

The Valley Swim Club has 300 stockholders, each holding one share of stock in the club. A share of club stock allows the shareholder's family to use the club's heated outdoor pool during the summer, upon payment of annual membership dues of \$175. The club has not issued any new stock in years, and only a few of the existing

shares come up for sale each year. The board of directors administers the sale of all stock. When a shareholder wants to sell, he or she turns the stock in to the board, which sells it to the person at the top of the waiting list. For the past few years, the length of the waiting list has remained relatively steady, at approximately 20 names.

However, during the past winter, two events occurred that have increased the demand for shares in the club. The winter was especially

severe, and subzero weather and heavy ice storms caused both the town and the county pools to buckle and crack. The problems were not discovered until maintenance crews began to ready the pools for the summer, and repairs cannot be completed until the fall. Also during the winter, the manager of the local country club had an argument with her board of directors and one night burned down the clubhouse. Although the pool itself was not damaged, the dressing room facilities, showers, and snack

bar were destroyed. As a result of these two events, the Valley Swim Club was inundated with applications to purchase shares. The waiting list suddenly grew to 250 people as the summer approached.

The board of directors of the swim club had refrained from issuing new shares in the past because there was never a very great demand, and the demand that did exist was usually absorbed within a year by stock turnover. In addition, the board has a real concern about overcrowding. It seemed like

the present membership was about right, and there were very few complaints about overcrowding, except on holidays like Memorial Day and the Fourth of July. However, at a recent board meeting, a number of new applicants had attended and asked the board to issue new shares. In addition, a number of current shareholders suggested that this might be an opportunity for the club to raise some capital for needed repairs and to improve some of the existing facilities. This was tempting to

the board. Although it had set the share price at \$500 in the past, the board could set it at a much higher level now. In addition, any new shares sold would result in almost total profit because the manager, lifeguard, and maintenance costs had already been budgeted for the summer and would not increase with additional members.

Before the board of directors could make a decision on whether to sell more shares and, if so, how many, the board members felt they needed

more information. Specifically, they would like to know the average number of people (family members, guests, etc.) that might use the pool each day during the summer. They would also like to know the number of days they could expect more than 500 people to use the pool from June through August, given the current number of shares.

The board of directors has the following daily attendance records for June through August from the previous summer; it thinks the figures

would provide accurate
estimates for the upcoming
summer:

139	380	193	399	177	238
-----	-----	-----	-----	-----	-----

273	367	378	197	161	224
-----	-----	-----	-----	-----	-----

172	359	461	273	308	368
-----	-----	-----	-----	-----	-----

275	463	242	213	256	541
-----	-----	-----	-----	-----	-----

337	578	177	303	391	235
-----	-----	-----	-----	-----	-----

402	287	245	262	400	218
-----	-----	-----	-----	-----	-----

487	247	390	447	224	271
198	356	284	399	239	259
310	322	417	275	274	232
347	419	474	241	205	317
393	516	194	190	361	369
421	478	207	243	411	361
595	303	215	277	419	
497	223	304	241	258	

341 315 331 384 130

291 258 407 246 195

The board has developed the following criteria for making a decision on whether to issue new shares:

- 1.** The expected number of days on which attendance would exceed 500 should be no more than 5 with the current membership.

- 2.** The current average daily attendance should be no more than 320.
- 3.** The average daily weekend (Saturday and Sunday) attendance should be no more than 500. (Weekend attendance is every sixth and seventh entry in each progression of seven entries in the preceding data.)

If these criteria are met, the club will issue one new share, at a price of \$1,000, for every two average attendees between

the current daily average and an upper limit of 400.

Should the club issue new shares? If so, how many will it issue, and how much additional revenue will it realize?

Chapter 12. Decision Analysis

[Page 515]

In the previous chapters dealing with linear programming, models were formulated and solved in order to aid the manager in making a decision. The solutions to the models were represented by values for *decision* variables. However, these linear programming models were all formulated under the

assumption that certainty existed. In other words, it was assumed that all the model coefficients, constraint values, and solution values were known with certainty and did not vary.

In actual practice, however, many decision-making situations occur under conditions of *uncertainty*. For example, the demand for a product may be not 100 units next week, but 50 or 200 units, depending on the state of the market (which is uncertain). Several decision-making

techniques are available to aid the decision maker in dealing with this type of decision situation in which there is uncertainty.

Decision situations can be categorized into two classes: situations in which probabilities *cannot* be assigned to future occurrences and situations in which probabilities *can* be assigned. In this chapter we will discuss each of these classes of decision situations separately and demonstrate the decision-making criterion most commonly associated with

each. Decision situations in which there are two or more decision makers who are in competition with each other are the subject of *game theory*, a topic included on the CD that accompanies this text.

The two categories of decision situation are probabilities that can be assigned to future occurrences and probabilities that cannot be assigned.

Components of Decision Making

A decision-making situation includes several components: the decisions themselves *and* the actual events that may occur in the future, known as **states of nature**. At the time a decision is made, the decision maker is uncertain which states of nature will occur in the future and has no control over them.

A state of nature is

*an actual event that
may occur in the
future.*

Suppose a distribution company is considering purchasing a computer to increase the number of orders it can process and thus increase its business. If economic conditions remain good, the company will realize a large increase in profit; however, if the economy takes a downturn, the company will

lose money. In this decision situation, the possible decisions are to purchase the computer and to not purchase the computer. The states of nature are *good* economic conditions and *bad* economic conditions. The state of nature that occurs will determine the outcome of the decision, and it is obvious that the decision maker has no control over which state will occur.

As another example, consider a concessions vendor who must decide whether to stock coffee for the concession stands at a

football game in November. If the weather is cold, most of the coffee will be sold, but if the weather is warm, very little coffee will be sold. The decision is to order or not to order coffee, and the states of nature are warm and cold weather.

To facilitate the analysis of these types of decision situations so that the best decisions result, they are organized into **payoff tables**. In general, a payoff table is a means of organizing and illustrating the payoffs from the different decisions, given the

various states of nature in a decision problem. A payoff table is constructed as shown in [Table 12.1](#).

Table 12.1. Payoff table

Decision	State of Nature	
	<i>a</i>	<i>b</i>
1	Payoff 1 <i>a</i>	Payoff 1 <i>b</i>
2	Payoff 2 <i>a</i>	Payoff 2 <i>b</i>

*Using a **payoff table** is a means of organizing a decision situation, including the payoffs from different decisions, given the various states of nature.*

Each decision, 1 or 2, in [Table 12.1](#) will result in an outcome, or *payoff*, for the particular state of nature that will occur in the future. Payoffs are

typically expressed in terms of profit revenues, or cost (although they can be expressed in terms of a variety of quantities). For example, if decision 1 is to purchase a computer and state of nature a is good economic conditions, payoff $1a$ could be \$100,000 in profit.

[Page 516]

It is often possible to assign probabilities to the states of nature to aid the decision maker in selecting the decision

that has the best outcome. However, in some cases the decision maker is not able to assign probabilities, and it is this type of decision-making situation that we will address first.

Decision Making Without Probabilities

The following example will illustrate the development of a payoff table without probabilities. An investor is to purchase one of three types of real estate, as illustrated in [Figure 12.1](#). The investor must decide among an apartment building, an office building, and a warehouse. The future states of nature that will determine how much profit the investor

will make are good economic conditions and poor economic conditions. The profits that will result from each decision in the event of each state of nature are shown in [Table 12.2](#).

[Page 517]

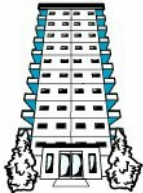
Figure 12.1. Decision situation with real estate investment alternatives

(This item is displayed on page 516 in the print version)

Office
building



Apartment
building



Warehouse



?



Table 12.2. Payoff table for the real estate investments

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS	POOR ECONOMIC CONDITIONS
Apartment building	\$50,000	\$30,000
Office building	100,000	40,000

Warehouse

30,000

10,000

Decision-Making Criteria

Once the decision situation has been organized into a payoff table, several criteria are available for making the actual decision. These decision criteria, which will be presented in this section, include maximax, maximin, minimax regret, Hurwicz, and equal

likelihood. On occasion these criteria will result in the same decision; however, often they will yield different decisions. The decision maker must select the criterion or combination of criteria that best suits his or her needs.

The Maximax Criterion

With the **maximax criterion**, the decision maker selects the decision that will result in the maximum of the maximum payoffs. (In fact, this is how

this criterion derives its name a maximum of a maximum.) The maximax criterion is very optimistic. The decision maker assumes that the most favorable state of nature for each decision alternative will occur. Thus, for example, using this criterion, the investor would optimistically assume that good economic conditions will prevail in the future.

*The **maximax criterion** results in the maximum of the maximum payoffs.*

The maximax criterion is applied in [Table 12.3](#). The decision maker first selects the maximum payoff for each decision. Notice that all three maximum payoffs occur under good economic conditions. Of the three maximum payoffs \$50,000, \$100,000, and \$30,000 the maximum is \$100,000; thus, the corresponding decision is to purchase the office building.

Table 12.3. Payoff table illustrating a maximax decision

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS	POOR ECONOMIC CONDITIONS
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

Maximum payoff



Although the decision to

purchase an office building will result in the largest payoff (\$100,000), such a decision completely ignores the possibility of a potential loss of \$40,000. The decision maker who uses the maximax criterion assumes a very optimistic future with respect to the state of nature.

Management Science

Application: Decision

Analysis at DuPont

DuPont has used decision analysis extensively since the mid-1960s to create, evaluate, and implement strategic alternatives within 10 of its businesses, each worldwide in scope, with over \$150 million in annual sales. As an example, one of these DuPont businesses was experiencing declining financial performance due to eroding prices and loss of market share to European and Japanese competitors. The business identified three possible strategies to evaluate from a number of alternatives. These strategies were (1) to continue with the current strategy, (2) to reestablish product leadership by

strengthening new product development efforts and product differentiation, and (3) to establish a low-cost market position by closing a plant and streamlining the product line to improve production efficiency. Uncertainties in the evaluation process derived from competitor strategies and from market size, share, and prices. The decision criterion was net present value for each of the three strategies, calculated using a Super Tree spreadsheet model. The ultimate decision was to strengthen product development and product differentiation. The selected strategy resulted in an approximate increase in value of \$175 million.



Source: F. Krumm and C. Rolle,
"Management and Application of
Decision and Risk Analysis in DuPont,"
Interfaces 22, no. 6
(November/December 1992): 8493.

Before the next criterion is presented, it should be pointed out that the maximax decision rule as presented here deals with *profit*. However, if the payoff table consisted of costs, the opposite selection would be indicated: the minimum of the minimum costs, or a *minimin* criterion. For the subsequent decision criteria we encounter, the same logic in the case of

costs can be used.

The Maximin Criterion

In contrast to the maximax criterion, which is very optimistic, the **maximin criterion** is pessimistic. With the maximin criterion, the decision maker selects the decision that will reflect the *maximum* of the *minimum* payoffs. For each decision alternative, the decision maker assumes that the minimum payoff will occur. Of these

minimum payoffs, the maximum is selected. The maximin criterion for our investment example is demonstrated in [Table 12.4](#).

Table 12.4. Payoff table illustrating a maximin decision

[\[View full size image\]](#)

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS	POOR ECONOMIC CONDITIONS
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

Maximum payoff

*The **maximin criterion** results in the maximum of the minimum payoff.*

The minimum payoffs for our example are \$30,000, \$40,000, and \$10,000. The maximum of these three payoffs is \$30,000; thus, the decision arrived at by using the maximin criterion would be to purchase the apartment building. This decision is relatively conservative because the alternatives considered include only the worst outcomes that could occur. The decision to purchase the office building as determined by the maximax criterion includes the possibility

of a large loss (\$40,000). The worst that can occur from the decision to purchase the apartment building, however, is *a gain of \$30,000*. On the other hand, the largest possible gain from purchasing the apartment building is much less than that of purchasing the office building (i.e., \$50,000 vs. \$100,000).

If [Table 12.4](#) contained costs instead of profits as the payoffs, the conservative approach would be to select the maximum cost for each decision. Then the decision that

resulted in the minimum of these costs would be selected.

The Minimax Regret Criterion

In our example, suppose the investor decided to purchase the warehouse, only to discover that economic conditions in the future were better than expected. Naturally, the investor would be disappointed that she had not purchased the office building because it would have resulted in the largest

payoff (\$100,000) under good economic conditions. In fact, the investor would *regret* the decision to purchase the warehouse, and the *degree of regret* would be \$70,000, the difference between the payoff for the investor's choice and the best choice.

Regret is the difference between the payoff from the best decision and all other decision payoffs.

This brief example demonstrates the principle underlying the decision criterion known as **minimax regret criterion**. With this decision criterion, the decision maker attempts to avoid regret by selecting the decision alternative that minimizes the maximum regret.

*The **minimax regret criterion** minimizes the maximum regret.*

To use the minimax regret criterion, a decision maker first selects the maximum payoff under each state of nature. For our example, the maximum payoff under good economic conditions is \$100,000, and the maximum payoff under poor economic conditions is \$30,000. All other payoffs under the respective states of nature are subtracted from these amounts, as follows:

Good Economic

Poor Economic

Conditions

$$\begin{array}{rcl} \$100,000 & = & \\ 50,000 & \$50,000 & \end{array}$$

Conditions

$$\begin{array}{rcl} \$30,000 & = & \$0 \\ 30,000 & & \end{array}$$

$$\begin{array}{rcl} \$100,000 & & \\ 100,000 & = \$0 & \end{array}$$

$$\begin{array}{rcl} \$30,000 & = & \\ (40,000) & \$70,000 & \end{array}$$

$$\begin{array}{rcl} \$100,000 & = & \\ 30,000 & \$70,000 & \end{array}$$

$$\begin{array}{rcl} \$30,000 & = & \\ 10,000 & \$20,000 & \end{array}$$

These values represent the regret that the decision maker would experience if a decision

were made that resulted in less than the maximum payoff. The values are summarized in a modified version of the payoff table known as a *regret table*, shown in [Table 12.5](#). (Such a table is sometimes referred to as an *opportunity loss table*, in which case the term *opportunity loss* is synonymous with *regret*.)

Table 12.5. Regret table

State of Nature		
Decision (Purchase)	GOOD ECONOMIC	POOR ECONOMIC

	CONDITIONS	CONDITIONS
Apartment building	\$50,000	\$ 0
Office building	0	70,000
Warehouse	70,000	20,000

[Page 520]

To make the decision according to the minimax regret criterion, the maximum regret for *each decision* must be determined.

The decision corresponding to the minimum of these regret values is then selected. This process is illustrated in [Table 12.6](#).

Table 12.6. Regret table illustrating the minimax regret decision

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS	POOR ECONOMIC CONDITIONS
Apartment building	\$50,000	\$ 0
Office building	0	70,000
Warehouse	70,000	20,000

The minimum regret value 

According to the minimax regret criterion, the decision should be to purchase the apartment building rather than the office building or the warehouse. This particular decision is based on the

philosophy that the investor will experience the least amount of regret by purchasing the apartment building. In other words, if the investor purchased either the office building or the warehouse, \$70,000 worth of regret could result; however, the purchase of the apartment building will result in, at most, \$50,000 in regret.

The Hurwicz Criterion

The **Hurwicz criterion** strikes

a compromise between the maximax and maximin criteria. The principle underlying this decision criterion is that the decision maker is neither totally optimistic (as the maximax criterion assumes) nor totally pessimistic (as the maximin criterion assumes). With the Hurwicz criterion, the decision payoffs are weighted by a **coefficient of optimism**, a measure of the decision maker's optimism. The coefficient of optimism, which we will define as α , is between zero and one (i.e., $0 \leq \alpha \leq 1.0$).

If $\alpha = 1.0$, then the decision maker is said to be completely optimistic; if $\alpha = 0$, then the decision maker is completely pessimistic. (Given this definition, if α is the coefficient of optimism, $1 - \alpha$ is the *coefficient of pessimism*.)

*The **Hurwicz criterion** is a compromise between the maximax and maximin criteria.*

*The **coefficient of optimism**, α , is a measure of the decision maker's optimism.*

The Hurwicz criterion requires that, for each decision alternative, the maximum payoff be multiplied by α and the minimum payoff be multiplied by $1 - \alpha$. For our investment example, if α equals .4 (i.e., the investor is slightly

pessimistic), $1 - \alpha = .6$, and the following values will result:

Decision

Values

Apartment building \$ 50,000(.4) +
30,000(.6) = \$38,000

Office building \$100,000(.4) -
40,000(.6) = \$16,000

Warehouse \$30,000(.4) +
10,000(.6) = \$18,000

*The **Hurwicz criterion** multiplies the best payoff by α , the coefficient of optimism, and the worst payoff by $1 - \alpha$, for each decision, and the best result is selected.*

The Hurwicz criterion specifies selection of the decision alternative corresponding to the maximum weighted value,

which is \$38,000 for this example. Thus, the decision would be to purchase the apartment building.

It should be pointed out that when $\alpha = 0$, the Hurwicz criterion is actually the maximin criterion; when $\alpha = 1.0$, it is the maximax criterion. A limitation of the Hurwicz criterion is the fact that α must be determined by the decision maker. It can be quite difficult for a decision maker to accurately determine his or her degree of optimism. Regardless of how the decision maker

determines α , it is still a completely *subjective* measure of the decision maker's degree of optimism. Therefore, the Hurwicz criterion is a completely subjective decision-making criterion.

[Page 521]

The Equal Likelihood Criterion

When the maximax criterion is applied to a decision situation, the decision maker implicitly

assumes that the most favorable state of nature for each decision will occur. Alternatively, when the maximin criterion is applied, the least favorable states of nature are assumed. The equal likelihood, or LaPlace, criterion weights each state of nature equally, thus assuming that the states of nature are equally likely to occur.

*The **equal likelihood criterion** multiplies the decision payoff for*

*each state of nature
by an equal weight.*

Because there are two states of nature in our example, we assign a weight of .50 to each one. Next, we multiply these weights by each payoff for each decision:

Decision

Values

\$ 50,000(.50) +

Apartment building $30,000(.50) =$
\$40,000

Office building $\$100,000(.50) -$
40,000(.50) =
\$30,000

Warehouse $\$ 30,000(.50) +$
10,000(.50) =
\$20,000

As with the Hurwicz criterion,
we select the decision that has
the maximum of these
weighted values. Because
\$40,000 is the highest

weighted value, the investor's decision would be to purchase the apartment building.

In applying the equal likelihood criterion, we are assuming a 50% chance, or .50 probability, that either state of nature will occur. Using this same basic logic, it is possible to weight the states of nature differently (i.e., unequally) in many decision problems. In other words, different probabilities can be assigned to each state of nature, indicating that one state is more likely to occur than another. The application of

different probabilities to the states of nature is the principle behind the decision criteria to be presented in the section on expected value.

Summary of Criteria Results

The decisions indicated by the decision criteria examined so far can be summarized as follows:

Criterion

***Decision
(Purchase)***

Maximax	Office building
Maximin	Apartment building
Minimax regret	Apartment building
Hurwicz	Apartment building
Equal likelihood	Apartment building

The decision to purchase the apartment building was

designated most often by the various decision criteria. Notice that the decision to purchase the warehouse was never indicated by any criterion. This is because the payoffs for an apartment building, under either set of future economic conditions, are always better than the payoffs for a warehouse. Thus, given any situation with these two alternatives (and any other choice, such as purchasing the office building), the decision to purchase an apartment building will always be made over the

decision to purchase a warehouse. In fact, the warehouse decision alternative could have been eliminated from consideration under each of our criteria. The alternative of purchasing a warehouse is said to be *dominated* by the alternative of purchasing an apartment building. In general, dominated decision alternatives can be removed from the payoff table and not considered when the various decision-making criteria are applied. This reduces the complexity of the decision analysis

somewhat. However, in our discussions throughout this chapter of the application of decision criteria, we will leave the dominated alternative in the payoff table for demonstration purposes.

*A **dominant** decision is one that has a better payoff than another decision under each state of nature.*

The use of several decision criteria often results in a mix of decisions, with no one decision being selected more than the others. The criterion or collection of criteria used and the resulting decision depend on the characteristics and philosophy of the decision maker. For example, the extremely optimistic decision maker might eschew the majority of the foregoing results and make the decision to purchase the office building

because the maximax criterion most closely reflects his or her personal decision-making philosophy.

The appropriate criterion is dependent on the risk personality and philosophy of the decision maker.

Solution of Decision-

Making Problems Without Probabilities with QM for Windows

QM for Windows includes a module to solve decision analysis problems. QM for Windows will be used to illustrate the use of the maximax, maximin, minimax regret, equal likelihood, and Hurwicz criteria for the real estate problem considered in this section. The problem data are input very easily. A summary of the input and

solution output for the maximax, maximin, and Hurwicz criteria is shown in [Exhibit 12.1](#). The decision with the equal likelihood criterion can be determined by using an alpha value for the Hurwicz criterion equal to the equal likelihood weight, which is .5 for our real estate investment example. The solution output with alpha equal to .5 is shown in [Exhibit 12.2](#). The decision with the minimax regret criterion is shown in [Exhibit 12.3](#).

Exhibit 12.1.

[\[View full size image\]](#)

Utilities

☒ Pools (seasonal)
 ☐ Cuts (seasonal)

Homeowner type

☐ ☐ ☐ ☐

Instructions

There are more results available in additional windows. These may be opened by using the WINDOW option in the Main Menu.

PP Decision Table Results

Real Estate Investment Example Solution

	Cost	Size	Row Min	Row Max	Expected
Investment	0	0			
Apartment Building	50,000	30,000	50,000	50,000	50,000
Office Building	100,000	40,000	100,000	100,000	100,000
Warehouse	30,000	10,000	30,000	30,000	30,000
		medium	30,000	100,000	30,000
			30,000	100,000	30,000

The maximum is 30,000 given by Apartment Building

The maximum is 100,000 given by Office Building

Exhibit 12.2.

[\[View full size image\]](#)

Equal likelihood weight

Objective

☒ Profit (maximize)
☐ Cost (minimize)

Decision Table

Instructions

There are some results available in additional windows. These may be opened by using the "SHOW" option in the Main Menu.

Decision Table Results

Real Estate Investment Decision Solution

	Good	Poor	Row Min	Row Max	Row Avg
Probabilities	0.5	0.5			
Apartment Building	50,000	30,000	30,000	50,000	40,000
Office Building	100,000	-40,000	-40,000	100,000	30,000
Warehouse	30,000	10,000	10,000	30,000	20,000
		Expected	30,000	100,000	40,000
			800,000	800,000	800,000

The maximum is 30,000 given by Apartment Building

The maximum is 100,000 given by Office Building

Exhibit 12.3.

\$ Regret or Opportunity Loss



Real Estate Investment Example Solution

	Good Regret	Poor Regret	Maximum Regret	Expected Regret
Probabilities	0	0		
Apartment Building	50,000	0	50,000	0
Office Building	0	70,000	70,000	0
Warehouse	70,000	20,000	70,000	0
Minimax regret			50,000	

[Page 523]

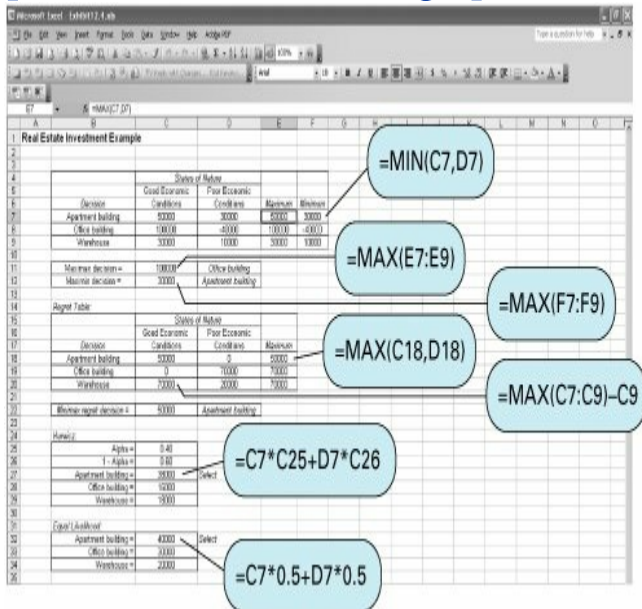
Solution of Decision-Making Problems

Without Probabilities with Excel

Excel can also be used to solve decision analysis problems using the decision-making criteria presented in this section. [Exhibit 12.4](#) illustrates the application of the maximax, minimax, minimax regret, Hurwicz, and equal likelihood criteria for our real estate investment example.

Exhibit 12.4.

[View full size image]



In cell E7 the formula **=MAX(C7,D7)** selects the maximum payoff outcome for the decision to purchase the apartment building. Next, in cell C11 the maximum of the maximum payoffs is determined with the formula **=MAX(E7:E9)**. The maximin decision is determined similarly.

In the regret table in [Exhibit 12.4](#), in cell C18 the formula **=MAX(C7:C9) - C7** computes the regret for the apartment building decision under good economic conditions, and then

the maximum regret for the apartment building is determined in cell E18, using the formula **=MAX(C18,D18)**. The minimax regret value is determined in cell C22 with the formula **=MIN(E18:E20)**.

The Hurwicz and equal likelihood decisions are determined using their respective formulas in cells **C27:C29** and **C32:C34**.

Decision Making with Probabilities

The decision-making criteria just presented were based on the assumption that no information regarding the likelihood of the states of nature was available. Thus, no *probabilities of occurrence* were assigned to the states of nature, except in the case of the equal likelihood criterion. In that case, by assuming that each state of nature was

equally likely and assigning a weight of .50 to each state of nature in our example, we were implicitly assigning a probability of .50 to the occurrence of each state of nature.

It is often possible for the decision maker to know enough about the future states of nature to assign probabilities to their occurrence. Given that probabilities can be assigned, several decision criteria are available to aid the decision maker. We will consider two of these criteria: *expected value*

and *expected opportunity loss* (although several others, including the *maximum likelihood criterion*, are available).

[Page 524]

Expected Value

To apply the concept of **expected value** as a decision-making criterion, the decision maker must first estimate the probability of occurrence of each state of nature. Once

these estimates have been made, the expected value for each decision alternative can be computed. The expected value is computed by multiplying each outcome (of a decision) by the probability of its occurrence and then summing these products. The expected value of a random variable x , written symbolically as $E(x)$, is computed as follows:

$$E(x) = \sum_{i=1}^n x_i P(x_i)$$

where

n = number of values of the random variable x

Expected value is computed by multiplying each decision outcome under each state of nature by the probability of its occurrence.

Using our real estate investment example, let us

suppose that, based on several economic forecasts, the investor is able to estimate a .60 probability that good economic conditions will prevail and a .40 probability that poor economic conditions will prevail. This new information is shown in [Table 12.7](#).

Table 12.7. Payoff table with probabilities for states of nature

Decision	State of Nature	
	GOOD ECONOMIC	POOR ECONOMIC

(Purchase)	CONDITIONS .60	CONDITIONS .40
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

The expected value (*EV*) for each decision is computed as follows:

$$EV(\text{apartment}) = \$50,000(.60) + 30,000(.40) = \$42,000$$

$$EV(\text{office}) = \$100,000(.60) + 40,000(.40) = \$44,000$$

$$EV(\text{warehouse}) = \$30,000(.60) + 10,000(.40) = \$22,000$$

The best decision is the one with the greatest expected value. Because the greatest expected value is \$44,000, the best decision is to purchase the office building. This does not

mean that \$44,000 will result if the investor purchases the office building; rather, it is assumed that one of the payoff values will result (either \$100,000 or \$40,000). The expected value means that if this decision situation occurred a large number of times, an *average* payoff of \$44,000 would result. Alternatively, if the payoffs were in terms of costs, the best decision would be the one with the lowest expected value.

Expected Opportunity

Loss

A decision criterion closely related to expected value is **expected opportunity loss**.

To use this criterion, we multiply the probabilities by the regret (i.e., opportunity loss) for each decision outcome rather than multiplying the decision outcomes by the probabilities of their occurrence, as we did for expected monetary value.

Expected opportunity loss is

the expected value of the regret for each decision.

[Page 525]

The concept of regret was introduced in our discussion of the minimax regret criterion. The regret values for each decision outcome in our example were shown in [Table 12.6](#). These values are repeated in [Table 12.8](#), with the

addition of the probabilities of occurrence for each state of nature.

**Table 12.8. Regret
(opportunity loss) table
with probabilities for states
of nature**

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS .60	POOR ECONOMIC CONDITIONS .40
Apartment building	\$50,000	\$ 0

Office building	0	70,000
Warehouse	70,000	20,000

The expected opportunity loss (*EOL*) for each decision is computed as follows:

$$EOL(\text{apartment}) = \$50,000(.60) + 0(.40) = \$30,000$$

$$EOL(\text{office}) = \$0(.60) + 70,000(.40) = \$28,000$$

$$EOL(\text{warehouse}) = \$70,000(.60) + 20,000(.40) = \$50,000$$

As with the minimax regret criterion, the best decision results from minimizing the regret, or, in this case, minimizing the *expected* regret or opportunity loss. Because \$28,000 is the minimum expected regret, the decision is to purchase the office building.

The expected value

*and expected
opportunity loss
criteria result in the
same decision.*

Notice that the decisions recommended by the expected value and expected opportunity loss criteria were the same to purchase the office building. This is not a coincidence because these two methods always result in the same decision. Thus, it is repetitious

to apply both methods to a decision situation when one of the two will suffice.

In addition, note that the decisions from the expected value and expected opportunity loss criteria are totally dependent on the probability estimates determined by the decision maker. Thus, if inaccurate probabilities are used, erroneous decisions will result. It is therefore important that the decision maker be as accurate as possible in determining the probability of each state of nature.

Solution of Expected Value Problems with QM for Windows

QM for Windows not only solves decision analysis problems without probabilities but also has the capability to solve problems using the expected value criterion. A summary of the input data and the solution output for our real estate example is shown in [Exhibit 12.5](#). Notice that the expected value results are included in the third column of this

Solution of Expected Value Problems with Excel and Excel QM

This type of expected value problem can also be solved by using an Excel spreadsheet. [Exhibit 12.6](#) shows our real estate investment example set up in a spreadsheet format. Cells E7, E8, and E9 include

the expected value formulas for this example. The expected value formula for the first decision, purchasing the apartment building, is embedded in cell E7 and is shown on the formula bar at the top of the spreadsheet.

Exhibit 12.6.

[\[View full size image\]](#)

Microsoft Excel - EdBRI72.4.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

100%

And

E7 =G7C7+D7D7

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Real Estate Investment Example													
2			States of Nature											
3			Good	Poor										
4			Conditions	Conditions	Expected									
5					Value									
6	Decision		0.80	0.40										
7	Apartment building	50000	30000		42000									
8	Office building	100000	-40000		44000									
9	Warehouse	30000	10000		22000									
10														

Expected value for apartment building

Excel QM is a set of spreadsheet macros that is included on the CD that accompanies this text, and it has a macro to solve decision analysis problems. Once activated, Excel QM is accessed by clicking on "QM" on the

menu bar at the top of the spreadsheet. Clicking on "Decision Analysis" will result in a Spreadsheet Initialization window. After entering several problem parameters, including the number of decisions and states of nature, and then clicking on "OK," the spreadsheet shown in [Exhibit 12.7](#) will result. Initially, this spreadsheet contains example values in cells **B8:C11**. [Exhibit 12.7](#) shows the spreadsheet with our problem data already typed in. The results are computed automatically as the

data are entered, using the cell formulas already embedded in the macro.

Exhibit 12.7.

[\[View full size image\]](#)

Click on "QM" to access the "Excel QM" menu.

Microsoft Excel - 1648112.7.xls

File Edit View Insert Format Tools Data QM Window Help 444/700

Type a question for help

100%

Formulas with Comments - Not Viewable

444

59 =SUMPRODUCT(B9:B10,C9:D9)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1	Real Estate Investment Example													
2														
3	Decision Tables													
4	Enter the profits in the main body of the data table. Enter probabilities in the first row if you want to compute the													
5	expected value.													
6	Data			Results										
7	Profit	State of Nature 1	State of Nature 2		EMV	Minimum	Maximum							
8	Probability	0.6	0.4											
9	Decision 1	50000	30000		42000	30000	50000							
10	Decision 2	100000	-40000		40000	-40000	100000							
11	Decision 3	30000	10000		22000	10000	30000							
12				Maximum	44000	30000	100000							
13														
14	Expected Value of Perfect Information													
15	Column best	100000	30000		72000	<Expected value under certainty								
16					44000	<Best expected value								
17					28000	<Expected value of perfect information								
18														
19	Regret													
20		State of Nature 1	State of Nature 2		Expected	Maximum								
21	Probability	0.6	0.4											
22	Decision 1	50000	0		30000	50000								
23	Decision 2	0	70000		28000	70000								
24	Decision 3	70000	20000		50000	70000								
25				Minimum	28000	50000								
26														

Expected Value of Perfect Information

It is often possible to purchase additional information regarding future events and thus make a better decision. For example, a real estate investor could hire an economic forecaster to perform an analysis of the economy to more accurately determine which economic condition will occur in the future. However,

the investor (or any decision maker) would be foolish to pay more for this information than he or she stands to gain in extra profit from having the information. That is, the information has some maximum value that represents the limit of what the decision maker would be willing to spend. This value of information can be computed as an expected value hence its name, the **expected value of perfect information** (also referred to as **EVPI**).

*The **expected value of perfect information** is the maximum amount a decision maker would pay for additional information.*

To compute the expected value of perfect information, we first look at the decisions under each state of nature. If we could obtain information that assured us which state of

nature was going to occur (i.e., perfect information), we could select the best decision for that state of nature. For example, in our real estate investment example, if we know for sure that good economic conditions will prevail, then we will decide to purchase the office building. Similarly, if we know for sure that poor economic conditions will occur, then we will decide to purchase the apartment building. These hypothetical "perfect" decisions are summarized in [Table 12.9](#).

Table 12.9. Pavoff table with

decisions, given perfect information

State of Nature

Decision (Purchase)	GOOD ECONOMIC CONDITIONS .60	POOR ECONOMIC CONDITIONS .40
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

The probabilities of each state of nature (i.e., .60 and .40) tell us that good economic conditions will prevail 60% of the time and poor economic conditions will prevail 40% of the time (if this decision situation is repeated many times). In other words, even though perfect information enables the investor to make the right decision, each state of nature will occur only a certain portion of the time. Thus, each of the decision outcomes

obtained using perfect information must be weighted by its respective probability:

$$\begin{aligned} & \$100,000(.60) + 30,000(.40) \\ & = \$72,000 \end{aligned}$$

The amount \$72,000 is the expected value of the decision, *given* perfect information, not the expected value *of* perfect information. The expected value of perfect information is the maximum amount that would be paid to gain information that would result in a decision better than the one made without perfect

information. Recall that the expected value decision without perfect information was to purchase an office building, and the expected value was computed as

$$\begin{aligned}EV(\text{office}) &= \$100,000(.60) \\ 40,000(.40) &= \$44,000\end{aligned}$$

The expected value of perfect information is computed by subtracting the expected value without perfect information (\$44,000) from the expected value given perfect information (\$72,000):

$$EVPI = \$72,000 - \$44,000 = \$28,000$$

EVPI equals the expected value, given perfect information, minus the expected value without perfect information.

[Page 528]

The expected value of perfect information, \$28,000, is the

maximum amount that the investor would pay to purchase perfect information from some other source, such as an economic forecaster. Of course, perfect information is rare and usually unobtainable. Typically, the decision maker would be willing to pay some amount less than \$28,000, depending on how accurate (i.e., close to perfection) the decision maker believes the information is.

It is interesting to note that the expected value of perfect information, \$28,000 for our example, is the same as the

expected opportunity loss (EOL)
for the decision selected, using
this later criterion:

$$EOL(\text{office}) = \$0(.60) + 70,000(.40) = \$28,000$$

*The expected value
of perfect information
equals the expected
opportunity loss for
the best decision.*

This will always be the case,

and logically so, because regret reflects the *difference between the best decision under a state of nature and the decision actually made*. This is actually the same thing determined by the expected value of perfect information.

Excel QM for decision analysis computes the expected value of perfect information, as shown in cell E17 at the bottom of the spreadsheet in [Exhibit 12.7](#).

The expected value of perfect information can also be determined by using Excel.

[Exhibit 12.8](#) shows the *EVPI* for

our real estate investment example.

Exhibit 12.8.

[\[View full size image\]](#)

The expected value, given perfect information, in Cell F12

The screenshot shows an Excel spreadsheet titled "Real Estate Investment Example". The spreadsheet is organized as follows:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	
1	Real Estate Investment Example														
2			States of Nature												
3			Good	Poor											
4			Conditions	Conditions	Expected										
5															
6	Decision		0.60	0.40	Value										
7	Apartment building		50000	30000	42000										
8	Office building		100000	-40000	44000										
9	Warehouse		30000	10000	22000										
10															
11			Expected value without perfect information =			44000									
12			Expected value given perfect information =			72000									
13			Expected value of perfect information =			28000									
14															

A callout box highlights the formula **=MAX(E7:E9)** used to calculate the expected value without perfect information.

=MAX(E7:E9)

$$=F_{12}-F_{11}$$

Decision Trees

Another useful technique for analyzing a decision situation is using a **decision tree**. A decision tree is a graphical diagram consisting of nodes and branches. In a decision tree the user computes the expected value of each outcome and makes a decision based on these expected values. The primary benefit of a decision tree is that it provides an illustration (or picture) of the decision-making process. This makes it easier to correctly compute the necessary expected values and

to understand the process of making the decision.

*A **decision tree** is a diagram consisting of square decision nodes, circle probability nodes, and branches representing decision alternatives.*

We will use our example of the real estate investor to

demonstrate the fundamentals of decision tree analysis. The various decisions, probabilities, and outcomes of this example, initially presented in [Table 12.7](#), are repeated in [Table 12.10](#). The decision tree for this example is shown in [Figure 12.2](#).

[Page 529]

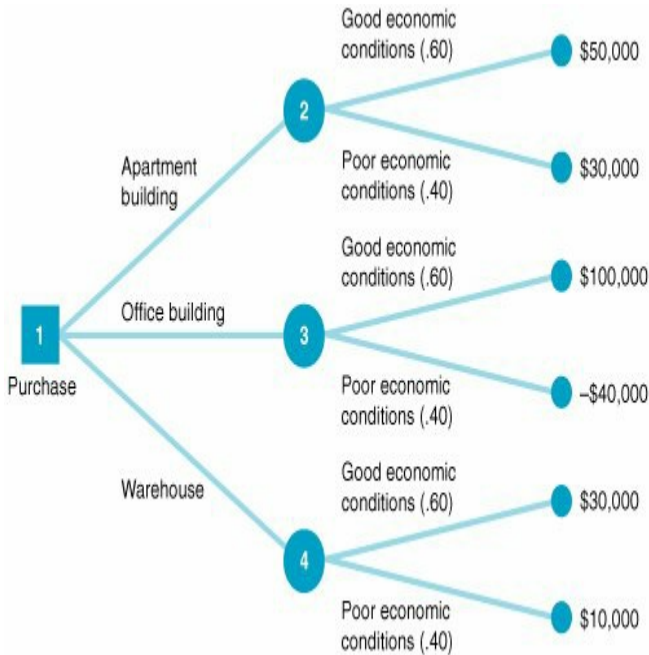
Table 12.10. Payoff table for real estate investment example

State of Nature

Decision (Purchase)	GOOD ECONOMIC CONDITIONS .60	POOR ECONOMIC CONDITIONS .40
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	-40,000
Warehouse	30,000	10,000

Figure 12.2. Decision

tree for real estate investment example



The circles (•) and the square (

■) in [Figure 12.2](#) are referred to as *nodes*. The square is a decision node, and the *branches* emanating from a decision node reflect the alternative decisions possible at that point. For example, in [Figure 12.2](#), node 1 signifies a decision to purchase an apartment building, an office building, or a warehouse. The circles are probability, or event, nodes, and the branches emanating from them indicate the states of nature that can occur: good economic conditions or poor economic

conditions.

The decision tree represents the sequence of events in a decision situation. First, one of the three decision choices is selected at node 1. Depending on the branch selected, the decision maker arrives at probability node 2, 3, or 4, where one of the states of nature will prevail, resulting in one of six possible payoffs.

The expected value is computed at each probability node.

Determining the best decision by using a decision tree involves computing the expected value at each probability node. This is accomplished by starting with the final outcomes (payoffs) and working backward through the decision tree toward node 1. First, the expected value of the payoffs is computed at each probability node:

$$EV(\text{node 2}) = .60(\$50,000) + .40(\$30,000) = \$42,000$$

$$EV(\text{node 3}) = .60(\$100,000) + .40(\$40,000) = \$44,000$$

$$EV(\text{node 4}) = .60(\$30,000) + .40(\$10,000) = \$22,000$$

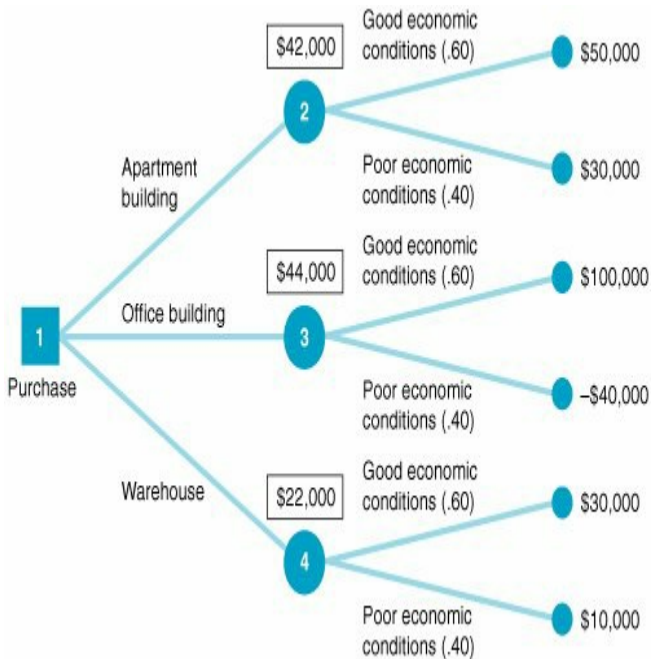
Branches with the greatest expected value are selected.

These values are now shown as the *expected* payoffs from each of the three branches emanating from node 1 in

[Figure 12.3](#). Each of these three expected values at nodes 2, 3, and 4 is the outcome of a possible decision that can occur at node 1. Moving toward node 1, we select the branch that comes from the probability node with the highest expected payoff. In [Figure 12.3](#), the branch corresponding to the highest payoff, \$44,000, is from node 1 to node 3. This branch represents the decision to purchase the office building. The decision to purchase the office building, with an expected payoff of \$44,000, is

the same result we achieved earlier by using the expected value criterion. In fact, when only one decision is to be made (i.e., there is not a series of decisions), the decision tree will always yield the same decision and expected payoff as the expected value criterion. As a result, in these decision situations a decision tree is not very useful. However, when a sequence or series of decisions is required, a decision tree can be very useful.

Figure 12.3. Decision tree with expected value at probability nodes



Decision Trees with QM for Windows

In QM for Windows, when the "Decision Analysis" module is accessed and you click on "New" to input a new problem, a menu appears that allows you to select either "Decision Tables" or "Decision Trees."

[Exhibit 12.9](#) shows the Decision Tree Results solution screen for our real estate investment example. Notice that QM for Windows requires that all nodes be numbered, including

the end, or terminal, nodes, which are shown as solid dots in [Figure 12.3](#). For QM for Windows, we have numbered these end nodes 5 through 10.

[Page 531]

Exhibit 12.9.

(This item is displayed on page 530 in the print version)

[\[View full size image\]](#)

Real Estate Investment Example Solution								
	Start Node	End Node	Branch Probability	Profit	Branch Use	End node	Node Type	Node Value
Start	0	1	0	0		1	Decision	44,000.0
Purchase apartment	1	2	0	0		2	Chance	42,000.0
Purchase office	1	3	0	0	Always	3	Chance	44,000.0
Purchase warehouse	1	4	0	0		4	Chance	22,000
Good conditions	2	5	.6	50,000		5	Final	50,000
Poor conditions	2	6	.4	30,000		6	Final	30,000
Good conditions	3	7	.6	100,000		7	Final	100,000
Poor conditions	3	8	.4	-40,000		8	Final	-40,000
Good conditions	4	9	.6	30,000		9	Final	30,000
Poor conditions	4	10	.4	10,000		10	Final	10,000

For the branch used columns the meanings are as follows:
Always - these are branches that should always be included
Possibly - these are branches that should be included if you get there and you **MIGHT** get there.
Backwards - these are branches that should be used if you get there **BUT** you should **NOT** get there.
 (They are used for the backwards pass from right to left).

Decision Trees with Excel and TreePlan

TreePlan is an Excel add-in program developed by Michael Middleton to construct and solve decision trees in an Excel spreadsheet format. Although Excel has the graphical and computational capability to develop decision trees, it is a difficult and slow process.

TreePlan is basically a decision tree template that greatly simplifies the process of setting up a decision tree in Excel.

The first step in using TreePlan is to gain access to it. The best way to go about this is to copy the TreePlan add-in file,

TreePlan.xla, from the CD accompanying this text onto your hard drive and then add it to the "Tools/Add-Ins" menu that you access at the top of your spreadsheet screen. Once you have added TreePlan to the "Tools" menu, you can invoke it by clicking on the "Decision Trees" menu item.

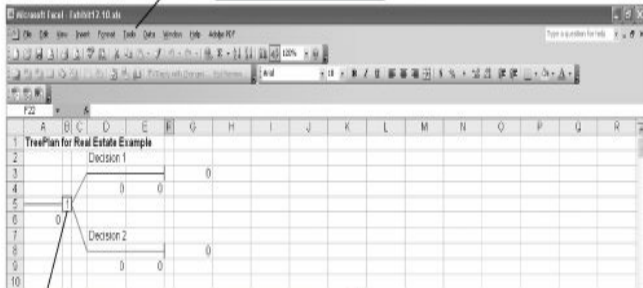
We will demonstrate how to use TreePlan with our real estate investment example shown in [Figure 12.3](#). The first step in using TreePlan is to generate a new tree on which to begin work. [Exhibit 12.10](#) shows a

new tree that we generated by positioning the cursor on cell B1 and then invoking the "Tools" menu and clicking on "Decision Trees." This results in a menu from which we click on "New Tree," which creates the decision tree shown in [Exhibit 12.10](#).

Exhibit 12.10.

[\[View full size image\]](#)

Invoke TreePlan from the "Tools" menu.



To create another branch, click B5, then the "Tools" menu, then "TreePlan," and select "Add a Branch" from the menu.

The decision tree in [Exhibit 12.10](#) uses the normal nodal convention we used in creating

the decision trees in [Figures 12.2](#) and [12.3](#) squares for decision nodes and circles for probability nodes (which TreePlan calls *event nodes*). However, this decision tree is only a starting point or template that we need to expand to replicate our example decision tree in [Figure 12.3](#).

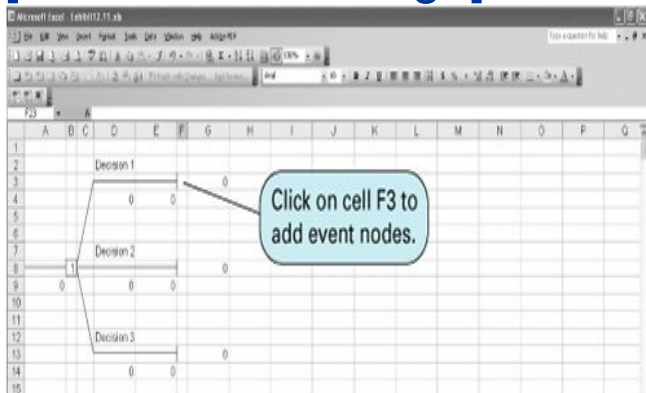
In [Figure 12.3](#), three branches emanate from the first decision node, reflecting the three investment decisions in our example. To create a third branch using TreePlan, click on

the decision node in cell B5 in [Exhibit 12.10](#) and then invoke TreePlan from the "Tools" menu. A window will appear, with several menu items, including "Add Branch." Select this menu item and click on "OK." This will create a third branch on our decision tree, as shown in [Exhibit 12.11](#).

[Page 532]

Exhibit 12.11.

[\[View full size image\]](#)



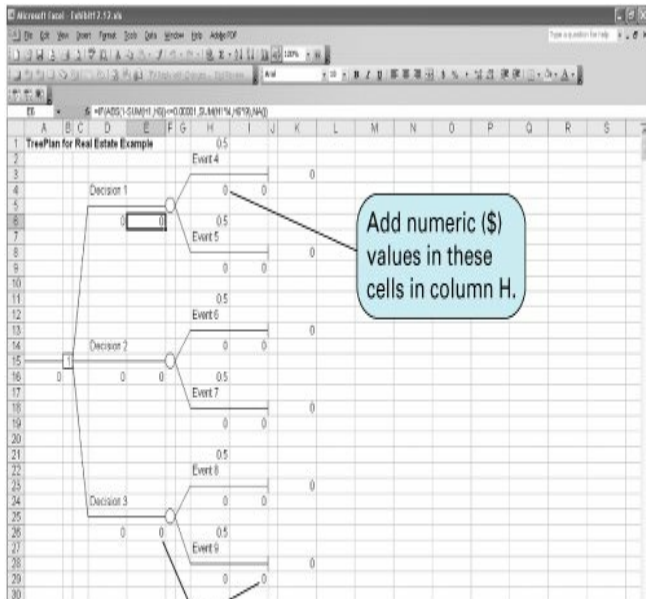
Next we need to expand our decision tree in [Exhibit 12.11](#) by adding probability nodes (2, 3, and 4 in [Figure 12.3](#)) and branches from these nodes for

our example. To add a new node, click on the end node in cell F3 in [Exhibit 12.11](#) and then invoke TreePlan from the "Tools" menu. From the menu window that appears, select "Change to Event Node" and then select "Two Branches" from the same menu and click on "OK." This process must be repeated two more times for the other two end nodes (in cells F8 and F13) to create our three probability nodes. The resulting decision tree is shown in [Exhibit 12.12](#), with the new probability nodes at cells F5,

F15, and F25 and with accompanying branches.

Exhibit 12.12.

[\[View full size image\]](#)



Add numeric (\$) values in these cells in column H.

These cells contain decision tree formulas; do not type in these cells in columns E and I.

The next step is to edit the decision tree labels and add the numeric data from our example. Generic descriptive labels are shown above each branch in [Exhibit 12.12](#) for example, "Decision 1" in cell D4 and "Event 4" in cell H2. We edit the labels the same way we would edit any spreadsheet. For example, if we click on cell D4, we can type in "Apartment Building" in place of "Decision 1," reflecting the decision corresponding to this branch in our example, as shown in [Figure 12.3](#). We can change the

other labels on the other branches the same way. The decision tree with the edited labels corresponding to our example is shown in [Exhibit 12.13](#).

[Page 533]

Exhibit 12.13.

[\[View full size image\]](#)

0 values below each branch for example, in cells D6 and E6 and in cells H9 and I4. The first 0 cell is where we type in the numeric value (i.e., \$ amount) for that branch. For our example, we would type in 50,000 in cell H4, 30,000 in H9, 100,000 in H14, and so on. These values are shown on the decision tree in [Exhibit 12.13](#). Likewise, we would type in the probabilities for the branches in the cells just above the branch H1, H6, H11, and so on. For example, we would type in 0.60 in cell H1 and 0.40 in cell

H6. These probabilities are also shown in [Exhibit 12.13](#).

However, we need to be very careful not to type anything into the second 0 branch cell for example, E6, I4, I9, E16, I14, I19, and so on. These cells automatically contain the decision tree formulas that compute the expected values at each node and select the best decision branches, so we do not want to type anything in these cells that would eliminate these formulas. For example, in [Exhibit 12.12](#) the formula in cell E6 is shown on the formula

bar at the top of the screen.
This is the expected value for that probability node.

The expected value for this decision tree and our example, \$44,000, is shown in cell A16 in [Exhibit 12.13](#).

Sequential Decision Trees

As noted earlier, when a decision situation requires only a single decision, an expected value payoff table will yield the same result as a decision tree.

However, a payoff table is usually limited to a single decision situation, as in our real estate investment example. If a decision situation requires a series of decisions, then a payoff table cannot be created, and a decision tree becomes the best method for decision analysis.

A sequential decision tree illustrates a situation requiring a series of decisions

To demonstrate the use of a decision tree for a sequence of decisions, we will alter our real estate investment example to encompass a 10-year period during which several decisions must be made. In this new example, the *first decision* facing the investor is whether to purchase an apartment building or land. If the investor purchases the apartment building, two states of nature are possible: Either the population of the town will

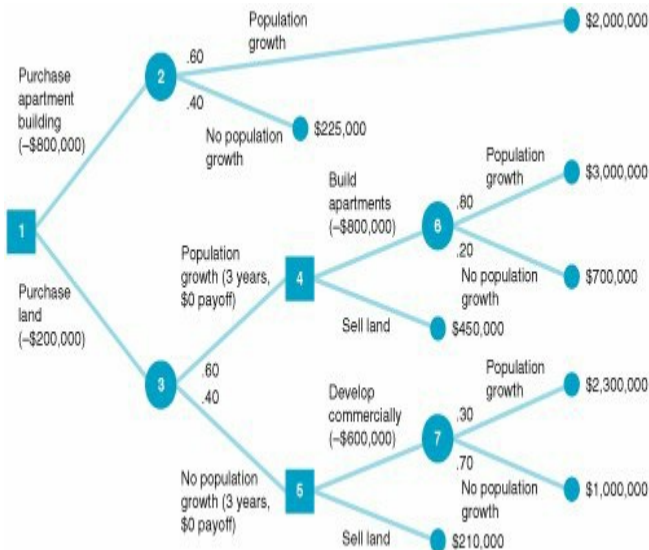
grow (with a probability of .60) or the population will not grow (with a probability of .40). Either state of nature will result in a payoff. On the other hand, if the investor chooses to purchase land, 3 years in the future another decision will have to be made regarding the development of the land. The decision tree for this example, shown in [Figure 12.4](#), contains all the pertinent data, including decisions, states of nature, probabilities, and payoffs.

Figure 12.4.

Sequential decision tree

(This item is displayed on page 535 in the print version)

[\[View full size image\]](#)



Management Science

Application: Evaluating

Electric Generator

Maintenance Schedules

Using Decision Tree Analysis

Electric utility companies plan annual outage periods for preventive maintenance on generators. The outages are typically part of 5- to-20-year master schedules. However, at Entergy Electric Systems these scheduled outages were traditionally based on averages that did not reflect short-term fluctuations in demand due to breakdowns and bad weather conditions. The master schedule had to be reviewed each week by outage planners who relied on their experience to determine

whether the schedule needed to be changed. A user-friendly software system was developed to assist planners at Entergy Electric Systems in making changes in their schedule. The system is based on decision tree analysis.

Each week in the master schedule is represented by a decision tree that is based on changes in customer demand, unexpected generator breakdowns, and delays in returning generators from planned outages. The numeric outcome of the decision tree is the average reserve margin of megawatts (MW) for a specific week. This value enables planners to determine whether customer demand will be met and whether the maintenance schedule planned for the week is acceptable. The planner's objective is to avoid negative power reserves by making changes in the

generators' maintenance schedule. The branches of the decision tree, their probabilities of occurrence, and the branch MW values are based on historical data. The new system has enabled Entergy Electric Systems to isolate high-risk weeks and to develop timely maintenance schedules on short notice. The new computerized system has reduced the maintenance schedules review time for as many as 260 weeks from several days to less than an hour.



Source: H. A. Taha and H. M. Wolf,
"Evaluation of Generator
Maintenance Schedules at Entergy
Electric Systems," *Interfaces* 26, no.
4 (July/August 1996): 5665.

At decision node 1 in [Figure 12.4](#), the decision choices are to purchase an apartment building and to purchase land. Notice that the cost of each venture (\$800,000 and \$200,000, respectively) is shown in parentheses. If the apartment building is purchased, two states of nature are possible at probability node 2: The town may exhibit population growth, with a probability of .60, or there may be no population growth or a decline, with a probability of .40. If the population grows,

the investor will achieve a payoff of \$2,000,000 over a 10-year period. (Note that this whole decision situation encompasses a 10-year time span.) However, if no population growth occurs, a payoff of only \$225,000 will result.

If the decision is to purchase land, two states of nature are possible at probability node 3. These two states of nature and their probabilities are identical to those at node 2; however, the payoffs are different. If population growth occurs *for a*

3-year period, no payoff will occur, but the investor will make another decision at node 4 regarding development of the land. At that point, either apartments will be built, at a cost of \$800,000, or the land will be sold, with a payoff of \$450,000. Notice that the decision situation at node 4 can occur only if population growth occurs first. If no population growth occurs at node 3, there is no payoff, and another decision situation becomes necessary at node 5: The land can be developed commercially

at a cost of \$600,000, or the land can be sold for \$210,000. (Notice that the sale of the land results in less profit if there is no population growth than if there is population growth.)

[Page 535]

If the decision at decision node 4 is to build apartments, two states of nature are possible: The population may grow, with a conditional probability of .80, or there may be no population growth, with a conditional probability of .20. The

probability of population growth is higher (and the probability of no growth is lower) than before because there has already been population growth for the first 3 years, as shown by the branch from node 3 to node 4. The payoffs for these two states of nature at the end of the 10-year period are \$3,000,000 and \$700,000, respectively, as shown in [Figure 12.4](#).

If the investor decides to develop the land commercially at node 5, then two states of nature can occur: Population growth can occur, with a

probability of .30 and an eventual payoff of \$2,300,000, or no population growth can occur, with a probability of .70 and a payoff of \$1,000,000. The probability of population growth is low (i.e., .30) because there has already been no population growth, as shown by the branch from node 3 to node 5.

This decision situation encompasses several sequential decisions that can be analyzed by using the decision tree approach outlined in our earlier (simpler) example. As before,

we start at the end of the decision tree and work backward toward a decision at node 1.

First, we must compute the expected values at nodes 6 and 7:

$$EV(\text{node 6}) = .80(\$3,000,000) + .20(\$700,000) = \$2,540,000$$

$$EV(\text{node 7}) = .30(\$2,300,000) + .70(\$1,000,000) = \$1,390,000$$

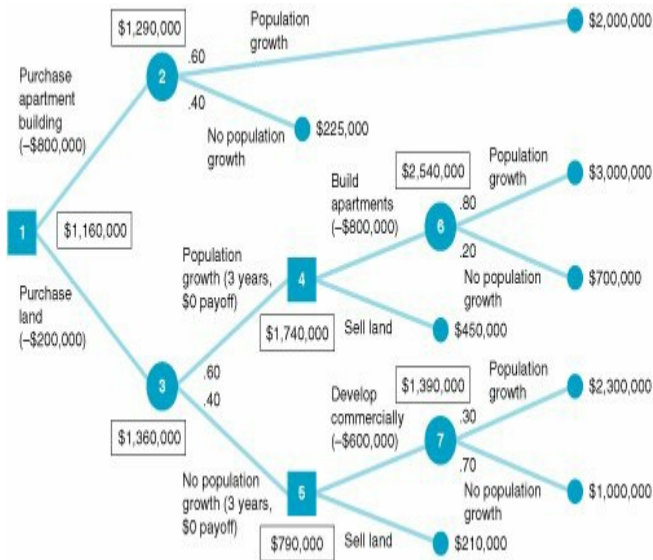
These expected values (and all other nodal values) are shown

in boxes in [Figure 12.5](#).

Figure 12.5.
Sequential decision
tree with nodal
expected values

(This item is displayed on page
536 in the print version)

[\[View full size image\]](#)



At decision nodes 4 and 5, we must make a decision. As with a normal payoff table, we make

the decision that results in the greatest expected value. At node 4 we have a choice between two values: \$1,740,000, the value derived by subtracting the cost of building an apartment building (\$800,000) from the expected payoff of \$2,540,000, *or* \$450,000, the expected value of selling the land computed with a probability of 1.0. The decision is to build the apartment building, and the value at node 4 is \$1,740,000.

This same process is repeated at node 5. The decisions at node 5 result in payoffs of \$790,000 (i.e., \$1,390,000 - 600,000 = \$790,000) and \$210,000. Because the value \$790,000 is higher, the decision is to develop the land commercially.

Next, we must compute the expected values at nodes 2 and 3:

$$EV(\text{node 2}) = .60(\$2,000,000) + .40(\$225,000) = \$1,290,000$$

$$EV(\text{node 3}) = .60(\$1,740,000)$$

$$+.40(\$790,000) = \$1,360,000$$

(Note that the expected value for node 3 is computed from the decision values previously determined at nodes 4 and 5.)

Now we must make the final decision for node 1. As before, we select the decision with the greatest expected value *after the cost of each decision is subtracted out*:

$$\begin{array}{l} \text{apartment } \$1,290,000 - 800,000 = \\ \text{building: } \$490,000 \end{array}$$

$$\text{land: } \$1,360,000 - 200,000 = \$1,160,000$$

Because the highest *net* expected value is \$1,160,000, the decision is to purchase land, and the payoff of the decision is \$1,160,000.

This example demonstrates the usefulness of decision trees for decision analysis. A decision tree allows the decision maker to see the logic of decision making because it provides a

picture of the decision process. Decision trees can be used for decision problems more complex than the preceding example without too much difficulty.

Sequential Decision Tree Analysis with QM for Windows

We have already demonstrated the capability of QM for Windows to perform decision tree analysis. For the sequential decision tree

example described in the preceding section and illustrated in [Figures 12.4](#) and [12.5](#), the program input and solution screen for QM for Windows is shown in [Exhibit 12.14](#).

[Page 537]

Exhibit 12.14.

[\[View full size image\]](#)

Real Estate Investment Example Solution								
	Start Node	End Node	Branch Probability	Profit	Branch Use	End node	Node Type	Node Value
Start	0	1	0	0		1	Decision	1,160,000.0
Purchase apartment	1	2	0	-800,000		2	Chance	1,290,000
Purchase land	1	3	0	-300,000	Always	3	Chance	1,300,000.0
Population growth	2	8	.6	2,000,000		8	Final	2,000,000
No population growth	2	9	.4	225,000		9	Final	225,000
Population growth	3	4	.6	0		4	Decision	1,740,000
No population growth	3	5	.4	0		5	Decision	790,000
Build apartments	4	6	0	-800,000	Possibly	6	Chance	2,540,000
Sell land	4	10	0	450,000		10	Final	450,000
Develop commercially	5	7	0	-600,000	Possibly	7	Chance	1,390,000
Sell land	5	11	0	210,000		11	Final	210,000
Population growth	6	12	.8	3,000,000		12	Final	3,000,000
No population growth	6	13	.2	700,000		13	Final	700,000
Population growth	7	14	.3	2,300,000		14	Final	2,300,000
No population growth	7	15	.7	1,000,000		15	Final	1,000,000

Expected value for the decision tree

Notice that the expected value for the decision tree, \$1,160,000, is given in the first row of the last column on the solution screen in [Exhibit](#)

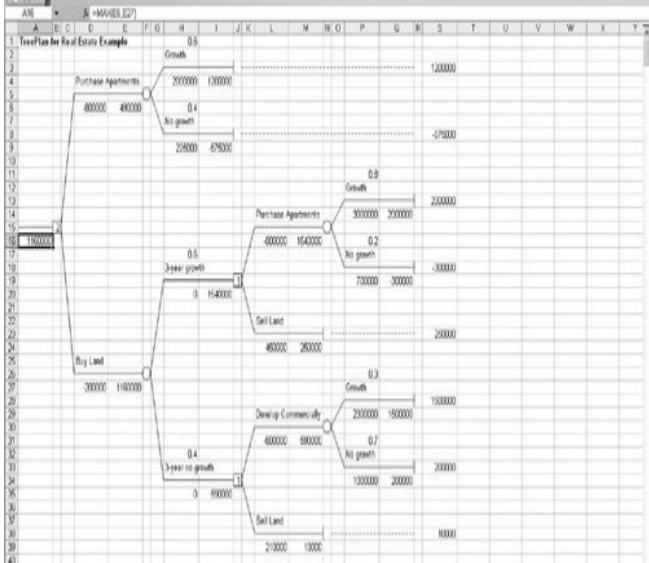
12.14.

Sequential Decision Tree Analysis with Excel and TreePlan

The sequential decision tree example shown in [Figure 12.5](#), developed and solved by using TreePlan, is shown in [Exhibit 12.15](#). Although this TreePlan decision tree is larger than the one we previously developed in [Exhibit 12.13](#), it was accomplished in exactly the same way.

Exhibit 12.15.

[\[View full size image\]](#)



Decision Analysis with Additional Information

Earlier in this chapter we discussed the concept of the *expected value of perfect information*. We noted that if perfect information could be obtained regarding which state of nature would occur in the future, the decision maker could obviously make better decisions. Although perfect information about the future is rare, it is often possible to gain

some amount of additional (imperfect) information that will improve decisions.

In this section we will present a process for using additional information in the decision-making process by applying **Bayesian analysis**. We will demonstrate this process using the real estate investment example employed throughout this chapter. Let's review this example briefly: A real estate investor is considering three alternative investments, which will occur under one of the two possible economic conditions

(states of nature) shown in [Table 12.11](#).

Table 12.11. Payoff table for the real estate investment example

Decision (Purchase)	State of Nature	
	GOOD ECONOMIC CONDITIONS .60	POOR ECONOMIC CONDITIONS .40
Apartment building	\$ 50,000	\$ 30,000
Office building	100,000	40,000

Warehouse

30,000

10,000

*In **Bayesian analysis** additional information is used to alter the marginal probability of the occurrence of an event.*

Recall that, using the expected value criterion, we found the best decision to be the purchase of the office building, with an expected value of \$44,000. We also computed the expected value of perfect information to be \$28,000. Therefore, the investor would be willing to pay up to \$28,000 for information about the states of nature, depending on how close to perfect the information was.

Now suppose that the investor has decided to hire a professional economic analyst

who will provide additional information about future economic conditions. The analyst is constantly researching the economy, and the results of this research are what the investor will be purchasing.

The economic analyst will provide the investor with a report predicting one of two outcomes. The report will be either positive, indicating that good economic conditions are most likely to prevail in the future, or negative, indicating that poor economic conditions

will probably occur. Based on the analyst's past record in forecasting future economic conditions, the investor has determined **conditional probabilities** of the different report outcomes, given the occurrence of each state of nature in the future. We will use the following notations to express these conditional probabilities:

g = good economic conditions

p = poor economic conditions

P = positive economic report

N = negative economic report

A conditional probability is the probability that an event will occur, given that another event has already occurred.

Management Science

Application: Decision

Analysis in the Electric Power Industry

Oglethorpe Power Corporation (OPC), a wholesale power generation and transmission cooperative, provides power to 34 consumer-owned distribution cooperatives in Georgia, representing 20% of the power in the state. Georgia Power Company meets the remaining power demand in Georgia. In general, Georgia has the ability to produce surplus power, whereas Florida, because of its rapidly growing population, must buy power from outside the state to meet its demand. In 1990, OPC learned that

Florida Power Corporation wanted to add a transmission line to Georgia capable of transmitting an additional 1,000 milliwatts (mw). OPC had to decide whether to add this additional capacity at a cost of around \$100 million, with annual savings of around \$20 million or more. OPC used decision analysis and, specifically, decision trees to help make its ultimate decision. Decision analysis has become a very popular form of quantitative analysis in the electric power industry in the United States. OPC's decision tree analysis included a series of decisions combined with uncertainties. The initial decision alternatives in the tree were whether to build a new transmission line alone, build it in a joint venture with Georgia Power, or not build a new line at all. Subsequent decisions included whether to upgrade existing facilities

to meet Florida's needs plus control of the new facilities. Uncertainties included construction costs, Florida's demand for power, competition from other power sellers to Florida, and the share of Florida power for which OPC would be able to contract. The full decision tree for this decision problem, including all combinations of decision nodes and probability nodes, includes almost 8,000 possible decision paths. The decision tree helped OPC understand its decision process better and enabled it to approach the problem so that it would understand its competitive situation with Florida Power Corporation to a greater extent before making a final decision.



Source: A. Borison, "Oglethorpe Power Corporation Decides about Investing in a Major Transmission System," *Interfaces* 25, no. 2 (March/April 1995): 2536.

The conditional probability of each report outcome, given the occurrence of each state of nature, follows:

$$P(P|g) = .80$$

$$P(N|g) = .20$$

$$P(P|p) = .10$$

$$P(N|p) = .90$$

For example, if the future economic conditions are, in fact, good (g), the probability that a positive report (P) will have been given by the

analyst, $P(P|g)$, is .80. The other three conditional probabilities can be interpreted similarly. Notice that these probabilities indicate that the analyst is a relatively accurate forecaster of future economic conditions.

The investor now has quite a bit of probabilistic information available not only the conditional probabilities of the report but also the *prior probabilities* that each state of nature will occur. These prior probabilities that good or poor economic conditions will occur

in the future are

$$P(g) = .60$$

$$P(p) = .40$$

[Page 540]

Given the conditional probabilities, the prior probabilities can be revised to form **posterior probabilities** by means of Bayes's rule. If we know the conditional probability that a positive report was presented, given that good economic conditions

prevail, $P(P|g)$, the posterior probability of good economic conditions, given a positive report, $P(g|P)$, can be determined using Bayes's rule, as follows:

$$\begin{aligned}P(g|P) &= \frac{P(P|g)P(g)}{P(P|g)P(g) + P(P|p)P(p)} \\&= \frac{(.80)(.60)}{(.80)(.60) + (.10)(.40)} \\&= .923\end{aligned}$$

A posterior probability is the altered marginal probability of an event, based on

*additional
information.*

The prior probability that good economic conditions will occur in the future is .60. However, by obtaining the additional information of a positive report from the analyst, the investor can revise the prior probability of good conditions to a .923 probability that good economic conditions will occur. The remaining posterior (revised)

probabilities are

$$P(g|N) = .250$$

$$P(p|P) = .077$$

$$P(p|N) = .750$$

Decision Trees with Posterior Probabilities

The original decision tree analysis of the real estate investment example is shown in [Figures 12.2](#) and [12.3](#). Using these decision trees, we

determined that the appropriate decision was the purchase of an office building, with an expected value of \$44,000. However, if the investor hires an economic analyst, the decision regarding which piece of real estate to invest in will not be made until after the analyst presents the report. This creates an additional stage in the decision-making process, which is shown in the decision tree in [Figure 12.6](#). This decision tree differs in two respects from the decision trees in [Figures 12.2](#)

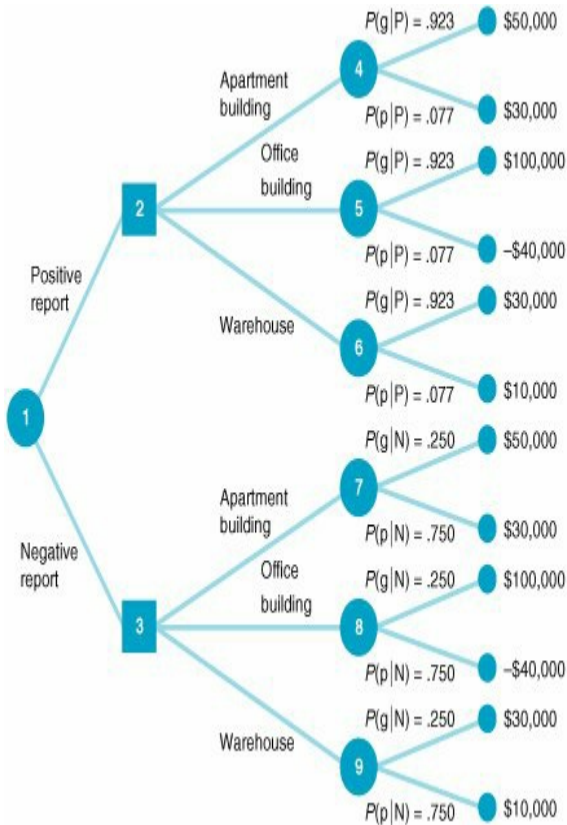
and [12.3](#). The first difference is that there are two new branches at the beginning of the decision tree that represent the two report outcomes. Notice, however, that given either report outcome, the decision alternatives, the possible states of nature, and the payoffs are the same as those in [Figures 12.2](#) and [12.3](#).

Figure 12.6. Decision tree with posterior probabilities

(This item is displayed on page

541 in the print version)

[\[View full size image\]](#)



The second difference is that the probabilities of each state of nature are no longer the prior probabilities given in [Figure 12.2](#); instead, they are the revised posterior probabilities computed in the previous section, using Bayes's rule. If the economic analyst issues a positive report, then the upper branch in [Figure 12.6](#) (from node 1 to node 2) will be taken. If an apartment building is purchased (the branch from node 2 to node 4),

the probability of good economic conditions is .923, whereas the probability of poor conditions is .077. These are the revised posterior probabilities of the economic conditions, given a positive report. However, before we can perform an expected value analysis using this decision tree, one more piece of probabilistic information must be determined the initial branch probabilities of positive and negative economic reports.

The probability of a positive report, $P(P)$, and of a negative

report, $P(N)$, can be determined according to the following logic. The probability that two dependent events, A and B, will both occur is

$$P(AB) = P(A|B)P(B)$$

If event A is a positive report and event B is good economic conditions, then according to the preceding formula,

$$P(Pg) = P(P|g)P(g)$$

Management Science

Application: Discount Fare

Allocation at American Airlines

The management of its reservation system, referred to as *yield management*, is a significant factor in American Airlines's profitability. At American Airlines, yield management consists of three functions: overbooking, discount fare allocation, and traffic management.

Overbooking is the practice of selling more reservations for a flight than there are available seats, to compensate for no-shows and cancellations. Discount allocation is the process of determining the number of discount fares to make

available on a flight. Too few discount fares will leave a flight with empty seats, while too many will limit the number of flights to schedule to maximize revenues.

The process of determining the allocation of discount fares for a flight is essentially a decision analysis model with a decision tree. The initial decision is to accept or reject a discount fare request. If the initial decision is to reject the request, there is a probability, p , that the seat will eventually be sold at full fare and a probability, $1 - p$, that the seat will not be sold at all. The estimate of p is updated several times before flight departure, depending on the forecast for seat demand and the number of available seats remaining. If the expected value of the decision to reject that is, $EV(\text{reject discount fare}) = (p)(\$ \text{full fare})$ is greater than

the discount fare, then the request is rejected.



Source: B. Smith, J. Leimkuhler, and R. Darrow, "Yield Management at American Airlines," *Interfaces* 22, no. 1 (January/February 1992): 831.

We can also determine the probability of a positive report and poor economic conditions the same way:

$$P(Pp) = P(P|p)P(p)$$

Next, we consider the two probabilities $P(Pg)$ and $P(Pp)$, also called *joint probabilities*. These are, respectively, the probability of a positive report and good economic conditions and the probability of a positive report and poor economic conditions. These two sets of

occurrences are **mutually exclusive** because both good and poor economic conditions cannot occur simultaneously in the immediate future.

Conditions will be either good or poor, but not both. To determine the probability of a positive report, we add the mutually exclusive probabilities of a positive report with good economic conditions and a positive report with poor economic conditions, as follows:

$$P(P) = P(P_g) + P(P_p)$$

*Events are **mutually exclusive** if only one can occur at a time.*

Now, if we substitute into this formula the relationships for $P(P_g)$ and $P(P_p)$ determined earlier, we have

$$P(P) = P(P|g)P(g) + P(P|p)P(p)$$

You might notice that the right-hand side of this equation is the denominator of the

Bayesian formula we used to compute $P(g|P)$ in the previous section. Using the conditional and prior probabilities that have already been established, we can determine that the probability of a positive report is

$$\begin{aligned} P(P) &= P(P|g)P(g) + P(P|p)P(p) \\ &= (.80)(.60) + (.10)(.40) = .52 \end{aligned}$$

Similarly, the probability of a negative report is

$$\begin{aligned} P(N) &= P(N|g)P(g) + P(N|p)P(p) \\ &= (.20)(.60) + (.90)(.40) = .48 \end{aligned}$$

$P(P)$ and $P(N)$ are also referred to as *marginal probabilities*.

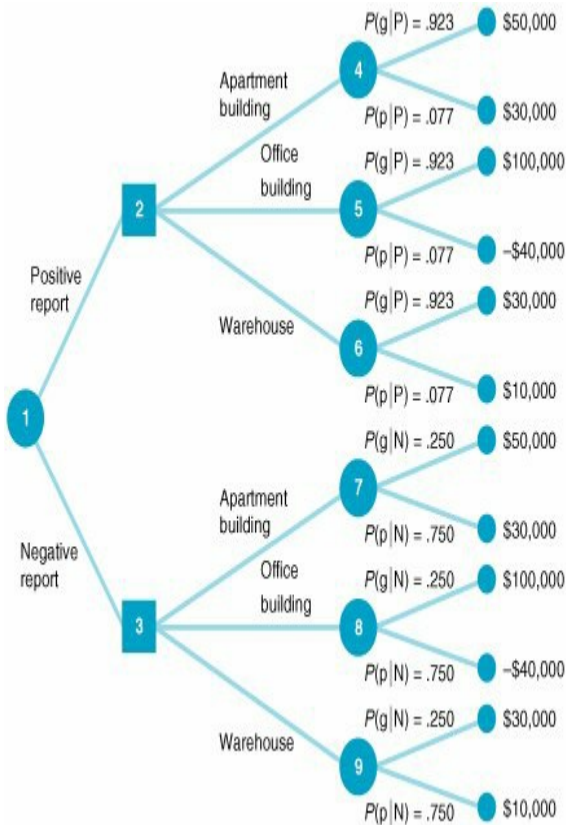
Now we have all the information needed to perform a decision tree analysis. The decision tree analysis for our example is shown in [Figure 12.7](#). To see how the decision tree analysis is conducted, consider the result at node 4 first. The value \$48,460 is the expected value of the purchase of an apartment building, given both states of nature. This expected value is computed as follows:

$$\begin{aligned}EV(\text{apartment building}) &= \\ \$50,000(.923) + 30,000(.077) \\ &= \$48,460\end{aligned}$$

Figure 12.7. Decision tree analysis for real estate investment example

(This item is displayed on page 543 in the print version)

[\[View full size image\]](#)



The expected values at nodes 5, 6, 7, 8, and 9 are computed similarly.

The investor will actually make the decision about the investment at nodes 2 and 3. It is assumed that the investor will make the best decision in each case. Thus, the decision at node 2 will be to purchase an office building, with an expected value of \$89,220; the decision at node 3 will be to purchase an apartment

building, with an expected value of \$35,000. These two results at nodes 2 and 3 are referred to as *decision strategies*. They represent a plan of decisions to be made, given either a positive or a negative report from the economic analyst.

The final step in the decision tree analysis is to compute the expected value of the decision strategy, given that an economic analysis is performed. This expected value, shown as \$63,194 at node 1 in [Figure 12.7](#), is computed as follows:

$$\begin{aligned}EV(\text{strategy}) &= \$89,220(.52) + 35,000(.48) \\ &= \$63,194\end{aligned}$$

This amount, \$63,194, is the expected value of the investor's decision strategy, given that a report forecasting future economic condition is generated by the economic analyst.

[Page 543]

Computing Posterior Probabilities with Tables

One of the difficulties that can occur with decision analysis with additional information is that as the size of the problem increases (i.e., as we add more decision alternatives and states of nature), the application of Bayes's rule to compute the posterior probabilities becomes more complex. In such cases, the posterior probabilities can be computed by using tables. This tabular approach will be demonstrated with our real estate investment example. The table for computing posterior probabilities for a

positive report and P (P) is initially set up as shown in [Table 12.12](#).

Table 12.12. Computation of joint probabilities

(1) State of Nature	(2) Prior Probability	(3) Conditional Probability	(4) Prior Probability x Conditional Probability (2) x (3)
Good conditions	$P(g) = .60$	$P(P g) = .80$	$P(Pg) =$
Poor conditions	$P(p) = .40$	$P(P p) = .10$	

$$\frac{P(P_p) = .0}{\sum = P(P) =}$$

The posterior probabilities for either state of nature (good or poor economic conditions), given a negative report, are computed similarly.

[Page 544]

No matter how large the decision analysis, the steps of

this tabular approach can be followed the same way as in this relatively small problem. This approach is more systematic than the direct application of Bayes's "rule," making it easier to compute the posterior probabilities for larger problems.

Computing Posterior Probabilities with Excel

The posterior probabilities computed in [Table 12.12](#) can also be computed by using

Excel. [Exhibit 12.16](#) shows [Table 12.12](#) set up in an Excel spreadsheet format, as well as the table for computing $P(N)$.

Exhibit 12.16.

[\[View full size image\]](#)

Microsoft Excel - Lab0812.16.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

Formulas: = 100% + 0

And: 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 61 62 63 64 65 66 67 68 69 70 71 72 73 74 75 76 77 78 79 80 81 82 83 84 85 86 87 88 89 90 91 92 93 94 95 96 97 98 99 100 101 102 103 104 105 106 107 108 109 110 111 112 113 114 115 116 117 118 119 120 121 122 123 124 125 126 127 128 129 130 131 132 133 134 135 136 137 138 139 140 141 142 143 144 145 146 147 148 149 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170 171 172 173 174 175 176 177 178 179 180 181 182 183 184 185 186 187 188 189 190 191 192 193 194 195 196 197 198 199 200 201 202 203 204 205 206 207 208 209 210 211 212 213 214 215 216 217 218 219 220 221 222 223 224 225 226 227 228 229 230 231 232 233 234 235 236 237 238 239 240 241 242 243 244 245 246 247 248 249 250 251 252 253 254 255 256 257 258 259 260 261 262 263 264 265 266 267 268 269 270 271 272 273 274 275 276 277 278 279 280 281 282 283 284 285 286 287 288 289 290 291 292 293 294 295 296 297 298 299 300 301 302 303 304 305 306 307 308 309 310 311 312 313 314 315 316 317 318 319 320 321 322 323 324 325 326 327 328 329 330 331 332 333 334 335 336 337 338 339 340 341 342 343 344 345 346 347 348 349 350 351 352 353 354 355 356 357 358 359 360 361 362 363 364 365 366 367 368 369 370 371 372 373 374 375 376 377 378 379 380 381 382 383 384 385 386 387 388 389 390 391 392 393 394 395 396 397 398 399 400 401 402 403 404 405 406 407 408 409 410 411 412 413 414 415 416 417 418 419 420 421 422 423 424 425 426 427 428 429 430 431 432 433 434 435 436 437 438 439 440 441 442 443 444 445 446 447 448 449 450 451 452 453 454 455 456 457 458 459 460 461 462 463 464 465 466 467 468 469 470 471 472 473 474 475 476 477 478 479 480 481 482 483 484 485 486 487 488 489 490 491 492 493 494 495 496 497 498 499 500 501 502 503 504 505 506 507 508 509 510 511 512 513 514 515 516 517 518 519 520 521 522 523 524 525 526 527 528 529 530 531 532 533 534 535 536 537 538 539 540 541 542 543 544 545 546 547 548 549 550 551 552 553 554 555 556 557 558 559 560 561 562 563 564 565 566 567 568 569 570 571 572 573 574 575 576 577 578 579 580 581 582 583 584 585 586 587 588 589 590 591 592 593 594 595 596 597 598 599 600 601 602 603 604 605 606 607 608 609 610 611 612 613 614 615 616 617 618 619 620 621 622 623 624 625 626 627 628 629 630 631 632 633 634 635 636 637 638 639 640 641 642 643 644 645 646 647 648 649 650 651 652 653 654 655 656 657 658 659 660 661 662 663 664 665 666 667 668 669 670 671 672 673 674 675 676 677 678 679 680 681 682 683 684 685 686 687 688 689 690 691 692 693 694 695 696 697 698 699 700 701 702 703 704 705 706 707 708 709 710 711 712 713 714 715 716 717 718 719 720 721 722 723 724 725 726 727 728 729 730 731 732 733 734 735 736 737 738 739 740 741 742 743 744 745 746 747 748 749 750 751 752 753 754 755 756 757 758 759 760 761 762 763 764 765 766 767 768 769 770 771 772 773 774 775 776 777 778 779 780 781 782 783 784 785 786 787 788 789 790 791 792 793 794 795 796 797 798 799 800 801 802 803 804 805 806 807 808 809 810 811 812 813 814 815 816 817 818 819 820 821 822 823 824 825 826 827 828 829 830 831 832 833 834 835 836 837 838 839 840 841 842 843 844 845 846 847 848 849 850 851 852 853 854 855 856 857 858 859 860 861 862 863 864 865 866 867 868 869 870 871 872 873 874 875 876 877 878 879 880 881 882 883 884 885 886 887 888 889 890 891 892 893 894 895 896 897 898 899 900 901 902 903 904 905 906 907 908 909 910 911 912 913 914 915 916 917 918 919 920 921 922 923 924 925 926 927 928 929 930 931 932 933 934 935 936 937 938 939 940 941 942 943 944 945 946 947 948 949 950 951 952 953 954 955 956 957 958 959 960 961 962 963 964 965 966 967 968 969 970 971 972 973 974 975 976 977 978 979 980 981 982 983 984 985 986 987 988 989 990 991 992 993 994 995 996 997 998 999 1000

1 Real Estate Investment Example - Posterior Probabilities

2

3

4 Posterior probability table for a positive report

(1)	(2)	(3)	(4)	(5)
States of Nature	Prior Probabilities	Conditional Probabilities	(2) X (3)	(4)/Sum(4)
Good conditions	0.8	0.80	0.48	0.622
Poor conditions	0.4	0.10	0.04	0.077
		$P(P)=$	0.52	

15 Posterior probability table for a negative report

(1)	(2)	(3)	(4)	(5)
States of Nature	Prior Probabilities	Conditional Probabilities	(2) X (3)	(4)/Sum(4)
Good conditions	0.8	0.20	0.16	0.260
Poor conditions	0.4	0.80	0.32	0.740
		$P(N)=$	0.48	

The Expected Value of Sample Information

Recall that we computed the expected value of our real estate investment example to be \$44,000 when we did not have any additional information. After obtaining the additional information provided by the economic analyst, we computed an expected value of \$63,194, using the decision tree in [Figure 12.7](#). The difference between these two expected values is called the **expected value of sample information (EVSI)**, and it is computed as follows:

$$EVSI = EV_{\text{with information}} - EV_{\text{without information}}$$

*The **expected value of sample information** is the difference between the expected value with and without additional information.*

For our example, the expected value of sample information is

$$EVSI = \$63,194 - \$44,000 = \$19,194$$

This means that the real estate investor would be willing to pay the economic analyst up to \$19,194 for an economic report that forecasted future economic conditions.

After we computed the expected value of the investment without additional information, we computed the *expected value of perfect information*, which equaled \$28,000. However, the expected value of the sample

information was only \$19,194. This is a logical result because it is rare that absolutely perfect information can be determined. Because the additional information that is obtained is less than perfect, it will be worth less to the decision maker. We can determine how close to perfect our sample information is by computing the **efficiency of sample information** as follows:

$$\text{efficiency} = EVSI \div EVPI = \\ \$19,194 / 28,000 = .69$$

*The **efficiency of sample information** is the ratio of the expected value of sample information to the expected value of perfect information.*

Thus, the analyst's economic report is viewed by the investor to be 69% as efficient as perfect information. In general, a high efficiency rating indicates that the information

is very good, or close to being perfect information, and a low rating indicates that the additional information is not very good. For our example, the efficiency of .68 is relatively high; thus, it is doubtful that the investor would seek additional information from an alternative source. (However, this is usually dependent on how much money the decision maker has available to purchase information.) If the efficiency had been lower, however, the investor might

seek additional information
elsewhere.

[Page 545]

Management Science

Application: Scheduling

Refueling at the Indian Point

3 Nuclear Power Plant

The New York Power Authority (NYPA) owns and operates 12 power projects that provide roughly one-fourth of all the electricity used in New York State. Approximately 20% of NYPA's electrical power (supplying Westchester County and New York City) is generated by the Indian Point 3 (IP3) nuclear power plant, located on the Hudson River. IP3 withdraws 840,000 gallons of water per minute from the river for cooling steam and then returns it to the river. When water is withdrawn from the river, fish, especially small fish and eggs, do

not always survive as they pass through the cooling system. The NYPA can reduce the negative effects by scheduling plant shutdowns to refuel IP3 when fish eggs and small fish are most abundant in the Hudson River. In the past, NYPA developed a refueling schedule according to a 10-year planning horizon, although unforeseen events often altered this schedule. There is uncertainty in the future about possible deregulation of the electric utility industry and its effect on the price of replacement power during refueling outages. The NYPA must consider these uncertainties in its attempt to provide low-cost power and minimize the environmental effects of refueling outages. The NYPA developed a decision analysis model to compare alternative strategies for refueling that balanced fish protection, the

cost of buying fuel, and the uncertainties of deregulation.

The key decisions of the model were the times of year that refueling outages should occur, which affects the level of fish protection, and the amount of fuel that should be ordered for the nuclear reactor core that allows for operation for a target number of days, which affects the cost of the operation. The cost consideration comprises three objectives: minimizing the cost of replacement electricity when IP3 is shut down, minimizing the amount of unused fuel at the end of an operating cycle, and minimizing the time that IP3 would operate at less than full power before refueling. The objective of fish protection was to minimize the sum of the average percentage reduction in the mortality rate caused by water removal at IP3

for five types of fish over the 10-year planning horizon. The three major uncertainties in the model were the cost of refueling, how long it takes IP3 to refuel and how well it operates, and when New York State is likely to deregulate the electric utility industry. The decision analysis model considered five strategies based on different schedules that reflected different time windows when fish were most vulnerable. The model showed that no strategy simultaneously minimized refueling cost while minimizing fish mortality rates. The strategy that was selected restricted the starting date for refueling to the third week in May, when fish protection would meet accepted standards, at a cost savings of \$10 million over the previous refueling schedule. The NYPA used the decision analysis model to

develop its refueling schedule for the 10-year period from 1999 to 2008.



Source: D. J. Dunning, S. Lockfort, Q. E. Ross, P. C. Beccue, and J. S. Stonebraker, "New York Power Authority Uses Decision Analysis to Schedule Refueling of Its Indian Point 3 Nuclear Power Plant," *Interfaces* 31, no. 5 (September/October 2001): 12135.

Utility

All the decision-making criteria presented so far in this chapter have been based on monetary value. In other words, decisions have been based on the potential dollar payoffs of the alternatives. However, there are certain decision situations in which individuals do not make decisions based on the expected dollar gain or loss.

For example, consider an individual who purchases automobile insurance. The decisions are to purchase and to not purchase, and the states of nature are *an accident* and *no accident*. The payoff table for this decision situation, including probabilities, is shown in [Table 12.13](#).

Table 12.13. Payoff table for auto insurance example

	State of Nature	
	No ACCIDENT	ACCIDENT .008
Decision		

.992

Purchase insurance	\$500	\$ 500
Do not purchase insurance	0	10,000

The dollar outcomes in [Table 12.13](#) are the *costs* associated with each outcome. The insurance costs \$500 whether there is an accident or no accident. If the insurance is not

purchased and there is no accident, then there is no cost at all. However, if an accident does occur, the individual will incur a cost of \$10,000.

The expected cost (EC) for each decision is

$$EC(\text{insurance}) = .992(\$500) + .008(\$500) = \$500$$
$$EC(\text{no insurance}) = .992(\$0) + .008(\$10,000) = \$80$$

Because the *lower* expected cost is \$80, the decision *should be* not to purchase insurance. However, people almost always purchase insurance (even when they are not legally required to

do so). This is true of all types of insurance, such as accident, life, or fire.

Why do people shun the greater *expected* dollar outcome in this type of situation? The answer is that people want to avoid a ruinous or painful situation. When faced with a relatively small dollar cost versus a disaster, people typically pay the small cost to avert the disaster. People who display this characteristic are referred to as **risk averters** because they avoid risky situations.

*People who forgo a high expected value to avoid a disaster with a low probability are **risk averters**.*

Alternatively, people who go to the track to wager on horse races, travel to Atlantic City to play roulette, or speculate in the commodities market decide to take risks even though the greatest *expected value* would occur if they simply held on to

the money. These people shun the greater expected value accruing from a sure thing (keeping their money) in order to take a chance on receiving a "bonanza." Such people are referred to as **risk takers**.

*People who take a chance on a bonanza with a very low probability of occurrence in lieu of a sure thing are **risk takers**.*

For both risk averters and risk takers (as well as those who are indifferent to risk), the decision criterion is something other than the expected dollar outcome. This alternative criterion is known as utility. Utility is a measure of the satisfaction derived from money. In our examples of risk averters and risk takers presented earlier, the utility derived from their decisions *exceeded* the expected dollar value. For example, the utility

to the average decision maker of having insurance is much greater than the utility of not having insurance.

Utility is a measure
of personal
satisfaction derived
from money.

As another example, consider two people, each of whom is offered \$100,000 to perform some particularly difficult and

strenuous task. One individual has an annual income of \$10,000; the other individual is a multimillionaire. It is reasonable to assume that the average person with an annual income of only \$10,000 would leap at the opportunity to earn \$100,000, whereas the multimillionaire would reject the offer. Obviously, \$100,000 has more *utility* (i.e., value) for one individual than for the other.

In general, the same incremental amount of money does not have the same

intrinsic value to every person. For individuals with a great deal of wealth, more money does not usually have as much intrinsic value as it does for individuals who have little money. In other words, although the dollar value is the same, the value as measured by utility is different, depending on how much wealth a person has. Thus, utility in this case is a measure of the pleasure or satisfaction an individual would receive from an incremental increase in wealth.

In some decision situations, decision makers attempt to assign a subjective value to utility. This value is typically measured in terms of units called **utiles**. For example, the \$100,000 offered to the two individuals may have a utility value of 100 utiles to the person with a low income and 0 utiles to the multimillionaire.

Utiles are units of subjective measures of utility.

In our automobile insurance example, the *expected utility* of purchasing insurance could be 1,000 utiles, and the expected utility of not purchasing insurance only 1 utile. These utility values are completely reversed from the [expected monetary values](#) computed from [Table 12.13](#), which explains the decision to purchase insurance.

As might be expected, it is usually very difficult to

measure utility and the number of utiles derived from a decision outcome. The process is a very subjective one in which the decision maker's psychological preferences must be determined. Thus, although the concept of utility is realistic and often portrays actual decision-making criteria more accurately than does expected monetary value, its application is difficult and, as such, somewhat limited.

Summary

The purpose of this chapter was to demonstrate the concepts and fundamentals of decision making when uncertainty exists. Within this context, several decision-making criteria were presented. The maximax, maximin, minimax regret, equal likelihood, and Hurwicz decision criteria were demonstrated for cases in which probabilities could not be attached to the occurrence of

outcomes. The expected value criterion and decision trees were discussed for cases in which probabilities could be assigned to the states of nature of a decision situation.

All the decision criteria presented in this chapter were demonstrated via rather simplified examples; actual decision-making situations are usually more complex.

Nevertheless, the process of analyzing decisions presented in this chapter is the logical method that most decision makers follow to make a

decision.

-

References

Baumol, W. J. *Economic Theory and Operations Analysis*, 4th ed. Upper Saddle River, NJ: Prentice Hall, 1977.

Bell, D. "Bidding for the S.S. *Kuniang*." *Interfaces* 14, no. 2 (March/April 1984):1723.

Dorfman, R., Samuelson, P. A., and Solow, R. M. *Linear Programming and Economic Analysis*. New York: McGraw-Hill, 1958.

Holloway, C. A. *Decision Making Under Uncertainty*. Upper Saddle River, NJ: Prentice Hall,

1979.

Howard, R. A. "An Assessment of Decision Analysis."

Operations Research 28, no. 1 (JanuaryFebruary 1980):427.

Keeney, R. L. "Decision Analysis: An Overview."

Operations Research 30, no. 5 (SeptemberOctober 1982):80338.

Luce, R. D., and Raiffa, H.

Games and Decisions. New York: John Wiley & Sons, 1957.

Von Neumann, J., and

Morgenstern, O. *Theory of Games and Economic Behavior*, 3rd ed. Princeton, NJ: Princeton University Press, 1953.

Williams, J. D. *The Compleat Strategyst*, rev. ed. New York: McGraw-Hill, 1966.

Example Problem Solution

The following example will illustrate the solution procedure for a decision analysis problem.

Problem Statement

T. Bone Puckett, a corporate raider, has acquired a textile company and is contemplating the future of one of its major

plants, located in South Carolina. Three alternative decisions are being considered: (1) expand the plant and produce lightweight, durable materials for possible sales to the military, a market with little foreign competition; (2) maintain the status quo at the plant, continuing production of textile goods that are subject to heavy foreign competition; or (3) sell the plant now. If one of the first two alternatives is chosen, the plant will still be sold at the end of a year. The amount of profit that could be

earned by selling the plant in a year depends on foreign market conditions, including the status of a trade embargo bill in Congress. The following payoff table describes this decision situation:

[Page 548]

Decision	State of Nature	
	<i>Good Foreign Competitive Conditions</i>	<i>Poor Foreign Competitive Conditions</i>
Expand	\$ 800,000	\$ 500,000

Maintain status quo	1,300,000	150,000
Sell now	320,000	320,000

- 1.** Determine the best decision by using the following decision criteria:
 - 1.** Maximax
 - 2.** Maximin
 - 3.** Minimax regret

4. Hurwicz ($\alpha = .3$)

5. Equal likelihood

- 2.** Assume that it is now possible to estimate a probability of .70 that good foreign competitive conditions will exist and a probability of .30 that poor conditions will exist. Determine the best decision by using expected value and expected opportunity loss.
- 3.** Compute the expected value of perfect

information.

- 4.** Develop a decision tree, with expected values at the probability nodes.
- 5.** T. Bone Puckett has hired a consulting firm to provide a report on future political and market situations. The report will be positive (P) or negative (N), indicating either a good (g) or poor (p) future foreign competitive situation. The conditional probability of each report outcome, given each state of nature, is

$$P(P|g) = .70$$

$$P(N|g) = .30$$

$$P(P|p) = .20$$

$$P(N|p) = .80$$

Determine the posterior probabilities by using Bayes's rule.

- 6.** Perform a decision tree analysis by using the posterior probability obtained in (e).

Solution

(part A): Determine Decisions Without Probabilities

Maximax:

Expand \$ 800,000

Status quo 1,300,000 Mā

Sell

320,000

Decision: Maintain statu quo.

[Page 549]

Maximin:

Expand

500,000^{\$} Ma

Status quo	-150,000
------------	----------

Sell	320,000
------	---------

Decision: Expand.

Minimax
regret:

Expand	\$500,000	Minimax
--------	-----------	---------

Step 1.	Status quo	650,000
	Sell	980,000

Decision: Expand.

Hurwicz
($\alpha = .3$):

\$800,00

Expand	500,000
	\$5

Status quo	\$1,300,000
	150,000
	\$2

Sell	\$ 320,000
	320,000(
	3

Decision: Expand.

Equal
likelihood:

	\$800,000
Expand	500,000
	\$6

	\$1,300,000
Status quo	150,000
	\$5

	\$320,000
Sell	320,000
	\$3

Decision: Expand.

**(part B): Determine
Decisions with *EV* and**

Expected
value:

	\$800,000
Expand	500,000
	\$7

Status quo	\$1,300,000
	150,000
	\$8

Sell	\$320,000
	320,000
	\$3

Step 2.

Decision: Maintain status quo.

Expected opportunity loss

Expand	\$500,000(. 0(.30) = \$3!
--------	------------------------------

Status quo	\$0 650,000 \$1
------------	-----------------------

Sell	\$980,000 180,000 \$7
------	-----------------------------

Decision: Maintain statu

quo.

(part C): Compute *EV*

expected	=
value given	1,300,00
perfect	+ 500,00
information	

= \$1,060

Step 3.

expected
 value =
 without \$1,300,000
 perfect 150,000(%)
 information

$$= \$865,000$$

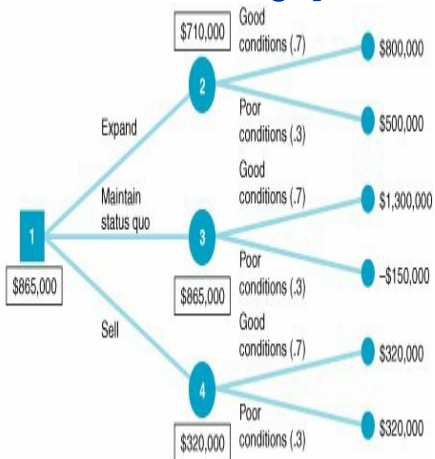
$$= \$1,060,000$$

$$EVPI = \$1,060,000 - \$865,000 = \$195,000$$

(part D): Develop a Decision Tree

[\[View full size image\]](#)

Step 4.



(part E): Determine Posterior Probabilities

$$\begin{aligned}P(g|P) &= \frac{P(P|g)P(g)}{P(P|g)P(g) + P(P|p)P(p)} \\&= \frac{(.70)(.70)}{(.70)(.70) + (.20)(.30)} \\&= .891\end{aligned}$$

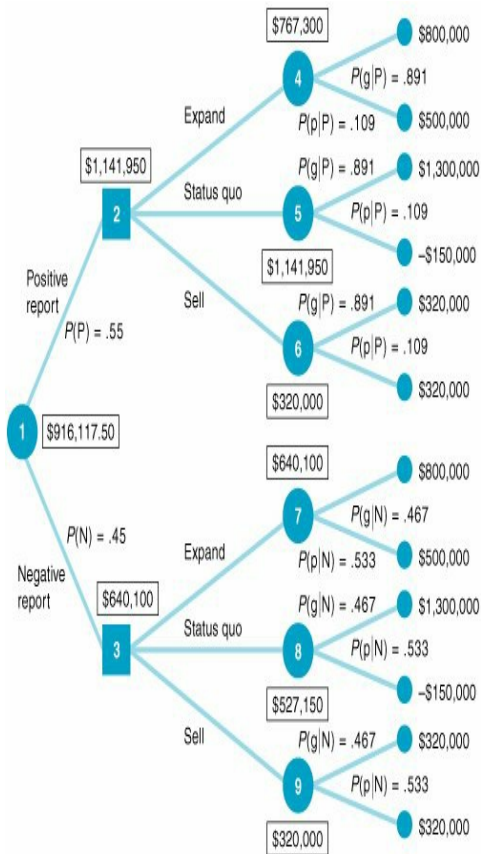
Step 5. $P(p|P) = .109$

$$\begin{aligned}P(g|N) &= \frac{P(N|g)P(g)}{P(N|g)P(g) + P(N|p)P(p)} \\&= \frac{(.30)(.70)}{(.30)(.70) + (.80)(.30)} \\&= .467\end{aligned}$$

$$P(p|N) = .533$$

(part F): Perform Decision Tree Analysis with Posterior Probabilities

Step 6.



Problems

A farmer in Iowa is considering either leasing some extra land or investing in savings certificates at the local bank. If weather conditions are good next year, the extra land will give the farmer an excellent harvest. However, if weather conditions are bad, the farmer will lose money. The savings certificates will result in the same return, regardless of the weather conditions. The return for each investment, given each type of weather condition, is shown in the following pay table:

Weather

1.

Decision

Good

Bad

Lease land

\$90,000

\$ 40,000

Buy savings
certificate

10,000

10,000

Select the best decision, using the following decision criteria:

1. Maximax

2. Maximin

The owner of the Burger Doodle Restaurant is considering two ways to expand operations: open a drive-up window or serve breakfast. The increase in profits resulting from these proposed expansions depends on whether a competitor opens a franchise down the street. The possible profits from each expansion in operations, given both future competitive situations, are shown in the following payoff table:

2. Decision	Competitor	
	Open	Not Open
Drive-up window	\$ 6,000	\$20,000
Breakfast	4,000	8,000

Select the best decision, using the following decision criteria.

1. Maximax

2. Maximin

Stevie Stone, a bellhop at the Royal Sundown Hotel in Atlanta, has been offered a management position. Although accepting the offer would assure him a job if there were a recession, if good economic conditions prevailed, he would actually make less money as a manager than as a bellhop (because of the large tips he gets as a bellhop). His salary during the next 5 years for each job, given each future economic condition, is

shown in the following payoff table:

Economic Condition

3.

Decision

Good

Recession

Bellhop

\$120,000

\$60,000

Manager

85,000

85,000

Select the best decision, using the following decision criteria.

1. Minimax regret

2. Hurwicz ($\alpha = .4$)

3. Equal likelihood

Brooke Bentley, a student in business administration, is trying to decide which management science course to take next quarter I, II, or III. "Steamboat" Fulton, "Death" Ray, and "Sadistic" Scott are the three management science professors who teach the courses. Brooke does not know who will teach what course. Brooke can expect a different grade in each of the courses, depending on who teaches next quarter, as shown in the following payoff table:

Professor

4. Course	Professor		
	Fulton	Ray	Scott

I

B

D

D

II

C

B

F

III

F

A

C

Determine the best course to take next quarter, using the following criteria.

1. Maximax

2. Maximin

[Page 553]

A farmer in Georgia must decide which crop to plant next year on his land: corn

peanuts, or soybeans. The return from each crop will be determined by whether a new trade bill with Russia passes the Senate. The profit the farmer will realize from each crop, given the two possible results on the trade bill, is shown in the following payoff table:

Crop	Trade Bill	
	Pass	Fail
Corn	\$35,000	\$8,000
5. Peanuts	18,000	12,000
Soybeans	22,000	20,000

Determine the best crop to plant, using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Minimax regret
- 4.** Hurwicz ($\alpha = .3$)
- 5.** Equal likelihood

A company must decide now which of three products to make next year to plan and order proper materials. The cost per unit of producing each product will be determined by whether a new union labor contract passes or fails. The cost per unit for each product, given each contract

result, is shown in the following payoff table:

		Contract Outcome	
Product		Pass	Fail
6.	1	\$7.50	\$6.00
	2	4.00	7.00
	3	6.50	3.00

Determine which product should be produced, using the following decision

criteria.

1. Minimin
2. Minimax

The owner of the Columbia Construction Company must decide between building housing development, constructing a shopping center, and leasing all the company's equipment to another company. The profit that will result from each alternative will be determined by whether material costs remain stable or increase. The profit from each alternative given the two possibilities for material costs, is shown in the following payoff table:

Material Costs

Decision

Stable

Increases

Houses

\$ 70,000

\$ 30,000

7.

Shopping center

105,000

20,000

Leasing

40,000

40,000

Determine the best decision, using the following decision criteria.

1. Maximax**2.** Maximin

3. Minimax regret
4. Hurwicz ($\alpha = .2$)
5. Equal likelihood

A local real estate investor in Orlando is considering three alternative investments: a motel, a restaurant, or a theater. Profits from the motel or restaurant will be affected by the availability of gasoline and the number of tourists; profits from the theater will be relatively stable under all conditions. The following payoff table shows the profit or loss that could result from each investment:

		Gasoline Availability		
Investment		Shortage	Stable	Surplus

Supply

8. Motel	\$ 8,000	\$15,000	\$20,0
Restaurant	2,000	8,000	6,00
Theater	6,000	6,000	5,00

Determine the best investment, using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Minimax regret
- 4.** Hurwicz ($\alpha = .4$)

5. Equal likelihood

A television network is attempting to decide during the summer which of the following three football games to televise on the Saturday following Thanksgiving Day: Alabama versus Auburn, Georgia versus Georgia Tech, or Army versus Navy. The estimated viewer ratings (millions of homes) for the games depend on the win/loss records of the six teams as shown in the following payoff table:

Number of Viewers (1,000,000s)			
Game	Both Teams Have Winning	One Team Has Winning Record; One Team Has Losing	Both Teams Have Losing

9.

Alabama
vs.
Auburn

Records

10.2

Losing
Record

7.3

Record

5.4

Georgia
vs.
Georgia
Tech

9.6

8.1

4.8

Army vs.
Navy

12.5

6.5

3.2

Determine the best game to televise,

using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Equal likelihood

Ann Tyler has come into an inheritance from her grandparents. She is attempting to decide among several investment alternatives. The return after 1 year is primarily dependent on the interest rate during the next year. The rate is currently 7%, and Ann anticipates that it will stay the same or go up or down by at most two points. The various investment alternatives plus their returns (\$10,000 given the interest rate changes, are shown in the following table:

Interest Rate

Investments	5%	6%	7%	8%	9%
-------------	----	----	----	----	----

Money market fund	2	3.1	4	4.3	5
-------------------	---	-----	---	-----	---

10. Stock growth fund	-3	-2	2.5	4	6
------------------------------	----	----	-----	---	---

Bond fund	6	5	3	3	2
-----------	---	---	---	---	---

Government fund	4	3.6	3.2	3	2.1
-----------------	---	-----	-----	---	-----

Risk fund	-9	-4.5	1.2	8.3	14
-----------	----	------	-----	-----	----

Savings bonds	3	3	3.2	3.4	3.6
---------------	---	---	-----	-----	-----

Determine the best investment, using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Equal likelihood

The Tech football coaching staff has six basic offensive plays it runs every game. Tech has an upcoming game against State on Saturday, and the Tech coaches know that State employs five different

defenses. The coaches have estimated the number of yards Tech will gain with each play against each defense, as shown in the following payoff table:

Play	Defense				
	54	63	Wide Tackle	Nickel	Blitz
Off tackle	3	-2	9	7	-1
Option	-1	8	-2	9	1
Toss sweep	6	16	-5	3	1

Draw	-2	4	3	10	-1
Pass	8	20	12	-7	-8
Screen	-5	-2	8	3	1

- 1.** If the coaches employ an offensive game plan, they will use the maximax criterion. What will be their best play?
- 2.** If the coaches employ a defensive plan, they will use the maximin criterion. What will be their best play?
- 3.** What will be their best offensive play if State is equally likely to use any

its five defenses?

Microcomp is a U.S.-based manufacturer of personal computers. It is planning to build a new manufacturing and distribution facility in either South Korea, China, Taiwan, the Philippines, or Mexico. It will take approximately 5 years to build the necessary infrastructure (roads, etc.), construct the new facility, and put it into operation. The eventual cost of the facility will differ between countries and will even vary within countries depending on the financial, labor, and political climate, including monetary exchange rates. The company has estimated the facility costs (in \$1,000,000s) in each country under three different future economic and political climates, as follows:

		Economic/Political Climate		
12.	Country	Decline	Same	Improvement
	South Korea	21.7	19.1	15.2
	China	19.0	18.5	17.6
	Taiwan	19.2	17.1	14.9
	Philippines	22.5	16.8	13.8
	Mexico	25.0	21.2	12.5

Determine the best decision, using the following decision criteria.

- 1.** Minimin
- 2.** Minimax
- 3.** Hurwicz ($\alpha = .4$)
- 4.** Equal likelihood

Place-Plus, a real estate development firm, is considering several alternative development projects. These include building and leasing an office park, purchasing a parcel of land and building office building to rent, buying and leasing a warehouse, building a strip mall, and building and selling condominiums. The financial success of these projects depends on interest rate movement in

the next 5 years. The various development projects and their 5-year financial return (in \$1,000,000s) given that interest rates will decline, remain stable, or increase, are shown in the following payoff table:

		Interest Rate		
Project		Decline	Stable	Increase
<u>13.</u>	Office park	\$0.5	\$1.7	\$4.5
	Office building	1.5	1.9	2.5
	Warehouse	1.7	1.4	1.0
	Mall	0.7	2.4	3.6

Condominiums	3.2	1.5	0.6
--------------	-----	-----	-----

Determine the best investment, using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Equal likelihood
- 4.** Hurwicz ($\alpha = .3$)

The Oakland Bombers professional basketball team just missed making the playoffs last season and believes it needs to sign only one very good free agent to make the playoffs next season. The team

is considering four players: Barry Byrd, Rayneal O'Neil, Marvin Johnson, and Michael Gordan. Each player differs according to position, ability, and attractiveness to fans. The payoffs (in \$1,000,000s) to the team for each player, based on the contract, profits from attendance, and team product sales for several different season outcomes, are provided in the following table:

		Season Outcome		
Player		Loser	Competitive	Make Playof
14.	Byrd	\$ 3.2	\$1.3	4.4
	O'Neil	5.1	1.8	6.3

Johnson	2.7	0.7	5.8
Gordan	6.3	1.6	9.6

[Page 557]

Determine the best decision, using the following decision criteria.

- 1.** Maximax
- 2.** Maximin
- 3.** Hurwicz ($\alpha = .60$)
- 4.** Equal likelihood

A machine shop owner is attempting to decide whether to purchase a new drill press, a lathe, or a grinder. The return from each will be determined by whether the company succeeds in getting a government military contract. The profit or loss from each purchase and the probabilities associated with each contract outcome are shown in the following payoff table:

<u>15.</u> Purchase	Contract .40	No Contract .60
Drill press	\$40,000	\$ -8,000
Lathe	20,000	4,000
Grinder	12,000	10,000

Compute the expected value for each purchase and select the best one.

A concessions manager at the Tech versus A&M football game must decide whether to have the vendors sell sun visors or umbrellas. There is a 30% chance of rain, a 15% chance of overcast skies, and a 55% chance of sunshine, according to the weather forecast in College Junction, where the game is to be held. The manager estimates that the following profits will result from each decision, given each set of weather conditions:

Weather Conditions

16. Decision	Rain .30	Overcast .15	Sunshir .55
Sun visors	\$ -500	\$ -200	\$1,500
Umbrellas	2,000	0	-900

- 1.** Compute the expected value for each decision and select the best one.
- 2.** Develop the opportunity loss table and compute the expected opportunity loss for each decision

Allen Abbott has a wide-curving, uphill

driveway leading to his garage. When there is a heavy snow, Allen hires a local carpenter, who shovels snow on the sidewalk in the winter, to shovel his driveway. The snow shoveler charges \$30 to shovel the driveway. Following is a probability distribution of the number of heavy snows each winter:

Heavy Snows	Probability
1	.13
2	.18
3	.26
4	.23

5

.10

6

.07

7

.03

1.00

[Page 558]

Allen is considering purchasing a new self-propelled snowblower for \$625 that would allow him, his wife, or his children to clear the driveway after a snow. Discuss what you think Allen's decision should be and why.

The Miramar Company is going to introduce one of three new products: a widget, a hummer, or a nimnot. The market conditions (favorable, stable, or unfavorable) will determine the profit or loss the company realizes, as shown in the following payoff table:

Market Conditions

Product	Favorable .2	Stable .7	Unfavorable .1
----------------	-----------------	--------------	-------------------

Widget	\$120,000	\$70,000	\$ 30,000
--------	-----------	----------	-----------

18.

Hummer	60,000	40,000	20,000
--------	--------	--------	--------

Nimnot	35,000	30,000	30,000
--------	--------	--------	--------

- 1.** Compute the expected value for each decision and select the best one.
- 2.** Develop the opportunity loss table and compute the expected opportunity loss for each product
- 3.** Determine how much the firm would be willing to pay to a market research firm to gain better information about future market conditions.

The financial success of the Downhill Ski Resort in the Blue Ridge Mountains is dependent on the amount of snowfall

during the winter months. If the snowfall averages more than 40 inches, the resort will be successful; if the snowfall is between 20 and 40 inches, the resort will receive a moderate financial return; and if the snowfall averages less than 20 inches, the resort will suffer a financial loss. The financial return and probability, given each level of snowfall, follow:

Snowfall Level (in.)	Financial Return
---------------------------------	-------------------------

19.

>40, .4	\$120,000
20-40, .2	40,000
<20, .4	40,000

A large hotel chain has offered to lease the resort for the winter for \$40,000. Compute the expected value to determine whether the resort should operate or lease. Explain your answer.

An investor must decide between two alternative investments: stocks and bonds. The return for each investment, given two future economic conditions, is shown in the following payoff table:

	Economic Conditions	
	Good	Bad
Investment		

Stocks

\$10,000 \$ 4,000

Bonds

7,000

2,000

What probability for each economic condition would make the investor indifferent to the choice between stock and bonds?

[Page 559]

In [Problem 10](#), Ann Tyler, with the help a financial newsletter and some library research, has been able to assign probabilities to each of the possible interest rates during the next year, as follows:

**Interest Rate
(%)**

Probability

5

.2

21.

6

.3

7

.3

8

.1

9

.1

Using expected value, determine her be

investment decision.

In [Problem 11](#) the Tech coaches have reviewed game films and have determined the following probabilities that State will use each of its defenses:

Defense	Probability
54	.40
63	.10
Wide tackle	.20
22. Nickel	.20

- 1.** Using expected value, rank Tech's plays from best to worst.
- 2.** During the actual game, a situation arises in which Tech has a third down and 10 yards to go, and the coaches are 60% certain State will blitz, with a 10% chance of any of the other four defenses. What play should Tech run? Is it likely the team will make the first down?

A global economist hired by Microcomputer, the U.S.-based computer manufacturer in [Problem 12](#), estimates that the probability that the economic and political climate overseas and in Mexico will

23. decline during the next 5 years is .40, the probability that it will remain approximately the same is .50, and the probability that it will improve is .10. Determine the best country to construct the new facility in and the expected value of perfect information.

In [Problem 13](#) the Place-Plus real estate development firm has hired an economist to assign a probability to each direction interest rates may take over the next 5 years. The economist has determined that there is a .50 probability that interest rates will decline, a .40 probability that

24. rates will remain stable, and a .10 probability that rates will increase.

1. Using expected value, determine the best project.
2. Determine the expected value of perfect information.

Fenton and Farrah Friendly, husband-and-wife car dealers, are soon going to open a new dealership. They have three offers from a foreign compact car company, from a U.S.-producer of full-sized cars, and from a truck company. The success of each type of dealership will depend on how much gasoline is going to be available during the next few years. The profit from each type of dealership, given the availability of gas, is shown in the following payoff table:

[Page 560]

25.

Gasoline Availability

Dealership	Shortage	Surplus
	.6	.4

Compact cars	\$300,000	\$150,0
Full-sized cars	100,000	600,0
Trucks	120,000	170,0

Determine which type of dealership the couple should purchase.

The Steak and Chop Butcher Shop purchases steak from a local meatpacking house. The meat is purchased on Monday at \$2.00 per pound, and the shop sells the steak for \$3.00 per pound. Any steak left over a

the end of the week is sold to a local z
for \$.50 per pound. The possible
demands for steak and the probability o
each are shown in the following table:

		Demand (lb.)	Probability
26.		20	.10
		21	.20
		22	.30
		23	.30
		24	.10

The shop must decide how much steak order in a week. Construct a payoff table for this decision situation and determine the amount of steak that should be ordered, using expected value.

The Loebuck Grocery must decide how many cases of milk to stock each week to meet demand. The probability distribution of demand during a week is shown in the following table:

Demand (cases)	Probability
15	.20

16	.25
----	-----

17	.40
----	-----

18	.15
----	-----

	1.00
--	------

27.

Each case costs the grocer \$10 and sells for \$12. Unsold cases are sold to a local farmer (who mixes the milk with feed for livestock) for \$2 per case. If there is a shortage, the grocer considers the cost of customer ill will and lost profit to be \$3 per case. The grocer must decide how many cases of milk to order each week.

- 1.** Construct the payoff table for this decision situation.
- 2.** Compute the expected value of each alternative amount of milk that could be stocked and select the best decision.
- 3.** Construct the opportunity loss table and determine the best decision.
- 4.** Compute the expected value of perfect information.

[Page 561]

The manager of the greeting card section of Mazey's department store is considering her order for a particular line of Christmas cards. The cost of each box of cards is \$3; each box will be sold for \$5 during the Christmas season. After

Christmas, the cards will be sold for \$2 box. The card section manager believes that all leftover cards can be sold at the price. The estimated demand during the Christmas season for the line of Christmas cards, with associated probabilities, is as follows:

		Demand (boxes)	Probability
28.		25	.10
		26	.15
		27	.30
		28	.20

29

.15

30

.10

- 1.** Develop the payoff table for this decision situation.
- 2.** Compute the expected value for each alternative and identify the best decision.
- 3.** Compute the expected value of perfect information.

The Palm Garden Greenhouse specializes in raising carnations that are sold to florists. Carnations are sold for \$3.00 per dozen; the cost of growing the carnations and distributing them to the

florists is \$2.00 per dozen. Any carnations left at the end of the day are sold to local restaurants and hotels for \$0.75 per dozen. The estimated cost of a customer ill will if demand is not met is \$1.00 per dozen. The expected daily demand (in dozens) for the carnations is as follows:

Daily Demand	Probability
20	.05
22	.10
24	.25
26	.30

28

.20

30

.10

1.00

- 1.** Develop the payoff table for this decision situation.
- 2.** Compute the expected value of each alternative number of (dozen) carnations that could be stocked and select the best decision.
- 3.** Construct the opportunity loss table and determine the best decision.

4. Compute the expected value of perfect information.

Assume that the probabilities of demand in [Problem 28](#) are no longer valid; the decision situation is now one without probabilities. Determine the best number of cards to stock, using the following decision criteria.

30.
 1. Maximin
 2. Maximax
 3. Hurwicz ($\alpha = .4$)
 4. Minimax regret

In [Problem 14](#), the Bombers'

management has determined the following probabilities of the occurrence each future season outcome for each player:

	Player	Probability		
		Loser	Competitive	Make Playof
31.	Byrd	.15	.55	.30
	O'Neil	.18	.26	.56
	Johnson	.21	.32	.47
	Gordan	.30	.25	.45

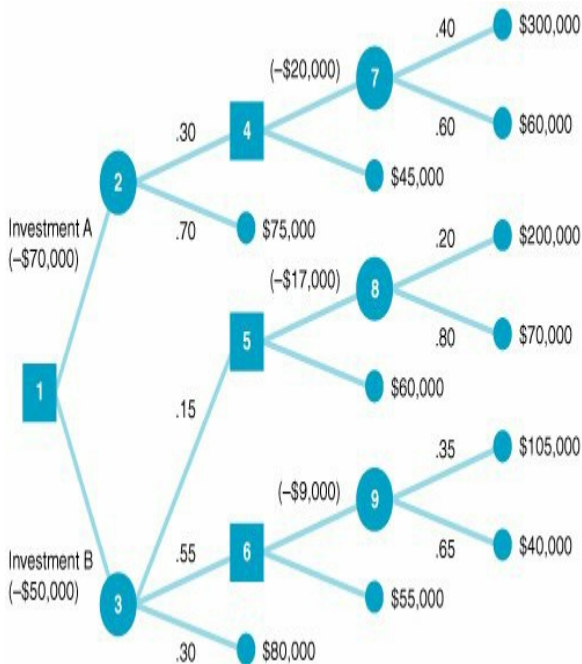
Compute the expected value for each player and indicate which player the team should try to sign.

Construct a decision tree for the decision situation described in [Problem 25](#) and indicate the best decision.

32.

Given the following sequential decision tree, determine which is the optimal investment, A or B:

33.



The management of First American Bar was concerned about the potential loss that might occur in the event of a phys

catastrophe such as a power failure or fire. The bank estimated that the loss from one of these incidents could be as much as \$100 million, including losses due to interrupted service and customer relations. One project the bank is considering is the installation of an emergency power generator at its operations headquarters. The cost of the emergency generator is \$800,000, and if it is installed, no losses from this type of incident will be incurred. However, if the generator is not installed, there is a 10% chance that a power outage will occur during the next year. If there is an outage there is a .05 probability that the resulting losses will be very large, or approximately \$80 million in lost earnings. Alternatively it is estimated that there is a .95 probability of only slight losses of around \$1 million. Using decision tree analysis, determine whether the bank should install the new power generator.

The Americo Oil Company is considering making a bid for a shale oil development contract to be awarded by the federal government. The company has decided to bid \$110 million. The company estimates that it has a 60% chance of winning the contract with this bid. If the firm wins the contract, it can choose one of three methods for getting the oil from the shale. It can develop a new method for oil extraction, use an existing (inefficient) process, or subcontract the processing to a number of smaller companies once the shale has been excavated. The results from these alternatives are as follows:

[Page 563]

Develop new process:

<i>Outcomes</i>	<i>Probability</i>	<i>Profit</i> (\$1,000,000)
Great success	.30	\$ 600
Moderate success	.60	300
Failure	.10	-100

35. Use present process:

<i>Outcomes</i>	<i>Probability</i>	<i>Profit</i> (\$1,000,000)
-----------------	--------------------	--------------------------------

Great success	.50	\$ 300
Moderate success	.30	200
Failure	.20	-40

Subcontract:

<i>Outcome</i>	<i>Probability</i>	<i>Profit</i> <i>(\$1,000,000)</i>
Moderate success	1.00	250

The cost of preparing the contract proposal is \$2 million. If the company does not make a bid, it will invest in an alternative venture with a guaranteed profit of \$30 million. Construct a sequential decision tree for this decision situation and determine whether the company should make a bid.

The machine shop owner in [Problem 15](#) is considering hiring a military consultant to ascertain whether the shop will get the government contract. The consultant is a former military officer who uses various personal contacts to find out such information. By talking to other shop owners who have hired the consultant,

- 36.** the owner has estimated a .70 probability that the consultant would present a favorable report, given that the contract is awarded to the shop, and a .80 probability that the consultant would present an unfavorable report, given that the contract is not awarded. Using decision tree analysis, determine the decision strategy the owner should follow, the expected value of this strategy, and the maximum fee the owner should pay the consultant.

The Miramar Company in [Problem 18](#) is considering contracting with a market research firm to do a survey to determine future market conditions. The results of the survey will indicate either positive or negative market conditions. There is a .60 probability of a positive report, given favorable conditions; a .30 probability of a positive report, given stable conditions.

and a .10 probability of a positive report given unfavorable conditions. There is a .90 probability of a negative report, given unfavorable conditions; a .70 probability of a positive report, given stable conditions; and a .40 probability, given favorable conditions. Using decision tree analysis *and* posterior probability tables, determine the decision strategy the company should follow, the expected value of the strategy, and the maximum amount the company should pay the market research firm for the survey results.

The Friendlys in [Problem 25](#) are considering hiring a petroleum analyst to determine the future availability of gasoline. The analyst will report that either a shortage or a surplus will occur. The probability that the analyst will indicate a shortage, given that a shortage actually occurs is .90; the probability th

38. the analyst will indicate a surplus, given that a surplus actually occurs is .70.

- 1.** Determine the decision strategy the Friendlys should follow, the expected value of this strategy, and the maximum amount the Friendlys should pay for the analyst's services.
- 2.** Compute the efficiency of the sample information for the Friendly car dealership.

Jeffrey Mogul is a Hollywood film producer, and he is currently evaluating a script by a new screenwriter and director, Betty Jo Thurston. Jeffrey knows that the probability of a film by a new director being a success is about .10 and that the probability it will flop is .90. The studio accounting department estimates that if this film is a hit, it will make \$25 million

39. profit, whereas if it is a box office failure it will lose \$8 million. Jeffrey would like to hire noted film critic Dick Roper to read the script and assess its chances of success. Roper is generally able to correctly predict a successful film 70% of the time and correctly predict an unsuccessful film 80% of the time. Roper wants a fee of \$1 million. Determine whether Roper should be hired, the strategy Mogul should follow if Roper is hired, and the expected value.

[Page 564]

Tech is playing State in the last conference game of the season. Tech is trailing State 21 to 14, with 7 seconds left in the game, when Tech scores a touchdown. Still trailing 21 to 20, Tech can either go for 2 points and win or go for 1 point to send the game into overtime. The conference championship

will be determined by the outcome of the game. If Tech wins, it will go to the Sugar Bowl, with a payoff of \$7.2 million; if it loses, it will go to the Gator Bowl, with a payoff of \$1.7 million. If Tech goes for 2 points, there is a 33% chance it will be successful and win (and a 67% chance it will fail and lose). If it goes for 1 point, there is a 0.98 probability of success and a tie and a 0.02 probability of failure. If the teams tie, they will play overtime, during which Tech believes it has only a 20% chance of winning because of fatigue.

- 40.**
1. Use decision tree analysis to determine whether Tech should go for 1 or 2 points.
 2. What would Tech's probability of winning the game in overtime have to be to make Tech indifferent to going for either 1 or 2 points?

Jay Seago is suing the manufacturer of his car for \$3.5 million because of a defect that he believes caused him to have an accident. The accident kept him out of work for a year. The company has offered him a settlement of \$700,000, which Jay would receive \$600,000 after attorneys' fees. His attorney has advised him that he has a 50% chance of winning his case. If he loses, he will incur attorneys' fees and court costs of \$75,000. If he wins, he is not guaranteed his full requested settlement. His attorney believes that there is a 50% chance he could receive the full settlement, in which case Jay would realize \$2 million after his attorney takes her cut, and a 50% chance that the jury will award him a lesser amount of \$1 million, of which Jay would get \$500,000.

41.

Using decision tree analysis, decide

whether Jay should proceed with his lawsuit against the manufacturer.

Tech has three health care plans for its faculty and staff to choose from, as follows:

Plan 1 monthly cost of \$32, with a \$500 deductible; the participants pay the first \$500 of medical costs for the year; the insurer pays 90% of all remaining expenses.

Plan 2 monthly cost of \$5 but a deductible of \$1,200, with the insurer paying 90% of medical expenses after the insurer pays the first \$1,200 in a year.

Plan 3 monthly cost of \$24, with no deductible; the participants pay 30% of expenses, with the remainder paid by the insurer.

Tracy McCoy, an administrative assistant in the management science department estimates that her annual medical expenses are defined by the following probability distribution:

42.

Annual Medical Expenses	Probability
--------------------------------	--------------------

\$ 100	.15
--------	-----

500	.30
-----	-----

1,500	.35
-------	-----

3,000	.10
-------	-----

5,000	.05
-------	-----

10,000

.05

[Page 565]

Determine which medical plan Tracy should select.

The Valley Wine Company purchases grapes from one of two nearby growers each season to produce a particular red wine. It purchases enough grapes to produce 3,000 bottles of the wine. Each grower supplies a certain portion of poor quality grapes, resulting in a percentage of bottles being used as fillers for cheap table wines, according to the following probability distribution:

Defective (%)	Probability of Percentage Defective	
	Grower A	Grower B
2	.15	.30
4	.20	.30
6	.25	.20
8	.30	.10
10	.10	.10

43.

The two growers charge different prices for their grapes and, because of differences in taste, the company charges different prices for its wine, depending on which grapes it uses. Following is the annual profit from the wine produced from each grower's grapes for each percentage defective:

Defective (%)	Profit	
	Grower A	Grower B
2	\$44,200	\$42,600
4	40,200	40,300

6	36,200	38,000
8	32,200	35,700
10	28,200	33,400

Use decision tree analysis to determine from which grower the company should purchase grapes.

Kroft Food Products is attempting to decide whether it should introduce a new line of salad dressings called Special Choices. The company can test market the salad dressings in selected geograph

areas or bypass the test market and introduce the product nationally. The cost of the test market is \$150,000. If the company conducts the test market, it must wait to see the results before deciding whether to introduce the salad dressings nationally. The probability of a positive test market result is estimated to be 0.6. Alternatively, the company can decide not to conduct the test market and go ahead and make the decision to introduce the dressings or not. If the salad dressings are introduced nationally and are a success, the company estimates that it will realize an annual profit of \$1.6 million, whereas if the dressings fail, it will incur a loss of \$700,000. The company believes the probability of success for the salad dressings is 0.50 if they are introduced without the test market. If the company does conduct the test market and it is positive, then the probability of

44.

successfully introducing the salad dressings increases to 0.8. If the test market is negative and the company introduces the salad dressings anyway, the probability of success drops to 0.30.

Using decision tree analysis, determine whether the company should conduct the test market.

In [Problem 44](#), determine the expected value of sample information (*EVSI*) (i.e. the test market value) and the expected value of perfect information (*EVPI*).

45.

Ellie Daniels has \$200,000 and is considering three mutual funds for investment: a global fund, an index fund, and an Internet stock fund. During the first year of investment, Ellie estimates that there is a .70 probability that the

market will go up and a .30 probability that the market will go down. Following are the returns on her \$200,000 investment at the end of the year under each market condition:

[Page 566]

Fund	Market Conditions	
	Up	Down
Global	\$25,000	\$ 8,000
Index	35,000	5,000
Internet	60,000	35,000

-
- 46.** At the end of the first year, Ellie will either reinvest the entire amount plus the return or sell and take the profit or loss. If she reinvests, she estimates that there is a .60 probability the market will go up and a .40 probability the market will go down. If Ellie reinvests in the global fund after the market has gone up, her return on her initial \$200,000 investment plus her \$25,000 return after 1 year will be \$45,000. If the market goes down, her loss will be \$15,000. If she reinvests after the market has gone down, her return will be \$34,000, and her loss will be \$17,000. Ellie reinvests in the index fund after the market has gone up, after 2 years her return will be \$65,000 if the market continues upward but only \$5,000 if the market goes down. Her return will be \$55,000 if she reinvests and the market

reverses itself and goes up after initially going down, and it will be \$5,000 if the market continues to go down. If Ellie invests in the Internet fund, she will make \$60,000 if the market goes up, but she will lose \$35,000 if it goes down. If she reinvests as the market continues upward, she will make an additional \$100,000; but if the market reverses and goes down, she will lose \$70,000. If she reinvests after the market has initially gone down, she will make \$65,000, but if the market continues to go down, she will lose an additional \$75,000.

Using decision tree analysis, determine which fund Ellie should invest in and its expected value.

Blue Ridge Power and Light is an electric utility company with a large fleet of vehicles, including automobiles, light trucks, and construction equipment. The

company is evaluating four alternative strategies for maintaining its vehicles at the lowest cost: (1) do no preventive maintenance at all and repair vehicle components when they fail; (2) take oil samples at regular intervals and perform whatever preventive maintenance is indicated by the oil analysis; (3) change the vehicle oil on a regular basis and perform repairs when needed; (4) change the oil at regular intervals, take oil samples regularly, and perform maintenance repairs as indicated by the sample analysis.

For autos and light trucks, strategy 1 (no preventive maintenance) costs nothing to implement and results in two possible outcomes: There is a .10 probability that a defective component will occur, requiring emergency maintenance at a cost of \$1,200, or there is a .90 probability that no defects will occur and no maintenance will be necessary.

Strategy 2 (take oil samples) costs \$200 to implement (i.e., take a sample), and there is a .10 probability that there will be a defective part and .90 probability that there will not be a defect. If there is actually a defective part, there is a .70 probability that the sample will correctly identify it, resulting in preventive maintenance at a cost of \$500. However, there is a .30 probability that the sample will not identify the defect and indicate that everything is okay, resulting in emergency maintenance later at a cost of \$1,200. On the other hand, if there are actually no defects, there is a .20 probability that the sample will erroneously indicate that there is a defect, resulting in unnecessary maintenance at a cost of \$250. There is an .80 probability that the sample will correctly indicate that there are no defects, resulting in no maintenance and no costs.

47.

Strategy 3 (changing the oil regularly) costs \$14.80 to implement and has two outcomes: a .04 probability of a defect component, which will require emergency maintenance at a cost of \$1,200, and a .96 probability that no defects will occur, resulting in no maintenance and no cost.

[Page 567]

Strategy 4 (changing the oil and sampling) costs \$34.80 to implement and results in the same probabilities of defects and no defects as strategy 3. If there is a defective component, there is a .70 probability that the sample will detect it and \$500 in preventive maintenance costs will be incurred. Alternatively, there is a .30 probability that the sample will not detect the defect, resulting in emergency maintenance at a cost of \$1,200. If there is no defect, there is a .20 probability that the sample will

indicate that there is a defect, resulting in an unnecessary maintenance cost of \$250, and there is an .80 probability that the sample will correctly indicate no defects, resulting in no cost.

Develop a decision strategy for Blue Ridge Power and Light and indicate the expected value of this strategy. [\[1\]](#)

[1] This problem is based on J. Mellichamp, D. Miller, and O.-J. Kwon, "The Southern Company Uses a Probability Model for Cost Justification of Oil Sample Analysis," *Interfaces* 23, no. 3 (May/June 1993): 11824.

In [Problem 47](#), the decision analysis is for automobiles and light trucks. Blue Ridge Power and Light would like to reformulate the problem for its heavy construction equipment. Emergency maintenance is much more expensive for heavy equipment, costing \$15,000. Required

48. preventive maintenance costs \$2,000, and unnecessary maintenance costs \$1,200. The cost of an oil change is \$100, and the cost of taking an oil sample and analyzing it is \$30. All the probabilities remain the same. Determine the strategy Blue Ridge Power and Light should use for its heavy equipment.

In [Problem 14](#), the management of the Oakland Bombers is considering hiring superscout Jerry McGuire to evaluate the team's chances for the coming season. McGuire will evaluate the team, assuming that it will sign one of the four free agents. The team's management has determined the probability of the team having a losing record with any of the four agents to be .21 by averaging the probabilities of losing for the four free agents in [Problem 31](#). The probability of the team having a competitive season

but not making the playoffs is developed similarly, and it is .35. The probability of the team making the playoffs is .44. The probability that McGuire will correctly predict that the team will have a losing season is .75, whereas the probability that he predicts a competitive season, given that it has a losing season, is .15 and the probability that he incorrectly predicts a playoff season, given that the team has a losing season, is .10. The probability that he successfully predicts a competitive season is .80, whereas the probability that he incorrectly predicts a losing season, given that the team is competitive, is .10, and the probability that he incorrectly predicts a playoff season, given the team has a competitive season, is .10. The probability that he correctly predicts a playoff season is .80, whereas the probability that he incorrectly predicts a losing season, given that the team makes the playoffs, is .05, and the

49.

probability that he predicts a competitive season, given the team makes the playoffs, is .10. Using decision tree analysis and posterior probabilities, determine the decision strategy the team should follow, the expected value of the strategy, and the maximum amount the team should pay for Jerry McGuire's predictions.

The Place-Plus real estate development firm in [Problem 24](#) is dissatisfied with the economist's estimate of the probabilities of future interest rate movement, so it is considering having a financial consulting firm provide a report on future interest rates. The consulting firm is able to cite a track record which shows that 80% of the time when interest rates had declined it had predicted they would, while 10% of the time when interest rates had declined the firm had predicted that they would

50.

remain stable and 10% of the time it has predicted that they would increase. The firm has been correct 70% of the time when rates have remained stable, whereas 10% of the time it has incorrectly predicted that rates would decrease, and 20% of the time it has incorrectly predicted that rates would increase. The firm has correctly predicted that interest rates would increase 90% of the time, whereas incorrectly predicting rates would decrease 2% and remain stable 8% of the time. Assuming that the consulting firm could supply an accurate report, determine how much Place-Plus should be willing to pay the consulting firm and how efficient the information would be.

A young couple has \$5,000 to invest in either savings bonds or a real estate deal. The expected return on each investment given good and bad economic conditions is shown in the following payoff table:

	Economic Condition	
	Good .6	Bad .4
Investment		
51. Savings bonds	\$ 1,000	\$1,000
Real estate	10,000	2,000

The expected value of investing in savin

bonds is \$1,000, and the expected value of the real estate investment is \$5,200. However, the couple decides to invest in savings bonds. Explain the couple's decision in terms of the utility they might associate with each investment.

Annie Hays recently sold a condominium she had bought and lived in while she was a college student over 15 years ago. She received \$150,000 for the condominium and is considering two investment alternatives. Annie can invest the entire amount in a bank money market for 1 year at 8% interest and thus receive \$162,000 at the end of a year, or she can invest in a speculative oil exploration project with a 50/50 chance of doubling her investment at the end of the year or losing everything.

- 1.** Determine which alternative Annie should invest in, according to the

expected value criterion, and indicate whether that is the alternative you would also select.

52.

2. Suppose Annie determined that the probability of success for the oil exploration investment would have to be .80 before she would be indifferent between the two alternatives. In other words, if the probability of success for the oil exploration investment is less than .80, Annie would invest in the money market, but if the probability of success is greater than .80, she would invest in the oil exploration. In this case, .80 is the utility value of the \$162,000 safe return. Determine the preferred alternative according to the expected utility value.

Case Problem

STEELEY ASSOCIATES VERSUS CONCORD FALLS

Steeley Associates, Inc., a property development firm, purchased an old house near the town square in Concord Falls, where State University is located. The old house was built in the mid-1800s, and Steeley Associates restored it. For almost a decade, Steeley has leased it to the university

for academic office space. The house is located on a wide lawn and has become a town landmark.

However, in 2003, the lease with the university expired, and Steeley Associates decided to build high-density student apartments on the site, using all the open space. The community was outraged and objected to the town council. The legal counsel for the town spoke with a representative from Steeley and hinted that if Steeley requested a permit, the town would probably reject it.

Steeley had reviewed the town building code and felt confident that its plan was within the guidelines, but that did not necessarily mean that it could win a lawsuit against the town to force the town to grant a permit.

The principals at Steeley Associates held a series of meetings to review their alternatives. They decided that they had three options: They could request the permit, they could sell the property, or they could request a permit for a low-density office building,

which the town had indicated it would not fight. Regarding the last two options, if Steeleley sells the house and property, it thinks it can get \$900,000. If it builds a new office building, its return will depend on town business growth in the future. It feels that there is a 70% chance of future growth, in which case Steeleley will see a return of \$1.3 million (over a 10-year planning horizon); if no growth (or erosion) occurs, it will make only \$200,000.

If Steeleley requests a permit for the apartments, a host of good

and bad outcomes are possible. The immediate good outcome is approval of its permit, which it estimates will result in a return of \$3 million. However, Steeley gives that result only a 10% chance of occurring.

Alternatively, Steeley thinks there is a 90% chance that the town will reject its application, which will result in another set of decisions.

Steeley can sell the property at that point. However, the rejection of the permit would undoubtedly decrease the value to potential buyers, and

Steeley estimates that it would get only \$700,000.

Alternatively, it could construct the office building and face the same potential outcomes it did earlier, namely, a 30% chance of no town growth and a \$200,000 return or a 70% chance of growth with a return of \$1.3 million. A third option is to sue the town. On the surface, Steeley's case looks good, but the town building code is vague, and a sympathetic judge could throw out its suit. Whether or not it wins, Steeley estimates its

possible legal fees to be \$300,000, and it feels it has only a 40% chance of winning. However, if Steeley does win, it estimates that the award will be approximately \$1 million, and it would also get its \$3 million return for building the apartments. Steeley also estimates that there is a 10% chance that the suit could linger on in the courts for such a long time that any future return would be negated during its planning horizon, and it would incur an additional \$200,000 in legal fees.

If Steeley loses the suit, it would then be faced with the same options of selling the property or constructing an office building. However, if the suit is carried this far into the future, it feels that the selling price it can ask will be somewhat dependent on the town's growth prospects at that time, which it feels it can estimate at only 5050. If the town is in a growth mode that far in the future, Steeley thinks that \$900,000 is a conservative

estimate of the potential sale price, whereas if the town is not growing, it thinks \$500,000 is a more likely estimate.

Finally, if Steeley constructs the office building, it feels that the chance of town growth is 50%, in which case the return would be only \$1.2 million. If no growth occurs, it conservatively estimates only a \$100,000 return.

- 1.** Perform a decision tree analysis of Steeley Associates's decision situation, using expected value, and indicate the

appropriate decision with these criteria.

- 2.** Indicate the decision you would make and explain your reasons.

Case Problem

TRANSFORMER REPLACEMENT AT MOUNTAIN STATES ELECTRIC SERVICE

Mountain States Electric Service is an electrical utility company serving several states in the Rocky Mountains region. It is considering replacing some of its equipment at a generating substation and is attempting to decide whether it should replace an older, existing PCB transformer. (PCB

is a toxic chemical known formally as polychlorinated biphenyl.) Even though the PCB generator meets all current regulations, if an incident occurred, such as a fire, and PCB contamination caused harm either to neighboring businesses or farms or to the environment, the company would be liable for damages. Recent court cases have shown that simply meeting utility regulations does not relieve a utility of liability if an incident causes harm to others. Also, courts have been

awarding large damages to individuals and businesses harmed by hazardous incidents.

If the utility replaces the PCB transformer, no PCB incidents will occur, and the only cost will be the cost of the transformer, \$85,000. Alternatively, if the company decides to keep the existing PCB transformer, then management estimates that there is a 50/50 chance of there being a high likelihood of an incident or a low likelihood of an incident. For the case in which there is a high likelihood that an incident will occur,

there is a .004 probability that a fire will occur sometime during the remaining life of the transformer and a .996 probability that no fire will occur. If a fire occurs, there is a .20 probability that it will be bad and the utility will incur a very high cost of approximately \$90 million for the cleanup, whereas there is an .80 probability that the fire will be minor and cleanup can be accomplished at a low cost of approximately \$8 million. If no fire occurs, no cleanup costs will occur. For the case in which

there is a low likelihood of an incident occurring, there is a .001 probability that a fire will occur during the life of the existing transformer and a .999 probability that a fire will not occur. If a fire does occur, the same probabilities exist for the incidence of high and low cleanup costs, as well as the same cleanup costs, as indicated for the previous case. Similarly, if no fire occurs, there is no cleanup cost.

Perform a decision tree analysis of this problem for Mountain States Electric Service and

indicate the recommended solution. Is this the decision you believe the company should make? Explain your reasons. [\[2\]](#)

[2] This case was adapted from W. Balson, J. Welsh, and D. Wilson, "Using Decision Analysis and Risk Analysis to Manage Utility Environmental Risk," *Interfaces* 22, no. 6 (November/December 1992): 12639.

Case Problem

THE CAROLINA COUGARS

The Carolina Cougars is a major league baseball expansion team beginning its third year of operation. The team had losing records in each of its first 2 years and finished near the bottom of its division. However, the team was young and generally competitive. The team's general manager, Frank Lane, and manager, Biff

Diamond, believe that with a few additional good players, the Cougars can become a contender for the division title and perhaps even for the pennant. They have prepared several proposals for free-agent acquisitions to present to the team's owner, Bruce Wayne.

Under one proposal the team would sign several good available free agents, including two pitchers, a good fielding shortstop, and two power-hitting outfielders for \$52 million in bonuses and annual salary. The second proposal is

less ambitious, costing \$20 million to sign a relief pitcher, a solid, good-hitting infielder, and one power-hitting outfielder. The final proposal would be to stand pat with the current team and continue to develop.

[Page 570]

General Manager Lane wants to lay out a possible season scenario for the owner so he can assess the long-run ramifications of each decision strategy. Because the only thing the owner understands is

money, Frank wants this analysis to be quantitative, indicating the money to be made or lost from each strategy. To help develop this analysis, Frank has hired his kids, Penny and Nathan, both management science graduates from Tech.

Penny and Nathan analyzed league data for the previous five seasons for attendance trends, logo sales (i.e., clothing, souvenirs, hats, etc.), player sales and trades, and revenues. In addition, they interviewed several other

owners, general managers, and league officials. They also analyzed the free agents that the team was considering signing.

Based on their analysis, Penny and Nathan feel that if the Cougars do not invest in any free agents, the team will have a 25% chance of contending for the division title and a 75% chance of being out of contention most of the season. If the team is a contender, there is a .70 probability that attendance will increase as the season progresses and the

team will have high attendance levels (between 1.5 million and 2.0 million) with profits of \$170 million from ticket sales, concessions, advertising sales, TV and radio sales, and logo sales. They estimate a .25 probability that the team's attendance will be mediocre (between 1.0 million and 1.5 million) with profits of \$115 million and a .05 probability that the team will suffer low attendance (less than 1.0 million) with profit of \$90 million. If the team is not a contender, Penny and Nathan

estimate that there is .05 probability of high attendance with profits of \$95 million, a .20 probability of medium attendance with profits of \$55 million, and a .75 probability of low attendance with profits of \$30 million.

If the team marginally invests in free agents at a cost of \$20 million, there is a 50/50 chance it will be a contender. If it is a contender, then later in the season it can either stand pat with its existing roster or buy or trade for players that could improve the team's chances of

winning the division. If the team stands pat, there is a .75 probability that attendance will be high and profits will be \$195 million. There is a .20 probability that attendance will be mediocre with profits of \$160 million and a .05 probability of low attendance and profits of \$120 million. Alternatively, if the team decides to buy or trade for players, it will cost \$8 million, and the probability of high attendance, with profits of \$200 million will be .80. The probability of mediocre

attendance with \$170 million in profits will be .15, and there will be a .05 probability of low attendance, with profits of \$125 million.

If the team is not in contention, then it will either stand pat or sell some of its players, earning approximately \$8 million in profit. If the team stands pat, there is a .12 probability of high attendance, with profits of \$110 million; a .28 probability of mediocre attendance, with profits of \$65 million; and a .60 probability of low attendance, with profits of \$40 million. If

the team sells players, the fans will likely lose interest at an even faster rate, and the probability of high attendance with profits of \$100 million will drop to .08, the probability of mediocre attendance with profits of \$60 million will be .22, and the probability of low attendance with profits of \$35 million will be .70.

The most ambitious free-agent strategy will increase the team's chances of being a contender to 65%. This strategy will also excite the fans most during the off-season

and boost ticket sales and advertising and logo sales early in the year. If the team does contend for the division title, then later in the season it will have to decide whether to invest in more players. If the Cougars stand pat, the probability of high attendance with profits of \$210 million will be .80, the probability of mediocre attendance with profits of \$170 million will be .15, and the probability of low attendance with profits of \$125 million will be .05. If the team buys players at a cost of \$10

million, then the probability of having high attendance with profits of \$220 million will increase to .83, the probability of mediocre attendance with profits of \$175 million will be .12, and the probability of low attendance with profits of \$130 million will be .05.

If the team is not in contention, it will either sell some players' contracts later in the season for profits of around \$12 million or stand pat. If it stays with its roster, the probability of high attendance with profits of \$110 million will be .15, the

probability of mediocre attendance with profits of \$70 million will be .30, and the probability of low attendance with profits of \$50 million will be .55. If the team sells players late in the season, there will be a .10 probability of high attendance with profits of \$105 million, a .30 probability of mediocre attendance with profits of \$65 million, and a .60 probability of low attendance with profits of \$45 million.

Assist Penny and Nathan in determining the best strategy to follow and its expected

value.

Case Problem

EVALUATING R&D PROJECTS AT WESTCOM SYSTEMS PRODUCTS COMPANY

WestCom Systems Products Company develops computer systems and software products for commercial sale. Each year it considers and evaluates a number of different R&D projects to undertake. It develops a road map for each project, in the form of a

standardized decision tree that identifies the different decision points in the R&D process from the initial decision to invest in a project's development through the actual commercialization of the final product.

The first decision point in the R&D process is whether to fund a proposed project for 1 year. If the decision is no, then there is no resulting cost; if the decision is yes, then the project proceeds at an incremental cost to the company. The company establishes specific short-term,

early technical milestones for its projects after 1 year. If the early milestones are achieved, the project proceeds to the next phase of project development; if the milestones are not achieved, the project is abandoned. In its planning process the company develops probability estimates of achieving and not achieving the early milestones. If the early milestones are achieved, the project is funded for further development during an extended time frame specific to a project. At the end of this

time frame, a project is evaluated according to a second set of (later) technical milestones. Again, the company attaches probability estimates for achieving and not achieving these later milestones. If the later milestones are not achieved, the project is abandoned.

[Page 571]

If the later milestones are achieved, technical uncertainties and problems have been overcome, and the

company next assesses the project's ability to meet its strategic business objectives. At this stage, the company wants to know if the eventual product coincides with the company's competencies and whether there appears to be an eventual, clear market for the product. It invests in a product "prelaunch" to ascertain the answers to these questions. The outcomes of the prelaunch are that either there is a strategic fit or there is not, and the company assigns probability estimates to each of

these two possible outcomes. If there is not a strategic fit at this point, the project is abandoned and the company loses its investment in the prelaunch process. If it is determined that there is a strategic fit, then three possible decisions result: (1) The company can invest in the product's launch, and a successful or unsuccessful outcome will result, each with an estimated probability of occurrence; (2) the company can delay the product's launch and at a later date decide

whether to launch or abandon; and (3) if it launches later, the outcomes are success or failure, each with an estimated probability of occurrence. Also, if the product launch is delayed, there is always a likelihood that the technology will become obsolete or dated in the near future, which tends to reduce the expected return.

The following table provides the various costs, event probabilities, and investment outcomes for five projects the company is considering:

<i>Decision Outcomes/Event</i>	<i>1</i>	<i>2</i>
Fund1 year	\$ 200,000	\$ 350,000
$P(\text{Early milestonesyes})$	.70	.67
$P(\text{Early milestonesno})$	.30	.33
Long-term funding	\$ 650,000	780,000
$P(\text{Late$		

milestonesyes)	.60	.56
$P(\text{Late milestonesno})$	.40	.44
Prelaunch funding	\$ 300,000	450,000
$P(\text{Strategic fityes})$	.80	.75
$P(\text{Strategic fitno})$	.20	.25
$P(\text{Investsuccess})$	.60	.65
$P(\text{Investfailure})$	.40	.35

$P(\text{Delay} \text{success})$	.80	.70	
$P(\text{Delay} \text{failure})$	.20	.30	
Invest success	\$7,300,000	8,000,000	4
Invest failure	2,000,000	3,500,000	1
Delay success	4,500,000	6,000,000	3
Delay failure	1,300,000	4,000,000	

Determine the expected value

for each project and then rank the projects accordingly for the company to consider. [\[3\]](#)

[3] This case was adapted from R. K. Perdue, W. J. McAllister, P. V. King, and B. G. Berkey, "Valuation of R and D projects Using Options Pricing and Decision Analysis Models," *Interfaces* 29, no. 6 (November/December 1999): 5774.

Chapter 13. Queuing Analysis

[Page 573]

Waiting in queues is one of the most common occurrences in everyone's life. Anyone who has gone shopping or to a movie has experienced the inconvenience of waiting in line to make purchases or buy a ticket. Not only do people spend a significant portion of their time waiting in lines, but products queue up in

production plants, machinery waits in line to be serviced, planes wait to take off and land, and so on. Because time is a valuable resource, the reduction of waiting time is an important topic of analysis.

The improvement of service with respect to waiting time has also become more important in recent years because of the increased emphasis on quality, especially in service-related operations. When customers go into a bank to take out a loan, cash a check, or make a deposit; take

their car into a dealer for service or repair; or shop at the grocery store, they increasingly equate quality service with rapid service. Aware of this, more and more companies are focusing on reducing waiting time as an important component of quality improvement. In general, companies are able to reduce waiting time and provide faster service by increasing their service capacity, which usually means adding more servers, such as more tellers at a bank, more mechanics at a car

dealership, or more checkout clerks at a grocery store. However, increasing service capacity in this manner has a monetary cost, and therein lies the basis of waiting line analysis: the trade-off between the cost of improved service and the cost of making customers wait.

Providing quick service is an important aspect of quality customer service.

Like decision analysis, **queuing analysis** is a probabilistic form of analysis, not a deterministic technique. Thus, the results of queuing analysis, referred to as operating characteristics, are probabilistic. These operating statistics (such as the average time a person must wait in line to be served) are used by the manager of the operation containing the queue to make decisions.

A number of different queuing

models exist to deal with different queuing systems. We will eventually discuss many of these queuing variations, but we will concentrate on two of the most common types of systems: the single-server system and the multiple-server system.

Elements of Waiting Line Analysis

Waiting lines form because people or things arrive at the servicing function, or server, faster than they can be served. However, this does not mean that the service operation is understaffed or does not have the overall capacity to handle the influx of customers. In fact, most businesses and organizations have sufficient serving capacity available to

handle their customers *in the long run*. Waiting lines result because customers do not arrive at a constant, evenly paced rate, nor are they all served in an equal amount of time. Customers arrive at random times, and the time required to serve them individually is not the same. Thus, a waiting line is continually increasing and decreasing in length (and is sometimes empty), and it approaches an average rate of customer arrivals and an average time to serve the

customer in the long run. For example, the checkout counters at a grocery store may have enough clerks to serve an average of 100 customers in an hour, and in any particular hour only 60 customers might arrive. However, at specific points in time during the hour, waiting lines may form because more than an average number of customers arrive, and they make more than an average number of purchases.

Operating characteristics are

average values for characteristics that describe the performance of a waiting line system.

Decisions about waiting lines and the management of waiting lines are based on these averages for customer arrivals and service times. They are used in queuing formulas to compute **operating characteristics**, such as the

average number of customers waiting in line and the average time a customer must wait in line. Different sets of formulas are used, depending on the type of waiting line system being investigated. For example, a bank drive-up teller window that has one bank clerk serving a single line of customers in cars is different from a single line of passengers at an airport ticket counter that is served by three or four airline agents. In the next section we present the different elements and components that

make up waiting lines before we look at queuing formulas in the following sections.

[Page 574]

The Single-Server Waiting Line System

A single server with a single waiting line is the simplest form of queuing system. As such, it will be used to demonstrate the fundamentals of a queuing system. As an example of this kind of system, consider Fast Shop Market.

Fast Shop Market has one checkout counter and one employee who operates the

cash register at the checkout counter. The combination of the cash register and the operator is the *server* (or service facility) in this queuing system; the customers who line up at the counter to pay for their selections form the *waiting line*, or **queue**. The configuration of this example queuing system is shown in [Figure 13.1](#).

Components of a waiting line system include arrivals, servers, and the waiting line structure.

Figure 13.1. The Fast Shop Market waiting line system

[\[View full size image\]](#)



The most important factors to consider in analyzing a queuing system such as the one in [Figure 13.1](#) are the following:

- 1.** The queue discipline (in what order customers are served)
- 2.** The nature of the calling population (where customers come from)

3. The arrival rate (how often customers arrive at the queue)
4. The service rate (how fast customers are served)

We will discuss each of these items as it relates to our example.

The Queue Discipline

The **queue discipline** is the order in which waiting customers are served.
Customers at Fast Shop Market

are served on a "first-come, first-served" basis. That is, the first person in line at the checkout counter is served first. This is the most common type of queue discipline. However, other disciplines are possible. For example, a machine operator might stack in-process parts beside a machine so that the last part is on top of the stack and will be selected first. This queue discipline is referred to as "last-in, first-out." Or the machine operator might simply reach into a box full of parts and

select one at random. In this case, the queue discipline is random. Often customers are scheduled for service according to a predetermined appointment, such as patients at a doctor's or dentist's office or diners at a restaurant where reservations are required. In this case, the customers are taken according to a prearranged schedule, regardless of when they arrive at the facility. One final example of the many types of queue disciplines is when customers are processed

alphabetically according to their last names, such as at school registration or at job interviews.

*The **queue discipline** is the order in which waiting customers are served.*

The Calling Population

The **calling population** is the source of the customers to the market, which in this case is assumed to be infinite. In other words, there is such a large number of possible customers in the area where the store is located that the number of potential customers is assumed to be infinite. Some queuing systems have finite calling populations. For example, the repair garage of a trucking firm that has 20 trucks has a finite calling population. The queue is the number of trucks waiting to be repaired, and the finite

calling population is the 20 trucks. However, queuing systems that have an assumed infinite calling population are more common.

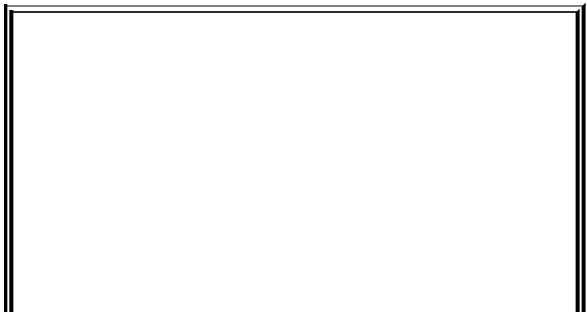
*The **calling population** is the source of customers; it may be infinite or finite.*

The Arrival Rate

The **arrival rate** is the rate at which customers arrive at the service facility during a specified period of time. This rate can be estimated from empirical data derived from studying the system or a similar system, or it can be an average of these empirical data. For example, if 100 customers arrive at a store checkout counter during a 10-hour day, we could say the arrival rate averages 10 customers per hour. However, although we might be able to determine a rate for arrivals by

counting the number of paying customers at the market during a 10-hour day, we would not know exactly when these customers would arrive on the premises. In other words, it might be that no customers would arrive during one hour and 20 customers would arrive during another hour. In general, these arrivals are assumed to be independent of each other and to vary randomly over time.

The **arrival rate** is the frequency at which customers arrive at a waiting line according to a probability distribution.



Time Out: For Agner Krarup Erlang

(This item is displayed on page 575 in
the print version)

The origin of queuing theory is found in telephone network congestion problems and the work of A. K. Erlang. Erlang (1878-1929), a Danish mathematician, was scientific adviser for the Copenhagen Telephone Company. In 1917 he published a paper outlining the development of telephone traffic theory in which he was able to determine the probability of the different numbers of calls waiting and of the waiting time when the system was in equilibrium. He assumed Poisson inputs (arrivals) from unlimited sources and

exponential holding times. Erlang's work provided the stimulus and formed the basis for the subsequent development of queuing theory.

Given these assumptions, it is further assumed that arrivals at a service facility conform to some probability distribution. Although arrivals could be

described by any distribution, it has been determined (through years of research and the practical experience of people in the field of queuing) that the number of arrivals per unit of time at a service facility can frequently be defined by a

Poisson distribution.

([Appendix C](#) at the end of this text contains a more detailed presentation of the Poisson distribution.)

*The arrival rate (λ)
is most frequently
described by a*

Poisson distribution.

The Service Rate

The **service rate** is the average number of customers who can be served during a specified period of time. For our Fast Shop Market example, 30 customers can be checked out (served) in 1 hour. A service rate is similar to an arrival rate

in that it is a random variable. In other words, such factors as different sizes of customer purchases, the amount of change the cashier must count out, and different forms of payment alter the number of persons that can be served over time. Again, it is possible that only 10 customers might be checked out during one hour and 40 customers might be checked out during the following hour.

*The **service rate** is the average number*

of customers who can be served during a time period.

The description of arrivals in terms of a *rate* and of service in terms of *time* is a convention that has developed in queuing theory. Like arrival rate, service time is assumed to be defined by a probability distribution. It has been determined by researchers in the field of queuing that service

times can frequently be defined by a *negative exponential probability* distribution.

([Appendix C](#) at the end of this text contains a more detailed presentation of the exponential distribution.) However, to analyze a queuing system, both arrivals and service must be in compatible units of measure. Thus, service time must be expressed as a service rate to correspond with an arrival rate.

The service time can often be described by the negative

*exponential
distribution.*

The Single-Server Model

The Fast Shop Market checkout counter is an example of a single-server queuing system with the following characteristics:

- 1.** An infinite calling population

2. A first-come, first-served queue discipline
3. Poisson arrival rate
4. Exponential service times

Assumptions of the basic single-server model.

These assumptions have been used to develop a model of a single-server queuing system.

However, the analytical derivation of even this simplest queuing model is relatively complex and lengthy. Thus, we will refrain from deriving the model in detail and will consider only the resulting queuing formulas. The reader must keep in mind, however, that these formulas are applicable only to queuing systems having the foregoing conditions.

Given that

λ = the arrival rate (average number of arrivals per time

period)

μ = the service rate (average number served per time period)

λ = mean arrival rate;

μ = mean service rate.

and that $\lambda < \mu$ (customers are served at a faster rate than they arrive), we can state the

following formulas for the operating characteristics of a single-server model.

Customers must be served faster than they arrive, or an infinitely large queue will build up.

[Page 577]

The probability that no customers are in the queuing

system (either in the queue or being served) is

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right)$$

*Basic single-server
queuing formulas.*

The probability that n customers are in the queuing system is

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \bullet P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

The average number of customers in the queuing system (i.e., the customers being serviced and in the waiting line) is

$$L = \frac{\lambda}{\mu - \lambda}$$

The average number of customers in the waiting line is

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

The average time a customer spends in the total queuing system (i.e., waiting and being served) is

$$W = \frac{1}{\mu - \lambda} = \frac{L}{\lambda}$$

The average time a customer spends waiting in the queue to be served is

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$

The probability that the server is busy (i.e., the probability that a customer has to wait),

known as the **utilization factor**, is

$$U = \frac{\lambda}{\mu}$$

The probability that the server is idle (i.e., the probability that a customer can be served) is

$$I = 1 - U = 1 - \frac{\lambda}{\mu}$$

This last term, $1 - (\lambda/\mu)$, is also equal to P_0 . That is, the probability of no customers in the queuing system is the same as the probability that the

server is idle.

We can compute these various operating characteristics for Fast Shop Market by simply substituting the average arrival and service rates into the foregoing formulas. For example, if

$\lambda = 24$ customers per hour
arrive at checkout counter

$\mu = 30$ customers per hour can
be checked out

then

$$\begin{aligned}P_0 &= \left(1 - \frac{\lambda}{\mu}\right) \\&= (1 - 24 / 30) \\&= .20 \text{ probability of no customers in the system}\end{aligned}$$

$$\begin{aligned}L &= \frac{\lambda}{\mu - \lambda} \\&= \frac{24}{30 - 24} \\&= 4 \text{ customers, on average, in the queuing system}\end{aligned}$$

$$\begin{aligned}L_q &= \frac{\lambda^2}{\mu(\mu - \lambda)} \\&= \frac{(24)^2}{30(30 - 24)} \\&= 3.2 \text{ customers, on average, in the waiting line}\end{aligned}$$

$$\begin{aligned}
 W &= \frac{1}{\mu - \lambda} \\
 &= \frac{1}{30 - 24} \\
 &= 0.167 \text{ hr. (10 min.) average time in the system per customer}
 \end{aligned}$$

$$\begin{aligned}
 W_q &= \frac{\lambda}{\mu(\mu - \lambda)} \\
 &= \frac{24}{30(30 - 24)} \\
 &= 0.133 \text{ hr. (8 min.) average time in the waiting line per customer}
 \end{aligned}$$

$$\begin{aligned}
 U &= \frac{\lambda}{\mu} \\
 &= \frac{24}{30} \\
 &= .80 \text{ probability that the server will be busy and} \\
 &\quad \text{the customer must wait}
 \end{aligned}$$

$$I = 1 - U$$

$$= 1 - .80$$

= .20 probability that the server will be idle and a customer can be served

Several important aspects of both the general model and this particular example will now be discussed in greater detail.

*Queuing system
operating statistics
are **steady state**, or
constant, over time.*

First, the operating characteristics are averages. Also, they are assumed to be **steady-state** averages. Steady state is a constant average level that a system realizes after a period of time. For a queuing system, the steady state is represented by the average operating statistics, also determined over a period of time.

[Page 579]

Related to this condition is the fact that the utilization factor,

U , must be less than one:

$$U < 1$$

or

$$\frac{\lambda}{\mu} < 1.0$$

and

$$\lambda < \mu$$

In other words, the ratio of the arrival rate to the service rate must be less than one, which also means *the service rate must be greater than the arrival rate* if this model is to

be used. The server must be able to serve customers faster than they come into the store in the long run, or the waiting line will grow to an infinite size, and the system will never reach a steady state.

The Effect of Operating Characteristics on Managerial Decisions

Now let us consider the operating characteristics of our example as they relate to management decisions. The

arrival rate of 24 customers per hour means that, on average, a customer arrives every 2.5 minutes (i.e., $1/24 \times 60$ minutes). This indicates that the store is very busy. Because of the nature of the store, customers purchase few items and expect quick service. Customers expect to spend a relatively large amount of time in a supermarket because typically they make larger purchases. But customers who shop at a convenience market do so, at least in part, because it is quicker than a

supermarket.

Given customers' expectations, the store's manager believes that it is unacceptable for a customer to wait 8 minutes and spend a total of 10 minutes in the queuing system (not including the actual shopping time). The manager wants to test several alternatives for reducing customer waiting time: (1) the addition of another employee to pack up the purchases and (2) the addition of a new checkout counter.

Alternative I: The Addition of an Employee

The addition of an extra employee will cost the store manager \$150 per week. With the help of the national office's marketing research group, the manager has determined that for each minute that average customer waiting time is reduced, the store avoids a loss in sales of \$75 per week. (That is, the store loses money when customers leave prior to shopping because of the long line or when customers do not return.)

If a new employee is hired, customers can be served in less time. In other words, the service rate, which is the number of customers served per time period, will *increase*. The previous service rate was

$\mu = 30$ customers served per hour

The addition of a new employee will increase the service rate to

$\mu = 40$ customers served per hour

It will be assumed that the

arrival rate will remain the same ($\lambda = 24$ per hour) because the increased service rate will not increase arrivals but instead will minimize the loss of customers. (However, it is not illogical to assume that an increase in service might increase arrivals in the long run.)

[Page 580]

Given the new λ and μ values, the operating characteristics can be recomputed as follows:

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = \left(1 - \frac{24}{40}\right)$$

= .40 probability of no customers in the system

$$L = \frac{\lambda}{\mu - \lambda} = \frac{24}{40 - 24}$$

= 1.5 customers, on average, in the queuing system

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(24)^2}{40(16)}$$

= 0.90 customer, on average, in the waiting line

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40 - 24}$$

= 0.063 hr. (3.75 min.) average time in the system per customer

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{24}{40(16)}$$

= 0.038 hr. (2.25 min.) average time in the waiting line per customer

$$U = \frac{\lambda}{\mu} = \frac{24}{40}$$

= .60 probability that the customer must wait

$$I = 1 - U = 1 - .60$$

= .40 probability that the server will be idle and a customer can be served

Remember that these operating characteristics are *averages* that result over a period of time; they are not absolutes. In other words, customers who arrive at the Fast Shop Market checkout counter will not find 0.90 customer in line. There could be no customers, or one, two, or three customers, for example. The value 0.90 is simply an average that occurs over time, as do the other operating characteristics.

The average waiting time per customer has been reduced from 8 minutes to 2.25

minutes, a significant amount. The savings (that is, the decrease in lost sales) is computed as follows:

$$8.00 \text{ min.} - 2.25 \text{ min.} = 5.75 \text{ min.}$$

$$5.75 \text{ min.} \times \$75/\text{min.} = \$431.25$$

Because the extra employee costs management \$150 per week, the total savings will be

$$\$431.25 - \$150 = \$281.25 \text{ per week}$$

The store manager would probably welcome this savings and consider the preceding operating statistics preferable to the previous ones for the condition where the store had only one employee.

Alternative II: The Addition of a New Checkout Counter

Next we will consider the manager's alternative of constructing a new checkout counter. The total cost of this project would be \$6,000, plus an extra \$200 per week for an additional cashier.

The new checkout counter would be opposite the present counter (so that the servers would have their backs to each other in an enclosed counter area). There would be several display cases and racks between the two lines so that customers waiting in line would not move back and forth between the lines. (Such movement, called *jockeying*, would invalidate the queuing formulas we already developed.) We will assume that the customers would divide themselves equally

between the two lines, so the arrival rate for each line would be half of the prior arrival rate for a single checkout counter. Thus, the new arrival rate for each checkout counter is

[Page 581]

$\lambda = 12$ customers per hour

and the service rate remains the same for each of the counters:

$\mu = 30$ customers served per hour

Substituting this new arrival rate and the service rate into our queuing formulas results in the following operating characteristics:

$$P_0 = .60 \text{ probability of no customers in the system}$$

$$L = 0.67 \text{ customer in the queuing system}$$

$$L_q = 0.27 \text{ customer in the waiting line}$$

$$W = 0.055 \text{ hr. (3.33 min.) per customer in the system}$$

0.022 hr. (1.33 min.) per customer
 W_q = in the waiting line

U = .40 probability that a customer must wait

I = .60 probability that a server will be idle and a customer can be served

Using the same sales savings of \$75 per week for each minute's reduction in waiting time, we find that the store would save

$$8.00 \text{ min.} - 1.33 = 6.67 \text{ min.}$$

$$6.67 \text{ min.} \times \$75/\text{min.} = \$500.00 \text{ per week}$$

Next we subtract the \$200 per week cost for the new cashier from this amount saved:

$$\$500 - \$200 = \$300$$

Because the capital outlay of this project is \$6,000, it would take 20 weeks ($\$6,000/\$300 =$

20 weeks) to recoup the initial cost (ignoring the possibility of interest on the \$6,000). Once the cost has been recovered, the store would save \$18.75 ($\$300.00 - \281.25) more per week by adding a new checkout counter rather than simply hiring an extra employee. However, we must not disregard the fact that during the 20-week cost recovery period, the \$281.25 savings incurred by simply hiring a new employee would be lost.

[Table 13.1](#) presents a summary of the operating characteristics

for each alternative. For the store manager, both of these alternatives seem preferable to the original conditions, which resulted in a lengthy waiting time of 8 minutes per customer. However, the manager might have a difficult time selecting between the two alternatives. It might be appropriate to consider other factors besides waiting time. For example, employee idle time is .40 with the first alternative and .60 with the second, which seems to be a significant difference. An additional factor is the loss

of space resulting from a new checkout counter.

Table 13.1. Operating characteristics for each alternative system

Operating Characteristics	Present System	Alternative I	Alternative II
L	4.00 customers	1.50 customers	0.6 customers
L_q	3.20 customers	0.90 customer	0.2 customers
W	10.00 min.	3.75 min.	3.3 min.

W_q	8.00 min.	2.25 min.	1.3
-------	-----------	-----------	-----

U	.80	.60	.40
-----	-----	-----	-----

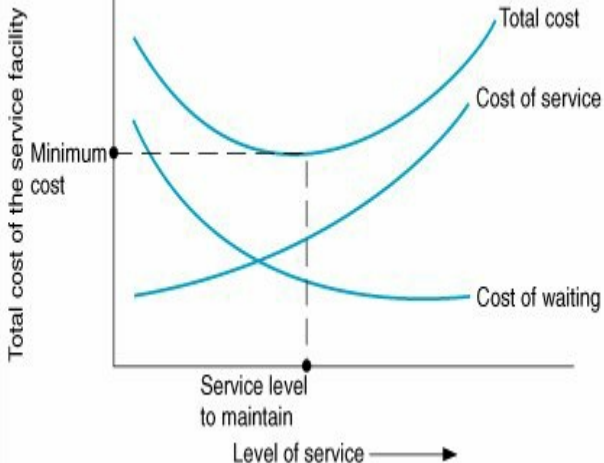
However, the final decision must be based on the manager's own experience and perceived needs. As we have noted previously, the results of queuing analysis provide information for decision making but do not result in an actual recommended decision, as an optimization model would.

Our two example alternatives illustrate the cost trade-offs associated with improved service. As the level of service increases, the corresponding cost of this service also increases. For example, when we added an extra employee in alternative I, the service was improved, and the cost of providing service also increased. But when the level of service was increased, the costs associated with customer waiting decreased. Maintaining

an appropriate level of service should minimize the sum of these two costs as much as possible. This cost trade-off relationship is summarized in [Figure 13.2](#). As the level of service increases, the cost of service goes up and the waiting cost goes down. The sum of these costs results in a total cost curve, and the level of service that should be maintained is the level where this total cost curve is at a minimum. (However, this does not mean we can determine an exact optimal minimum cost

solution because the service and waiting characteristics we can determine are averages and thus uncertain.)

Figure 13.2. Cost trade-off for service levels



As the level of service improves, the

*cost of service
increases.*

Computer Solution of the Single-Server Model with Excel and Excel QM

Excel spreadsheets can be used to solve queuing problems, although someone must enter all the queuing formulas into spreadsheet cells. [Exhibit 13.1](#)

shows a spreadsheet set up to solve the single-server model for the original Fast Shop Market example. Notice that the arrival rate is entered in cell D3, the service rate is entered in cell D4, and our single-server model queuing formulas are entered in cells D6 to D10. For example, cell D7 includes the formula for L_q , average number of customers in the system, shown on the formula bar at the top of the spreadsheet.

(This item is displayed on page 582 in the print version)

Formula for L_q , average number in queue

Microsoft Excel - [Book1].xlsx

File Home Insert Formulas Data Review Help

Formulas

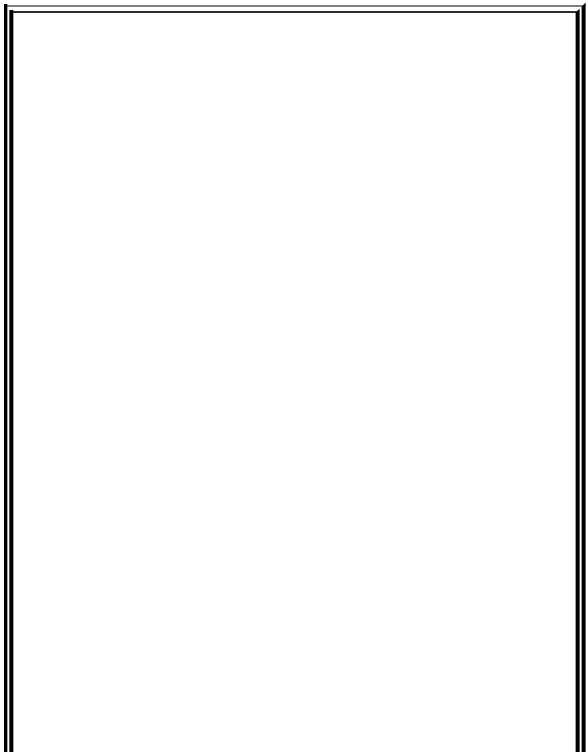
fx = (D3)/(D4*(D4-D3))

Fast Shop Market Example

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Fast Shop Market Example															
2																
3			Arrival rate =	24	customers per hour											
4			Service rate =	30	customers per hour											
5																
6			Average number in the system (L) =	4.0	customers											
7			Average number in the queue (Lq) =	3.2	customers											
8			Average time in the system (W) =	10	minutes											
9			Average time in the queue (Wq) =	8	minutes											
10			Utilization factor (U) =	0.8												
11																

$L = 1 / (D4 - D3)$

$Wq = D3 / (D4 * (D4 - D3))$



Management Science

Application: Reducing

Perceived Waiting Time at Bank of America

In a market research study, Bank of America discovered that when a customer stands in line, waiting for teller service for more than about 3 minutes, a gap develops between the actual waiting time and the customer's perceived waiting time. For example, if customers wait for 2 minutes, they feel like they've been waiting for 2 minutes; however, if they have been waiting for 5 minutes, it may seem more like 10 minutes to them. Bank of America also learned from prior studies that long waits have a direct relationship

to customer satisfaction.

Furthermore, the bank knew from previous psychological studies that if people were distracted with non-boring activities, time would seem to pass more quickly. As a result, Bank of America undertook an experiment to see if customer's perceived waiting time could be reduced if it placed televisions above the tellers in a bank branch lobby to entertain customers waiting in line. The bank installed televisions tuned to CNN in a typical bank branch and measured actual versus perceived waiting times. The results showed that the amount of waiting time customers overestimated dropped from 32% to 15% when compared to a bank without television. To measure the financial impact of these results, the bank employed a customer satisfaction index (based on a

customer survey)); every one-point improvement in the index results in \$1.40 in increased annual revenue per household customer. The experimental results indicated that the projected reductions in perceived waiting times would result in a 5.9-point increase in the customer satisfaction index. As a result, a bank branch with 10,000 household customers could expect an increase in annual revenue of \$82,600, while a bank with only a few thousand household customers could expect to recoup the estimated \$10,000 one-time cost of installing televisions in less than a year.



Source: S. Thomke, "R&D Comes to Services: Bank of America's Pathbreaking Experiments," *Harvard Business Review* 81, (April 4, 2003): 709.

Excel QM includes a spreadsheet macro for waiting line (i.e., queuing) analysis. Excel QM includes all the different queuing models that will be presented in this chapter. The queuing macro in Excel QM is one of the more useful ones in this text because some of the queuing formulas we will introduce are complex and often difficult to set up manually in spreadsheet cells.

After Excel QM has been activated, the "Waiting Line Analysis" menu is accessed from the "QM" menu on the menu bar at the top of the spreadsheet. This will result in a Spreadsheet Initialization window, which will enable us to title the problem and specify some of the spreadsheet output. [Exhibit 13.2](#) shows the Excel QM spreadsheet for our Fast Shop Market single-server model. When this spreadsheet first appears, cells **B7:B8** contain example data values, and the results relate to these

arrival and service rates. Thus, the first step is to enter the arrival and service rates for our Fast Shop Market example in cells **B7:B8**. The results will be computed automatically, using the queuing formulas already embedded in the macro.

[Page 584]

Exhibit 13.2.

[\[View full size image\]](#)

Click on "QM" to access Excel QM queuing model macros.

Microsoft Excel - Table12.2.xls

File Edit View Insert Format Tools Data Window Help 480x500

1 Fast Shop Market Example

2

3 Waiting Lines: M/M/1 (Single Server Model)

4 The arrival RATE and service RATE both must be rates and use the same time unit. Given a time such as 10 minutes, convert it to a rate such as 6 per hour.

5

6 Data Results

7 Arrival rate (λ)	24	Average server utilization (ρ)	0.8
8 Service rate (μ)	30	Average number of customers in the queue (L_q)	3.2
		Average number of customers in the system (L_s)	4
		Average waiting time in the queue (W_q)	0.133
		Average time in the system (W_s)	0.167
		Probability (% of time) system is empty (P_0)	0.2

Initially these cells contain "example" values—input problem arrival and service rates.

Computer Solution of the Single-Server Model with QM for Windows

QM for Windows has a module that is capable of performing queuing analysis. As an illustration of the computerized analysis of the single-server queuing system, we will use QM for Windows to analyze the Fast Shop Market example. The model input and solution output are shown in [Exhibit 13.3](#).

Exhibit 13.3.

[\[View full size image\]](#)

Fast Shop Market Example Solution

Parameter	Value	Parameter	Value	Minutes	Seconds
MM/1 (exponential service)		Average server utilization	.8		
Arrival rate(λ)	24	Average number in the queue(L_q)	3.2		
Service rate(μ)	30	Average number in the system(L)	4		
Number of servers	1	Average time in the queue(W_q)	.1333	8	400
		Average time in the system(W)	.1667	10	600

Undefined and Constant Service Times

Sometimes it cannot be assumed that a waiting line system has an arrival rate that is Poisson distributed or service times that are exponentially distributed. For example, many manufacturing operations use automated equipment or robots that have constant service times. Thus, the single-server model with Poisson arrivals and constant service times is a

queuing variation that is of particular interest to manufacturing operations.

Constant service times occur with machinery and automated equipment.

Constant service times are a special case of the single-server model with

undefined service times.

The constant service time model is actually a special case of a more general variation of the single-server model in which service times cannot be assumed to be exponentially distributed. As such, service times are said to be *general*, or *undefined*. The basic queuing formulas for the operating characteristics of the undefined

service time model are as follows:

[Page 585]

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$L_q = \frac{\lambda^2 \sigma^2 + (\lambda / \mu)^2}{2(1 - \lambda / \mu)}$$

$$L = L_q + \frac{\lambda}{\mu}$$

$$W_q = \frac{L_q}{\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

$$U = \frac{\lambda}{\mu}$$

The key formula for undefined service times is for L_q , the number of customers in the waiting line. In this formula, μ and σ are the mean and standard deviation, respectively, for any general probability distribution with independent service times.

For the Poisson distribution, the mean is equal to the variance. This same relationship is true for Poisson service rates if service times are exponentially distributed. Thus, if we let $\sigma = \mu$ in the preceding formula for

L_q for undefined service times, it becomes the same as our basic formula with exponential service times. In fact, all the queuing formulas become the same as in the basic single-server model.

As an example of the single-server model with undefined service times, consider a business firm with a single fax machine. Employees arrive randomly to use the fax machine, at an average rate of 20 per hour, according to a Poisson distribution. The time

an employee spends using the machine is not defined by any probability distribution but has a mean of 2 minutes and a standard deviation of 4 minutes. The operating characteristics for this system are computed as follows:

$$P_0 = 1 - \frac{\lambda}{\mu}$$

$$P_0 = 1 - \frac{20}{30} = .33 \text{ probability that no one is using the machine}$$

$$L_q = \frac{\lambda^2 \sigma^2 + (\lambda / \mu)^2}{2(1 - \lambda / \mu)} = \frac{(20)^2 (1 / 15)^2 + (20 / 30)^2}{2(1 - (20 / 30))}$$

= 3.33 employees waiting in line

$$L = L_q + \frac{\lambda}{\mu} = 3.33 + (20 / 30)$$

= 4.0 employees in line and using the machine

$$W_q = \frac{L_q}{\lambda} = \frac{3.33}{20}$$

= 0.1665 hr. (10 min.) waiting in line

$$W = W_q + \frac{1}{\mu} = 0.1665 + \left(\frac{1}{30} \right)$$

= 0.1998 hr. (12 min.) in the system

$$U = \lambda / \mu = 20 / 30 = 67\% \text{ machine utilization}$$

[Page 586]

In the constant

*service time model,
there is no variability
in service times; $\sigma = 0$.*

In the case of constant service times, there is no variability in service times (i.e., service time is the same constant value for each customer); thus, $\sigma = 0$. Substituting $\sigma = 0$ into the undefined service time formula for L_q results in the following formula for L_q with constant

service times:

$$\begin{aligned}L_q &= \frac{\lambda^2 \sigma^2 + (\lambda / \mu)^2}{2(1 - (\lambda / \mu))} \\&= \frac{\lambda^2 (0)^2 + (\lambda / \mu)^2}{2(1 - (\lambda / \mu))} \\&= \frac{(\lambda / \mu)^2}{2(1 - (\lambda / \mu))} \\L_q &= \frac{\lambda^2}{2\mu(\mu - \lambda)}\end{aligned}$$

Notice that this new formula for L_q for constant service times is simply our basic single-server formula for L_q divided by two. All the remaining formulas for L , W_q , and W are the same as

the single-server formulas using this formula for L_q .

The following example illustrates the single-server model with constant service times. The Petroco Service Station has an automatic car wash, and motorists purchasing gas at the station receive a discounted car wash, depending on the number of gallons of gas they buy. The car wash can accommodate one car at a time, and it requires a constant time of 4.5 minutes for a wash. Cars arrive at the car wash at

an average rate of 10 per hour (Poisson distributed). The service station manager wants to determine the average length of the waiting line and the average waiting time at the car wash.

First, determine λ and μ such that they are expressed as rates:

$$\lambda = 10 \text{ cars per hr.}$$

$$\mu = 60/4.5 = 13.3 \text{ cars per hr.}$$

Substituting λ and μ into the queuing formulas for constant

service time,

$$L_q = \frac{\lambda^2}{2\mu(\mu - \lambda)} = \frac{(10)^2}{2(13.3)(13.3 - 10)} \\ = 1.14 \text{ cars waiting}$$

$$W_q = \frac{L_q}{\lambda} = \frac{1.14}{10} \\ = 0.114 \text{ hr. (6.84 min.) waiting in line}$$

Computer Solution of the Constant Service Time Model with Excel

[Exhibit 13.4](#) shows an Excel spreadsheet set up to solve our Petroco Service Station

example with constant service times. As in [Exhibit 13.1](#) for the Fast Shop Market example, the queuing formulas are embedded in individual cells. For example, the formula for L_q , the average number in the queue, is entered in cell D6 and is shown on the formula bar at the top of the spreadsheet.

[Page 587]

Exhibit 13.4.

[\[View full size image\]](#)

Average number
in the queue, L_q

Microsoft Excel - Exhibit13.4.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Formulas: =D8*(D4/D3)*60

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Petroco Service Station Example With Constant Service Times															
2																
3			Arrival rate =	10												
4			Service rate =	13.3												
5																
6			Average number in the queue (L_q) =	1.14												
7			Average number in the system (L) =	1.09												
8			Average time in the queue (W_q) =	6.84												
9			Average time in the system (W) =	11.35												
10			Utilization factor (ρ) =	0.75												
11																

$$=D8+(1/D4)*60$$

$$=(D6/D3)*60$$

Computer Solution of the Undefined and Constant

Service Time Models with QM for Windows

QM for Windows can be used to solve the single-server model with undefined (referred to as "general") and constant service times similarly to the single-server model in the previous section. [Exhibit 13.5](#) shows the QM for Windows solution output screen for our fax machine example with undefined service times (on page [585](#)).

Exhibit 13.5.

[\[View full size image\]](#)

Fax Machine Example Solution					
Parameter	Value	Parameter	Value	Minutes	Seconds
M/G/1 (general service times)		Average server utilization	.6667		
Arrival rate(λ)	20	Average number in the queue(L_q)	3.3601		
Service rate(μ)	30	Average number in the system(L)	4.0267		
Number of servers	1	Average time in the queue(W_q)	.168	10.0802	604.8121
Standard deviation	.067	Average time in the system(W)	.2013	12.0802	724.8121

Finite Queue Length

For some waiting line systems, the length of the queue may be limited by the physical area in which the queue forms; space may permit only a limited number of customers to enter the queue. Such a waiting line is referred to as a **finite queue** and results in another variation of the single-phase, single-channel queuing model.

*In a **finite queue**,*

the length of the queue is limited.

The basic single-server model must be modified to consider the finite queuing system. It should be noted that for this case, the service rate does not have to exceed the arrival rate ($\mu > \lambda$) in order to obtain steady-state conditions. The resulting operating characteristics, where M is the maximum number in the

system, are as follows:

$$P_0 = \frac{1 - (\lambda / \mu)}{1 - (\lambda / \mu)^{M+1}}$$

$$P_n = (P_0) \left(\frac{\lambda}{\mu} \right)^n \text{ for } n \leq M$$

$$L = \frac{\lambda / \mu}{1 - (\lambda / \mu)} - \frac{(M+1)(\lambda / \mu)^{M+1}}{1 - (\lambda / \mu)^{M+1}}$$

[Page 588]

Because P_n is the probability of n units in the system, if we define M as the maximum number allowed in the system, then P_M (the value of P_n for $n = M$) is the probability that a

customer will not join the system. The remaining equations are

$$L_q = L - \frac{\lambda(1 - P_M)}{\mu}$$

$$W = \frac{L}{\lambda(1 - P_M)}$$

$$W_q = W - \frac{1}{\mu}$$

As an example of the single-server model with finite queue, consider Metro Quick Lube, a one-bay service facility located next to a busy highway in an urban area. The facility has space for only one vehicle in

service and three vehicles lined up to wait for service. There is no space for cars to line up on the busy adjacent highway, so if the waiting line is full (three cars), prospective customers must drive on.

The mean time between arrivals for customers seeking lubrication service is 3 minutes. The mean time required to perform the lube operation is 2 minutes. Both the interarrival times and the service times are exponentially distributed. As stated previously, the maximum number of vehicles

in the system is four. The operating characteristics are computed as follows:

$$\lambda = 20$$

$$\mu = 30$$

$$M = 4$$

First, we will compute the probability that the system is full and the customer must drive on, P_M . However, this first requires the determination of P_0 , as follows:

$$P_0 = \frac{1 - (\lambda / \mu)}{1 - (\lambda / \mu)^{M+1}} = \frac{1 - (20 / 30)}{1 - (20 / 30)^5}$$

= .38 probability of no cars in the system

$$P_M = (P_0) \left(\frac{\lambda}{\mu} \right)^{n=M} = (.38) \left(\frac{20}{30} \right)^4$$

= .076 probability that four cars are in the system, it is full,
and a customer must drive on

Next, to compute the average queue length, L_q , the average number of cars in the system, L , must be computed, as follows:

$$L = \frac{\lambda / \mu}{1 - (\lambda / \mu)} - \frac{(M + 1)(\lambda / \mu)^{M+1}}{1 - (\lambda / \mu)^{M+1}}$$

$$L = \frac{20 / 30}{1 - (20 / 30)} - \frac{(5)(20 / 30)^5}{1 - (20 / 30)^5}$$

= 1.24 cars in the system

$$L_q = L - \frac{\lambda(1 - P_M)}{\mu} = 1.24 - \frac{20(1 - .076)}{30}$$

= 0.62 cars waiting

[Page 589]

Management Science

Application: Providing

Telephone Order Service in the Retail Catalog Business

In the TIME OUT box for A. K. Erlang on page [575](#) it was noted that the origin of queuing analysis was in telephone congestion problems in the early 1900s. Today queuing analysis is still an important tool in analyzing telephone service in catalog phone sales, one of the largest retail businesses in the United States. In a recent year, one of the largest catalog sales companies, L.L.Bean, with over \$1 billion in annual sales, received more than 12 million customer calls. During the peak holiday season, its 3,000 customer

service representatives took more than 140,000 customer calls on its busiest days. Lands' End, the 15th largest mail-order company in the United States, with sales of over \$1.3 billion, handles more than 15 million phone calls each year. On an average day, its 300-plus phone lines handle between 40,000 and 50,000 calls, and during the weeks prior to Christmas, it expands to more than 1,100 phone lines, to handle more than 100,000 phone calls daily. One of the key factors in maintaining a successful catalog phone order system is to provide prompt service; if customers have to wait too long before talking to a customer service representative, they may hang up and not call back. Catalog companies such as L.L.Bean and Lands' End often use queuing analysis to make a number of decisions related to order

processing, including the number of telephone trunk lines and customer service representatives they need during various hours of the day and days of the year, the amount of computing capacity they need to handle call volume, the number of workstations and the amount of equipment needed, and the number of full-time and part-time customer service representatives to hire and train.



To compute the average waiting time, W_q , the average time in the system, W , must be computed first:

$$W = \frac{L}{\lambda(1 - P_M)} = \frac{1.24}{20(1 - .076)}$$

= 0.067 hr. (4.03 min.) waiting in the system

$$W_q = W - \frac{1}{\mu}$$
$$= 0.067 - \frac{1}{30}$$

= 0.033 hr. (2.03 min.) waiting in line

Computer Solution of the Finite Queue Model with Excel

The Excel spreadsheet solution to the Metro Quick Lube example problem with a finite queue is shown in [Exhibit 13.6](#). Notice that the formula for P_0 in cell D7 is shown on the formula bar at the top of the spreadsheet. The formula for L in cell D9 is shown in the callout box.

Exhibit 13.6.

(This item is displayed on page 590 in the print version)

[\[View full size image\]](#)

Formula for P_0 in cell D7

Microsoft Excel - Exhibit13.6.xls

File Edit View Insert Format Tools Data Window Help

Formulas: =1.0304/(1-0.304/(0.304*(D5+1)))

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	Metro Quick Lube (Example With Finite Queue)															
2																
3			Arrival rate =	20	cars per hour											
4			Service rate =	30	cars per hour											
5			M =	4												
6																
7			P_0 =	0.334												
8			P_m =	0.076												
9			Average number in the system (L) =	1.24	cars											
10			Average number in the queue (Lq) =	0.53	cars											
11			Average time in the system (W) =	4.03	minutes											
12			Average time in the queue (Wq) =	2.03	minutes											
13																

Formula for P_0 in cell D7:

$$= ((D3/D4)/(1 - (D3/D4))) - ((D5+1)*(D3/D4)^{(D5+1)})/(1 - (D3/D4)^{(D5+1)})$$

Note that Excel QM also has a spreadsheet macro for the finite queue model that can be accessed similarly to the single-server model in [Exhibit 13.2](#).

[Page 590]

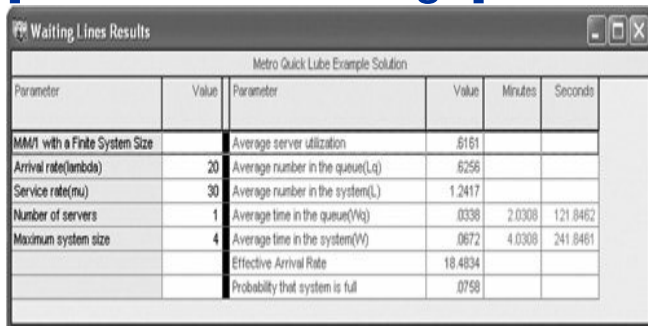
Computer Solution of the Finite Queue Model with QM for Windows

The single-server model with finite queue can be solved with QM for Windows. [Exhibit 13.7](#)

shows the model solution screen for our Metro Quick Lube example.

Exhibit 13.7.

[\[View full size image\]](#)



Metro Quick Lube Example Solution					
Parameter	Value	Parameter	Value	Minutes	Seconds
MM/1 with a Finite System Size		Average server utilization	.6161		
Arrival rate(λ)	20	Average number in the queue(L_q)	6256		
Service rate(μ)	30	Average number in the system(L)	1.2417		
Number of servers	1	Average time in the queue(W_q)	.0338	2.0308	121.8462
Maximum system size	4	Average time in the system(W)	.0672	4.0308	241.8461
		Effective Arrival Rate	18.4834		
		Probability that system is full	.0758		

Finite Calling Population

For some waiting line systems there is a specific, limited number of potential customers that can arrive at the service facility. This is referred to as a **finite calling population**. As an example of this type of system, consider the Wheelco Manufacturing Company.

*With a **finite calling population** the customers from*

*which arrivals
originate are limited.*

Wheelco Manufacturing operates a shop that includes 20 machines. Due to the type of work performed in the shop, there is a lot of wear and tear on the machines, and they require frequent repair. When a machine breaks down, it is tagged for repair, with the date of the breakdown noted and a repair person is called. The

company has one senior repair person and an assistant. They repair the machines in the same order in which they break down (a first-in, first-out queue discipline). Machines break down according to a Poisson distribution, and the service times are exponentially distributed.

The finite calling population for this example is the 20 machines in the shop, which we will designate as N .

The single-server model with a Poisson arrival and exponential service times and a finite calling population has the following set of formulas for determining operating characteristics. λ in this model is the arrival rate for each member of the population:

$$P_0 = \frac{1}{\sum_{n=0}^N \frac{N!}{(N-n)!} \left(\frac{\lambda}{\mu}\right)^n}, \text{ where } N = \text{population size}$$

$$P_n = \frac{N!}{(N-n)!} \left(\frac{\lambda}{\mu}\right)^n P_0, \text{ where } n = 1, 2, \dots, N$$

$$L_q = N - \left(\frac{\lambda + \mu}{\lambda}\right) (1 - P_0)$$

$$L = L_q + (1 - P_0)$$

$$W_q = \frac{L_q}{(N - L)\lambda}$$

$$W = W_q + \frac{1}{\mu}$$

The formulas for P_0 and P_n are both relatively complex and can be cumbersome to compute manually. As a result, tables

are often used to compute these values, given λ and μ . Alternatively, the queuing module in QM for Windows has the capability to solve the finite calling population model, which we will demonstrate.

Returning to our example of the Wheelco Manufacturing Company, each machine operates an average of 200 hours before breaking down and a repair person is called. The average time to repair a machine is 3.6 hours. The breakdown rate is Poisson distributed, and the service

time is exponentially distributed. The company would like an analysis of machine idle time due to breakdowns to determine whether the present repair staff is sufficient.

Using the formulas we developed for the single-server model with finite calling population, we can determine the operating characteristics for the machine repair system, as follows:

$$\lambda = 1 / 200 \text{ hr.} = .005 \text{ per hr.}$$

$$\mu = 1 / 3.6 \text{ hr.} = .2778 \text{ per hr.}$$

$$N = 20 \text{ machines}$$

$$P_0 = \frac{1}{\sum_{n=0}^N \frac{N!}{(N-n)!} \left(\frac{\lambda}{\mu} \right)^n}$$

$$= \frac{1}{\sum_{n=0}^{20} \frac{20!}{(20-n)!} \left(\frac{.005}{.2778} \right)^n}$$

$$P_0 = .649$$

$$L_q = N - \left(\frac{\lambda + \mu}{\lambda} \right) (1 - P_0) = 20 - \left(\frac{.005 + .2778}{.005} \right) (1 - .652)$$

$$= .169 \text{ machines waiting}$$

$$L = L_q + (1 - P_0) = .169 + (1 - .652)$$

$$= .520 \text{ machines in the system}$$

$$W_q = \frac{L_q}{(N - L)\lambda} = \frac{.169}{(20 - .520)(.005)}$$

$$= 1.74 \text{ hr. waiting for repair}$$

$$W = W_q + \frac{1}{\mu} = 1.74 + \frac{1}{.2778}$$

$$= 5.33 \text{ hr. in the system}$$

These results show that the repair person and assistant are busy 35% of the time repairing machines. Of the 20 machines, an average of .52, or 2.6%, are broken down, waiting for repair, or under repair. Each broken-down machine is idle (broken down, waiting for repair, or under repair) an average of 5.33 hours. Thus the system seems adequate.

Computer Solution of the Finite Calling Population Model with Excel and Excel QM

The finite population queuing model can be tedious to solve by using an Excel spreadsheet because of the complexity of entering the formula for P_0 in the spreadsheet. An array must be created for the N components required by the summation in the denominator of the formula for P_0 . [Exhibit](#)

13.8 shows the Excel spreadsheet for the Wheelco Manufacturing Company example, with this summation array in cells **F4:G26**.

Exhibit 13.8.

[\[View full size image\]](#)

This is a model in which the spreadsheet macro in Excel QM is especially useful. The finite population model is accessed from the "QM" menu on the menu bar at the top of the spreadsheet. After selecting "Spreadsheet Initialization," the resulting spreadsheet for our Wheelco Manufacturing Company example is shown in [Exhibit 13.9](#). As in our other Excel QM examples, the spreadsheet initially includes example data values. The first step is to enter the parameters for our own example in cells

B7:B9 as shown in [Exhibit 13.9](#).

[Page 593]

Exhibit 13.9.

[\[View full size image\]](#)

Click on "QM" to access the macro for the finite population model.

Microsoft Excel - Exhibit13.3.xls

File Edit View Insert Format Tools Data QM Window Help

100%

And

Wheelco Manufacturing Company Example

Waiting Lines M/M/1 with a finite population

The arrival rate is for each member of the population. If they go for service every 20 minutes then order 3 (per hour)

Data		Results	
Arrival rate (λ) per customer	0.005	Average server utilization(ρ)	0.2500
Service rate (μ)	0.2778	Average number of customers in the queue(L_q)	0.1000
Population size (N)	20	Average number of customers in the system(L)	0.5196
		Average waiting time in the queue(W_q)	1.7340
		Average time in the system(W)	5.2040
		Probability (% of time) system is empty (P_0)	0.6494
		Effective arrival rate	0.0974

Enter problem data in cells B7:B9.

Computer Solution of the Finite Calling Population

Model with QM for Windows

QM for Windows is especially useful for solving the finite calling population model because the formulas are complex and their manual solution can be tedious and time-consuming. [Exhibit 13.10](#) includes the QM for Windows solution screen for the Wheelco Manufacturing Company example.

Exhibit 13.10.

[\[View full size image\]](#)

Waiting Lines Results					
Wheelco Manufacturing Company Example Solution					
Parameter	Value	Parameter	Value	Minutes	Seconds
MM/1 with a Finite Population		Average server utilization	.3506		
Arrival PER CUSTOMER	.005	Average number in the queue(Lq)	.169		
Service rate(μ)	.2778	Average number in the system(L)	.5196		
Number of servers	1	Average time in the queue(Wq)	1.7349	104.0937	6,245.62
Population size	20	Average time in the system(W)	5.3346	320.0764	19,204.58
		Effective Arrival Rate	.0974		
		Probability that customer waits	.3333		

The Multiple-Server Waiting Line

Slightly more complex than the single-server queuing system is the single waiting line being serviced by more than one server (i.e., multiple servers). Examples of this type of waiting line include an airline ticket and check-in counter where passengers line up in a single line, waiting for one of several agents for service, and a post office line, where customers in

a single line wait for service from several postal clerks.

Figure 13.3 illustrates this type of **multiple-server queuing system**.

*In **multiple-server models**, two or more independent servers in parallel serve a single waiting line.*

As an example of this type of system, consider the customer

service department of the Biggs Department Store. The customer service department of the store has a waiting room in which chairs are placed along the wall, in effect forming a single waiting line. Customers come to this area with questions or complaints or to clarify matters regarding credit card bills. The customers are served by three store representatives, each located in a partitioned stall. Customers are served on a first-come, first-served basis.

**Figure 13.3. A
multiple-server
waiting line**



United States Post Office
www.usps.com



The store management wants to analyze this queuing system because excessive waiting times can make customers angry enough to shop at other stores. Typically, customers who come to this area have some problem and thus are impatient anyway. Waiting a long time serves only to increase their impatience.

First, the queuing formulas for a multiple-server queuing system will be presented. These

formulas, like single-server model formulas, have been developed on the assumption of a *first-come, first-served queue discipline, Poisson arrivals, exponential service times, and an infinite calling population*. The parameters of the multiple-server model are as follows:

λ = the arrival rate (average number of arrivals per time period)

μ = the service rate (average number served per time period) per server (channel)

c = the number of servers

$c\mu$ = the mean effective service rate for the system, which must exceed the arrival rate

The formulas for the operating characteristics of the multiple-server model are as follows.

$c\mu > \lambda$: the total number of servers must be able to serve customers faster than they arrive.

The probability that there are no customers in the system (all servers are idle) is

$$P_0 = \frac{1}{\left[\sum_{n=0}^{c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{c\mu}{c\mu - \lambda} \right)}$$

[Page 595]

The probability of n customers in the queuing system is

$$P_n = \frac{1}{c!c^{n-c}} \left(\frac{\lambda}{\mu} \right)^n P_0, \text{ for } n > c; \quad P_n = \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n P_0, \text{ for } n \leq c$$

The average number of customers in the queuing system is

$$L = \frac{\lambda \mu (\lambda / \mu)^c}{(c-1)! (c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu}$$

The average time a customer spends in the queuing system (waiting and being served) is

$$W = \frac{L}{\lambda}$$

The average number of customers in the queue is

$$L_q = L - \frac{\lambda}{\mu}$$

The average time a customer spends in the queue, waiting to be served, is

$$W_q = W - \frac{1}{\mu} = \frac{L_q}{\lambda}$$

The probability that a customer arriving in the system must wait for service (i.e., the probability that all the servers are busy) is

$$P_w = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \frac{c\mu}{c\mu - \lambda} P_0$$

Notice in the foregoing formulas that if $c = 1$ (i.e., if

there is one server), then these formulas become the single-server formulas presented previously in this chapter.

Returning to our example, let us assume that a survey of the customer service department for a 12-month period shows that the arrival rate and service rate are as follows:

$\lambda = 10$ customers per hr. arrive at the service department

$\mu = 4$ customers per hr. can be served by each store representative

In addition, recall that this is a three-server queuing system; therefore,

$c = 3$ store representatives

[Page 596]

Using the multiple-server model formulas, we can compute the following operating characteristics for the service department:

$$P_0 = \frac{1}{\left[\sum_{n=0}^{n=c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{c\mu}{c\mu - \lambda} \right)}$$

$$= \frac{1}{\left[\frac{1}{0!} \left(\frac{10}{4} \right)^0 + \frac{1}{1!} \left(\frac{10}{4} \right)^1 + \frac{1}{2!} \left(\frac{10}{4} \right)^2 \right] + \frac{1}{3!} \left(\frac{10}{4} \right)^3 \frac{3(4)}{3(4) - 10}}$$

= 0.045 probability that no customers are in the service department

$$L = \frac{\lambda\mu(\lambda / \mu)^c}{(c-1)!(c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu}$$

$$= \frac{(10)(4)(10 / 4)^3}{(3-1)! [3(4) - 10]^2} (.045) + \frac{10}{4}$$

= 6 customers, on average, in the service department

$$W = \frac{L}{\lambda}$$

$$= \frac{6}{10}$$

= 0.60 hr. (36 min.) average time in the service department per customer

$$L_q = L - \frac{\lambda}{\mu}$$

$$= 6 - \frac{10}{4}$$

= 3.5 customers, on average, waiting to be served

$$W_q = \frac{L_q}{\lambda}$$

$$= \frac{3.5}{10}$$

= 0.35 hr. (21 min.) average time waiting in line per customer

$$P_w = \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \frac{c\mu}{c\mu - \lambda} P_0$$

$$= \frac{1}{3!} \left(\frac{10}{4} \right)^3 \frac{3(4)}{3(4) - 10} (.045)$$

= .703 probability that a customer must wait for service (i.e., that there are three or more customers in the system)

The department store's management has observed that customers are frustrated by the relatively long waiting time of 21 minutes and the .703 probability of waiting. To try to improve matters, management has decided to consider the addition of an extra service representative. The operating characteristics for this system must be recomputed with $c = 4$ service representatives.

Substituting this value along with λ and μ into our queuing formulas results in the following operating characteristics:

$P_0 = .073$ probability that no customers are in the service department

$L = 3.0$ customers, on average, in the service department

$W = 0.30$ hr. (18 min.) average time in the service department per customer

$L_q = 0.5$ customer, on average, waiting

to be served

$W_q = 0.05$ hr. (3 min.) average time
waiting in line per customer

$P_w = .31$ probability that a customer must
wait for service

As in our previous example of the single-server system, the queuing operating characteristics provide input into the decision-making process, and the decision criteria are the waiting costs

and service costs. The department store management would have to consider the cost of the extra service representative, as compared to the dramatic decrease in customer waiting time from 21 minutes to 3 minutes, in making a decision.

Computer Solution of the Multiple-Server Model with Excel and Excel QM

The multiple-server model is somewhat cumbersome to set

up in a spreadsheet format because of the necessity to enter some of the complex queuing formulas for this model into spreadsheet cells. [Exhibit 13.11](#) shows a spreadsheet set up for our Biggs Department Store multiple-server example.

Exhibit 13.11.

[\[View full size image\]](#)

you can see, it is long and complex. The specific summation terms in P_0 for our example are entered directly into the formula. Thus, for a larger problem with a larger number of servers, additional summation terms would have to be entered. The term FACT() takes the factorial of a number or numbers in a cell. For example, FACT(1) is 1! Notice also that two of the other more complex formulas in this model for P_w and L are shown in callout boxes attached to [Exhibit 13.11](#).

Solution of the multiple-server model, as with some of the other more complex queuing models, can be tedious using Excel spreadsheets. Excel QM is particularly useful for solving these more complex queuing models. [Exhibit 13.12](#) shows the Excel QM spreadsheet for our Biggs Department Store example. As with other queuing models we have solved with an Excel QM queuing macro, all the formulas are already embedded in this spreadsheet; all that is required in this case is that the arrival and service

rates and number of servers be entered in cells **B7:B9**; the operating characteristics in cells **E7:E12** are computed automatically.

[Page 598]

Exhibit 13.12.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 13.12.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

Formulas: =SUM(B1:B10) 110% 100%

Tools: Format Painter AutoFill Fill Color Background Color Text Color Font Color

Window: Exhibit 13.12.xls

Cell: B1

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	Biggs Department Store Example With Multiple Servers												
2													
3	Waiting Lines	M/M/3											
4	The arrival RATE and service RATE both must be rates and use the same time unit. Given a time such as 10 minutes, convert it to a rate such as 6 per hour												
5													
6	Data												
7	Arrival rate (λ)	10			Results								
8	Service rate (μ)	4			Average server utilization(ρ)	0.8333							
9	Number of servers(s)	3			Average number of customers in the queue(L_q)	3.6112							
10					Average number of customers in the system(L)	6.0112							
11					Average waiting time in the queue(W_q)	0.3611							
12					Average time in the system(W)	0.6011							
13					Probability (% of time) system is empty (P_0)	0.0443							

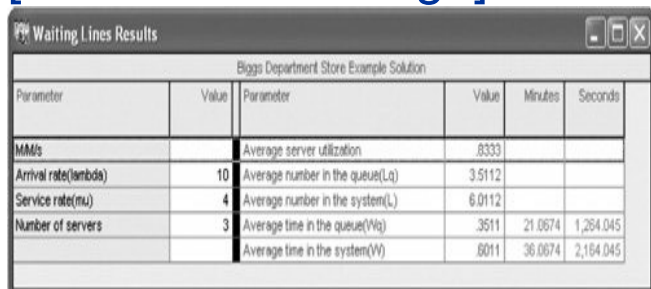
Computer Solution of the Multiple-Server Model with QM for Windows

[Exhibit 13.13](#) shows the QM for

Windows solution screen for the Biggs Department Store multiple-server model example.

Exhibit 13.13.

[\[View full size image\]](#)



The screenshot shows a Windows application window titled "Waiting Lines Results". Inside the window, there is a table titled "Biggs Department Store Example Solution". The table has six columns: "Parameter", "Value", "Parameter", "Value", "Minutes", and "Seconds". The table contains five rows of data. The first row shows "MM/s" with a value of 10, "Average server utilization" with a value of .8333, and empty cells for minutes and seconds. The second row shows "Arrival rate(λ)" with a value of 4, "Average number in the queue(L_q)" with a value of 3.5112, and empty cells for minutes and seconds. The third row shows "Service rate(μ)" with a value of 3, "Average number in the system(L)" with a value of 6.0112, and empty cells for minutes and seconds. The fourth row shows "Number of servers" with a value of 3, "Average time in the queue(W_q)" with a value of .3511, 21.0674 minutes, and 1,264.045 seconds. The fifth row shows empty cells for the first two columns, "Average time in the system(W)" with a value of .6011, 36.0674 minutes, and 2,164.045 seconds.

Parameter	Value	Parameter	Value	Minutes	Seconds
MM/s	10	Average server utilization	.8333		
Arrival rate(λ)	4	Average number in the queue(L_q)	3.5112		
Service rate(μ)	3	Average number in the system(L)	6.0112		
Number of servers	3	Average time in the queue(W_q)	.3511	21.0674	1,264.045
		Average time in the system(W)	.6011	36.0674	2,164.045

Additional Types of Queuing Systems

The *single queue with a single server* and the *single queue with multiple servers* are two of the most common types of queuing systems. However, there are two other general categories of queuing systems: *the single queue with single servers in sequence* and the *single queue with multiple servers in sequence*. [Figure 13.4](#) presents a schematic of

each of these two systems.

Figure 13.4. Single queues with single and multiple servers in sequence

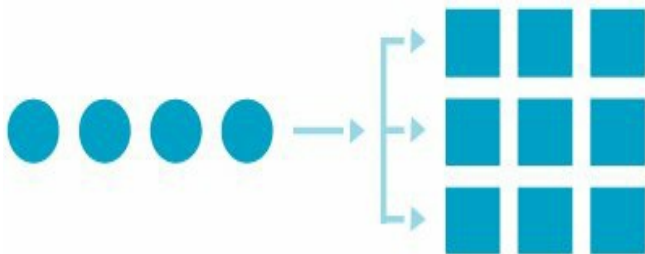
(This item is displayed on page 599 in the print version)

Queue

Servers



Single queue with single servers in sequence



Single queue with multiple servers in sequence

An example of a queuing system that has a single queue leading into a sequence of

single servers is the personnel office of a company where job applicants line up to apply for a specific job. All the applicants wait in one area and are called alphabetically. The application process consists of moving from one interview to the next in a single sequence to take tests, answer questions, fill out forms, and so on. Another example of this type of system is an assembly line, in which products are queued up prior to being worked on by a sequenced line of machines.

If, in the personnel office

example, an extra sequence of interviews were added, the result would be a queuing system with a single queue and multiple servers in sequence. Likewise, if products were lined up in a single queue prior to being worked on by machines in any one of three assembly lines, the result would be a sequence of multiple servers.

[Page 599]

These are four general categories of queuing systems. Other items that can contribute

to the variety of possible queuing systems include the following:

- Queuing systems in which customers *balk* from entering the system or leave the line if it is too long (i.e., *renege*)
- Servers who provide service on some basis other than first-come, first-served (such as alphabetical order or appointment, as in a doctor's office)

- Service times that are not exponentially distributed or are undefined or constant
- Arrival rates that are not Poisson distributed
- Jockeying (i.e., movement between queues), which can often occur when there are multiple servers, each preceded by a separate queue (such as in a bank with several tellers or in a grocery store with several cash registers)

Summary

The various forms of queuing systems can make queuing a potentially complex field of analysis. However, because queues are encountered often in our everyday life, the analysis of queues is an important and widely explored area of management science. We have considered only the fundamentals of a few basic types of queuing systems; a number of other analysis

models have been developed to analyze the more complex queuing systems.

Some queuing situations, however, are so complex that it is impossible to develop an analytical model. When these situations occur, an alternative form of analysis is *simulation*, in which the real-life queuing system is simulated via a computerized mathematical model. The operating characteristics are determined by observing the simulated queuing system. This alternative technique,

simulation, is the subject of the next chapter.

References

Buffa, E. S. *Operations Management: Problems and Models*, 3rd ed. New York: John Wiley & Sons, 1972.

Feller, W. *An Introduction to Probability Theory and Its Applications*. Vol. 1, 3rd ed. New York: John Wiley & Sons, 1968.

Hillier, F., and Lieberman, G. J. *Introduction to Operations Research*, 4th ed. San Francisco: Holden-Day, 1986.

Morse, P. M. *Queues, Inventories, and Maintenance*.

New York: John Wiley & Sons,
1958.

Saaty, T. L. *Elements of
Queuing Theory*. New York:
McGraw-Hill, 1961.

Example Problem Solution

The following example illustrates the analysis of both a single-server and a multiple-server queuing system, including the determination of the operating characteristics for each system.

Problem Statement

The new accounts loan officer

of the Citizens Northern Savings Bank interviews all customers for new accounts. The customers desiring to open new accounts arrive at the rate of 4 per hour, according to a Poisson distribution, and the accounts officer spends an average of 12 minutes with each customer, setting up a new account.

- 1.** Determine the operating characteristics (P_0 , L , L_q , W , W_q , and P_w) for this system.

2. Add an additional accounts officer to the system described in this problem so that it is now a multiple-server queuing system with two channels and determine the operating characteristics required in part A.

Solution

Determine Operating Characteristics for the Single-Server System

Step 1.

$\lambda = 4$ customers per hr. arrive

$\mu = 5$ customers per hr. are served

$$P_0 = \left(1 - \frac{\lambda}{\mu}\right) = \left(1 - \frac{4}{5}\right)$$

= .20 probability of no customers in the system

$$L = \frac{\lambda}{\mu - \lambda} = \frac{4}{5 - 4}$$

= 4 customers, on average, in the queuing system

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{(4)^2}{5(5 - 4)}$$

= 3.2 customers, on average, in the waiting line

$$W = \frac{1}{\mu - \lambda} = \frac{1}{5 - 4}$$

= 1 hr. on average, in the system

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{(4)}{5(5 - 4)}$$

= 0.80 hr. (48 min.) average time in the waiting line

$$P_w = \frac{\lambda}{\mu} = \frac{4}{5}$$

= .80 probability that the new accounts officer is busy and a customer must wait

Determine the Operating Characteristics for the Multiple-Server System

$\lambda = 4$ customers per hr.
arrive

$\mu = 5$ customers per hr.
are served

$c = 2$ servers

Step 2.

$$\begin{aligned}P_0 &= \frac{1}{\left[\sum_{n=0}^{n=c-1} \frac{1}{n!} \left(\frac{\lambda}{\mu} \right)^n \right] + \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \left(\frac{c\mu}{c\mu - \lambda} \right)} \\&= \frac{1}{\left[\frac{1}{0!} \left(\frac{4}{5} \right)^0 + \frac{1}{1!} \left(\frac{4}{5} \right)^1 \right] + \frac{1}{2!} \left(\frac{4}{5} \right)^2 \frac{(2)(5)}{(2)(5) - 4}} \\&= .429 \text{ probability that no customers are in the system}\end{aligned}$$

$$\begin{aligned}L &= \frac{\lambda \mu \left(\frac{\lambda}{\mu} \right)^c}{(c-1)! (c\mu - \lambda)^2} P_0 + \frac{\lambda}{\mu} \\&= \frac{(4)(5) \left(\frac{4}{5} \right)^2}{1! [(2)(5) - (4)]^2} (.429) + \frac{4}{5} \\&= 0.952 \text{ customer, on average, in the system}\end{aligned}$$

$$\begin{aligned}L_q &= L - \frac{\lambda}{\mu} = .952 - \frac{4}{5} \\&= 0.152 \text{ customer, on average, waiting to be served}\end{aligned}$$

$$\begin{aligned}W &= \frac{L}{\lambda} = \frac{.952}{4} \\&= 0.238 \text{ hr. (14.3 min.) average time in the system}\end{aligned}$$

$$\begin{aligned}W_q &= \frac{L_q}{\lambda} = \frac{.152}{4} \\&= 0.038 \text{ hr. (2.3 min.) average time spent waiting in line}\end{aligned}$$

$$\begin{aligned}P_w &= \frac{1}{c!} \left(\frac{\lambda}{\mu} \right)^c \frac{c\mu}{c\mu - \lambda} P_0 = \frac{1}{2!} \left(\frac{4}{5} \right)^2 \frac{(2)(5)}{(2)(5) - 4} (.429) \\&= .229 \text{ probability that a customer must wait for service}\end{aligned}$$

Problems

For each of the following queuing systems, indicate whether it is a single- or multiple-server model, the queue discipline, and whether its calling population is infinite or finite.

- 1.** Hair salon
- 2.** Bank
- 3.** Laundromat
- 1.** **4.** Doctor's office
- 5.** Adviser's office

- 6.** Airport runway
- 7.** Service station
- 8.** Copy center
- 9.** Team trainer
- 10.** Mainframe computer

[Page 602]

In the Fast Shop Market example in this chapter, Alternative II was to add a new checkout counter at the **2.** market. This alternative was analyzed using the single-server model. Why was the multiple-server model not used?

1. Is the following

statement true or false? "The single-phase, single-channel model with Poisson arrivals and undefined service times will always have larger (i.e., greater) operating characteristic values (i.e., W , W_q , L , L_q) than the same model with exponentially distributed service times." Explain your answer.

3.

2. Is the following statement true or false? "The single-phase, single-channel model with Poisson arrivals and constant service times will always have smaller (i.e., lower) operating characteristic values (i.e., W , W_q , L , L_q) than the same model with exponentially distributed service times." Explain your answer.

- Why do waiting lines form at a service facility even though there may be more than enough service
- 4.** capacity to meet normal demand in the long run?

- 5.** Provide an example of when a first-in, first-out (FIFO) rule for queue discipline would not be appropriate.

- Under what conditions will the single-server queuing model with Poisson arrivals and undefined
- 6.** service times provide the same operating characteristics as the basic model with exponentially distributed service times?

A single-server queuing system

with an infinite calling population and a first-come, first-served queue discipline has the following arrival and service rates:

7.

$\lambda = 16$ customers per hour

$\mu = 24$ customers per hour

Determine P_0 , P_3 , L , L_q , W , W_q , and U .

8.

The ticket booth on the Tech campus is operated by one person, who is selling tickets for the annual Tech versus State football game on Saturday. The ticket seller can serve an average of 12 customers per hour; on average, 10 customers arrive to purchase tickets each hour. Determine the average time a ticket buyer must wait and the portion of time the

ticket seller is busy.

The Petroco Service Station has one pump for regular unleaded gas, which (with an attendant) can service 10 customers per hour. Cars arrive at the regular unleaded pump at a rate of 6 per hour.

- 9.** Determine the average queue length, the average time a car is in the system, and the average time a car must wait. If the arrival rate increases to 12 cars per hour, what will be the effect on the average queue length?

The Dynaco Manufacturing Company produces a particular product in an assembly line operation. One of the machines on the line is a drill press that has a single assembly line feeding into it.

- A partially completed unit arrives at the press to be worked on every 7.5 minutes, on average. The machine operator can process an average of 10 parts per hour. Determine the average number of parts waiting to be worked on, the percentage of time the operator is working, and the percentage of time the machine is idle.
- 10.**

- The management of Dynaco Manufacturing Company (see [Problem 10](#)) likes to have its operators working 90% of the time. What must the assembly line arrival rate be for the operators to be as busy as management would like?
- 11.**

The Peachtree Airport in Atlanta

serves light aircraft. It has a single runway and one air traffic controller to land planes. It takes an airplane 12 minutes to land and clear the runway. Planes arrive at the airport at the rate of four per hour.

- 1.** Determine the average number of planes that will stack up, waiting to land.
- 2.** Find the average time a plane must wait in line before it can land.

12.

[Page 603]

- 3.** Calculate the average time it takes a plane to clear the runway once it has notified the airport that it is in the vicinity and wants to land.

4. The FAA has a rule that an air traffic controller can, on average, land planes a maximum of 45 minutes out of every hour. There must be 15 minutes of idle time available to relieve the tension. Will this airport have to hire an extra air traffic controller?

The First American Bank of Rapid City has one outside drive-up teller. It takes the teller an average of 4 minutes to serve a bank customer. Customers arrive at the drive-up window at a rate of 12 per hour. The bank operations officer is currently analyzing the possibility of adding a second drive-up window, at an annual cost of

13. \$20,000. It is assumed that

arriving cars would be equally divided between both windows. The operations officer estimates that each minute's reduction in customer waiting time would increase the bank's revenue by \$2,000 annually. Should the second drive-up window be installed?

During registration at State University every semester, students in the college of business must have their courses approved by the college adviser. It takes the adviser an average of 2 minutes to approve each schedule, and students arrive at the adviser's office at the rate of 28 per hour.

- 1.** Compute L , L_q , W , W_q , and U .

14.

- 2.** The dean of the college has received a number of complaints from students about the length of time they must wait to have their schedules approved. The dean feels that waiting 10.00 minutes to get a schedule approved is not unreasonable. Each assistant the dean assigns to the adviser's office will reduce the average time required to approve a schedule by 0.25 minute, down to a minimum time of 1.00 minute to approve a schedule. How many assistants should the dean assign to the adviser?

All trucks traveling on Interstate 40 between Albuquerque and Amarillo

are required to stop at a weigh station. Trucks arrive at the weigh station at a rate of 200 per 8-hour day, and the station can weigh, on the average, 220 trucks per day.

- 1.** Determine the average number of trucks waiting, the average time spent waiting and being weighed at the weigh station by each truck, and the average waiting time before being weighed for each truck.

- 2.** If the truck drivers find out they must remain at the weigh station longer than 15 minutes, on average, they will start taking a different route or traveling at night, thus depriving the state of taxes. The state of New Mexico estimates that it loses

15.

\$10,000 in taxes per year for each extra minute trucks must remain at the weigh station. A new set of scales would have the same service capacity as the present set of scales, and it is assumed that arriving trucks would line up equally behind the two sets of scales. It would cost \$50,000 per year to operate the new scales. Should the state install the new set of scales?

- In [Problem 15](#), suppose passing truck drivers look to see how many trucks are waiting to be weighed at the weigh station. If
- 16.** they see four or more trucks in line, they will pass by the station and risk being caught and ticketed. What is the probability that a truck

will pass by the station?

In [Problem 14](#), the dean of the college of business at State University is considering the addition of a second adviser in the college advising office to serve students waiting to have their schedules approved. This new

17. adviser could serve the same number of students per hour as the present adviser. Determine L , L_q , W , and W_q for this altered advising system. As a student, would you recommend adding the adviser?

The Dynaco Manufacturing Company has an assembly line that feeds two drill presses. As

18. partially completed products come off the line, they are lined up to be worked on as drill presses become available. The units arrive at the workstation (containing both presses) at the rate of 100 per hour. Each press operator can process an average of 60 units per hour. Compute L , L_q , W , and W_q .
-

[Page 604]

The Acme Machine Shop has five machines that periodically break down and require service. The average time between breakdowns is 4 days, distributed according to an exponential distribution. The average time to repair a machine is 1 day, distributed according to an exponential distribution. One mechanic repairs the machines in

the order in which they break down.

19.

- 1.** Determine the probability of the mechanic being idle.
- 2.** Determine the mean number of machines waiting to be repaired.
- 3.** Determine the mean time machines wait to be repaired.
- 4.** Determine the probability that three machines are not operating (are being repaired or waiting to be repaired).

McBurger's fast-food restaurant has a drive-through window with a single server who takes orders from an intercom and also is the cashier. The window operator is

- assisted by other employees who prepare the orders. Customers
- 20.** arrive at the ordering station prior to the drive-through window every 4.5 minutes (Poisson distributed), and the service time is 2.8 minutes (exponentially distributed). Determine the average length of the waiting line and the waiting time. Discuss the quality of the service.

- Game World, a video game arcade at Tanglewood Mall, has just installed a new virtual reality battle game. It requires exactly 2.7 minutes to play. Customers arrive, on average, every 2.9 minutes
- 21.** (Poisson distributed) to play the game. How long will the line of customers waiting to play the game be, and how long, on

average, must a customer wait?

22. Drivers who come to get their licenses at the Department of Motor Vehicles have their photograph taken by an automated machine that develops the photograph onto the license card and laminates the complete license. The machine requires a constant time of 4.5 minutes to prepare a completed license. If drivers arrive at the machine at the mean rate of 10 per hour (Poisson distributed), determine the average length of the waiting line and the average waiting time.

A vending machine at City Airport dispenses hot coffee, hot chocolate, or hot tea, in a constant

service time of 20 seconds.

- 23.** Customers arrive at the vending machine at a mean rate of 60 per hour (Poisson distributed). Determine the average length of the waiting line and the average time a customer must wait.

Norfolk, Virginia, a major seaport on the East Coast, has a ship coal-loading facility. Coal trucks filled with coal arrive at the port facility at the mean rate of 149 per day (Poisson distributed). The facility operates 24 hours a day. The coal trucks are unloaded one at a time, on a first-come, first-served basis, by automated mechanical equipment that empties the trucks in a constant time of 8 minutes per truck, regardless of truck size. The port authority is negotiating with a

coal company for an additional 30 trucks per day. However, the coal company will not use this port facility unless the port authority can assure it that its coal trucks will not have to wait to be unloaded at the port facility for more than 12 hours per truck, on average. Can the port authority provide this assurance?

The Bay City Police Department has eight patrol cars that are on constant call 24 hours per day. A patrol car requires repairs every 20 days, on average, according to an exponential distribution. When a patrol car is in need of repair, it is driven into the motor pool, which has a repair person on duty at all times. The average time required to repair a patrol car is 18 hours

(exponentially distributed). Determine the average time a patrol car is not available for use and the average number of patrol cars out of service at any one time. Indicate whether the repair service seems adequate.

- The Rowntown Cab Company has four cabs that are on duty during normal business hours. The cab company dispatcher receives requests for service every 8 minutes, on average, according to an exponential distribution. The
- 26.** average time to complete a trip is 20 minutes (exponentially distributed). Determine the average number of customers waiting for service and the average time a customer must wait for a cab.

The Riverton Post Office has four stations for service. Customers line up in single file for service on a FIFO basis. The mean arrival rate is 40 per hour, Poisson distributed, and the mean service time per server is 4 minutes, exponentially distributed. Compute the operating characteristics for this operation.

- 27.** Indicate whether the operation appears to be satisfactory in terms of the following: (a) postal workers' (servers') idle time; (b) customer waiting time and/or the number waiting for service; and (c) the percentage of the time a customer can walk in and get served without waiting at all.

- In [Problem 18](#), the Dynaco Company has found that if more than three units (average) are waiting to be processed at any one workstation, then too much money is being tied up in work-in-process inventory (i.e., units waiting to be processed). The
- 28.** company estimates that (on average) each unit waiting to be processed costs \$50 per day. Alternatively, operating a third press would cost \$150 per day. Should the company operate a third press at this workstation?

Cakes baked by the Freshfood Bakery are transported from the ovens to be packaged by one of three wrappers. Each wrapper can wrap an average of 200 cakes per

29. hour. The cakes are brought to the wrappers at the rate of 500 per hour. If a cake sits longer than 5 minutes before being wrapped, it will not be fresh enough to meet the bakery's quality control standards. Does the bakery need to hire another wrapper?

The associate dean of the college of business at Tech is determining which of two copiers he should lease for the college's administrative suite. A regular copier leases for \$8 per hour, and it takes an employee an average of 6 minutes (exponentially distributed) to complete a copying job. A deluxe, high-speed copier leases for \$16 per hour, and it requires an average of 3 minutes to complete a copying job.

30.

Employees arrive at the copying machine at a rate of 7 per hour (Poisson distributed), and an employee's time is valued at \$10 per hour. Determine which copier the college should lease.

The Quick Wash 24-hour Laundromat has 16 washing machines. A machine breaks down every 20 days (exponentially distributed). The repair service with which the laundromat contracts takes an average of 1 day to repair a machine (exponentially distributed). A washing machine averages \$5 per hour in revenue. The laundromat is considering a new repair service that guarantees repairs in 0.50 days, but it charges \$10 more per hour than the current repair service. Should the

31.

laundromat switch to the new repair service?

The Regency Hotel has enough space at its entrance for six taxicabs to line up, wait for guests, and then for loading passengers. Cabs arrive at the hotel every 8 minutes; if a taxi drives by the hotel and the line is full, then it must drive on. Hotel guests require a taxi every 5 minutes, on average. It takes a cab driver an

32. average of 3.5 minutes to load passengers and luggage and leave the hotel (exponentially distributed).

- 1.** What is the average time a cab must wait for a fare?
- 2.** What is the probability that the line will be full when a cab

drives by, causing it to drive on?

The local Burger Doodle fast-food restaurant has a drive-through window. Customers in cars arrive at the window at the rate of 10 per hour (Poisson distributed). It requires an average of 4 minutes (exponentially distributed) to take and fill an order. The restaurant chain has a service goal of an average waiting time of 3 minutes.

1. Will the current system meet the restaurant's service goal?
2. If the restaurant is not meeting its service goal, it can add a second drive-through window that will reduce the service time per customer to 2.5 minutes. Will

33.

the additional window enable the restaurant to meet its service goal?

- 3.** During the 2-hour lunch period, the arrival rate of drive-in customers increases to 20 per hour. Will the two-window system be able to achieve the restaurant's service goal during this rush period?

[Page 606]

From 3:00 P.M. to 8:00 P.M. the local K-Star Supermarket has a steady stream of customers. Customers finish shopping and arrive at the checkout area at a rate of 70 per hour (Poisson distributed). It is assumed that

when customers arrive at the cash registers, they will divide themselves relatively evenly so that all the checkout lines contain

34. an equal number. The average checkout time at a register is 7 minutes (exponentially distributed). The store manager's service goal is for customers to be out of the store within 12 minutes (on average) after they complete their shopping and arrive at a cash register. How many cash registers must the store have open to achieve the manager's service goal?

Customers arrive to check in at the exclusive and expensive Regency Hotel's lobby at a rate of 40 per hour (Poisson distributed). The hotel normally has three clerks

35.

available at the desk to check in guests. The average time for a clerk to check in a guest is 4 minutes (exponentially distributed). Clerks at the Regency are paid \$24 per hour, and the hotel assigns a goodwill cost of \$2 per minute for the time a guest must wait in line. Determine whether the present check-in system is cost-effective. If it is not, recommend what hotel management should do.

36.

The Riverview Clinic has two general practitioners who see patients daily. An average of six patients arrive at the clinic per hour. Each doctor spends an average of 15 minutes with a patient. The patients wait in a waiting area until one of the two doctors is able to see them.

However, because patients typically do not feel well when they come to the clinic, the doctors do not believe it is good practice to have a patient wait longer than an average of 15 minutes. Should this clinic add a third doctor, and, if so, will this alleviate the waiting problem?

The Footrite Shoe Company is going to open a new branch at a mall, and company managers are attempting to determine how many salespeople to hire. Based on an analysis of mall traffic, the company estimates that customers will arrive at the store at a rate of 10 per hour, and from

37. past experience at its other branches, the company knows that salespeople can serve an

average of 6 customers per hour. How many salespeople should the company hire to uphold a company policy that on average the probability of a customer having to wait for service be no more than .30?

- When customers arrive at Gilley's Ice Cream Shop, they take a number and wait to be called to purchase ice cream from one of the counter servers. From experience in past summers, the store's staff knows that customers arrive at a rate of 40 per hour on summer days between 3:00 P.M. and 10:00 P.M., and a server can
- 38.** serve 15 customers per hour on average. Gilley's wants to make sure that customers wait no longer than 10 minutes for service.

Gilley's is contemplating keeping three servers behind the ice cream counter during the peak summer hours. Will this number be adequate to meet the waiting time policy?

Moore's television repair service receives an average of six TV sets per 8-hour day to be repaired. The service manager would like to be able to tell customers that they can expect 1-day service. What **39.** average repair time per set will the repair shop have to achieve to provide 1-day service, on average? (Assume that the arrival rate is Poisson distributed and repair times are exponentially distributed.)

- In [Problem 39](#), suppose that Moore's television repair service cannot accommodate more than
- 40.** 30 TV sets at a time. What is the probability that the number of TV sets on hand (under repair and waiting for service) will exceed the shop capacity?

- Maggie Attaberry is a nurse on the evening shift from 10:00 P.M. to 6:00 A.M. at Community Hospital. She has 15 patients for whom she is responsible in her area. She averages two calls from each of her patients every evening, on average (Poisson distributed), and she must spend an average of 10 minutes (negative exponential distribution) with each patient who calls. Nurse Attaberry has indicated
- 41.** to her shift supervisor that,

although she has not kept records, she believes her patients must wait about 10 minutes, on average, for her to respond, and she has requested that her supervisor assign a second nurse to her area. The supervisor believes 10 minutes is too long for a patient to wait, but she does not want her nurses to be idle more than 40% of the time. Determine what the supervisor should do.

[Page 607]

The Escargot is a small French restaurant with six waiters and waitresses. The average service time at the restaurant for a table (of any size) is 85 minutes (Poisson distributed). The restaurant does not take

reservations, and parties arrive for
42. dinner (and stay and wait) every 18 minutes (negative exponential distribution). The restaurant owner is concerned that a lengthy waiting time might hurt its business in the long run. What are the current waiting time and queue length for the restaurant? Discuss the business implications of the current waiting time and any actions the restaurant owner might take.

Hudson Valley Books is a small, independent publisher of fiction and nonfiction books. Each week the publisher receives an average of seven unsolicited manuscripts to review (Poisson distributed). The publisher has 12 freelance reviewers in the area who read and evaluate manuscripts. It takes a

43. reviewer an average of 10 days (exponentially distributed) to read a manuscript and write a brief synopsis. (Reviewers work on their own, 7 days a week.) Determine how long the publisher must wait, on average, to receive a reviewer's manuscript evaluation, how many manuscripts are waiting to be reviewed, and how busy the reviewers are.

Amanda Fall is starting up a new house painting business, Fall Colors. She has been advertising in the local newspaper for several months. Based on inquiries and informal surveys of the local housing market, she anticipates that she will get painting jobs at the rate of four per week (Poisson distributed). Amanda has also

determined that it will take a four-person team of painters an average of 0.7 week (exponentially distributed) for a typical painting job.

1. Determine the number of teams of painters Amanda needs to hire so that customers will have to wait no longer than 2 weeks to get their houses painted.
2. If the average price for a painting job is \$1,700 and Amanda pays a team of painters \$500 per week, will she make any money?

Partially completed products arrive at a workstation in a manufacturing operation at a mean rate of 40 per hour (Poisson

distributed). The processing time at the workstation averages 1.2 minutes per unit (exponentially distributed). The manufacturing company estimates that each unit

45. of work-in-process inventory at the workstation costs \$31 per day (on average). However, the company can add extra employees and reduce the processing time to 0.90 minute per unit, at a cost of \$52 per day. Determine whether the company should continue the present operation or add extra employees.

The Atlantic Coast Shipping Company has a warehouse terminal in Spartanburg, South Carolina. The capacity of each terminal dock is three trucks. As trucks enter the terminal, the

drivers receive numbers; and when one of the three dock spaces becomes available, the truck with the lowest number enters the vacant dock. Truck arrivals are Poisson distributed, and the unloading and loading times (service times) are exponentially distributed. The average arrival rate at the terminal is five trucks per hour, and the average service rate per dock is two trucks per hour (30 minutes per truck).

- 1.** Compute L , L_q , W , and W_q .
- 2.** The management of the shipping company is considering adding extra employees and equipment to improve the average service time per terminal dock to 25 minutes per truck. It would cost the company \$18,000

per year to achieve this improved service.

46.

Management estimates that it will increase its profit by \$750 per year for each minute it is able to reduce a truck's waiting time.

Determine whether management should make the investment.

- 3.** Now suppose that the managers of the shipping company have decided that truck waiting time from part a is excessive and they want to reduce the waiting time. They have determined that there are two alternatives available for reducing the waiting time. They can add a fourth dock, or they can add extra employees and equipment at the existing docks, which will

reduce the average service time per location from the original 30 minutes per truck to 23 minutes per truck. The costs of these alternatives are approximately equal. Management desires to implement the alternative that reduces waiting time by the greatest amount. Which alternative should be selected? (Computer solution is suggested.)

[Page 608]

The Waterfall Buffet in the lower level of the National Art Gallery serves food cafeteria-style daily to visitors and employees. The buffet is self-service. From 7:00 A.M. to 9:00 A.M. customers arrive at the

47.

buffet at a rate of 10 per minute; from 9:00 A.M. to noon, at 4 per minute; from noon to 2:00 P.M., at 14 per minute; and from 2:00 P.M. to closing at 5:00 P.M., at 8 per minute. All the customers take about the same amount of time to serve themselves from the buffet. Once a customer goes through the buffet, it takes 0.4 minute to pay the cashier. The gallery does not want a customer to have to wait longer than 4 minutes to pay. How many cashiers should be working at each of the four times during the day?

The Clip Joint is a hairstyling salon at University Mall. Four stylists are always available to serve customers on a first-come, first-served basis. Customers arrive at

an average rate of five per hour, and the stylists spend an average of 35 minutes on each customer.

- 48.**
- 1.** Determine the average number of customers in the salon, the average time a customer must wait, and the average number waiting to be served.
 - 2.** The salon manager is considering adding a fifth stylist. Would this have a significant impact on waiting time?

The Delacroix Inn in Alexandria is a small, exclusive hotel with 20 rooms. Guests can call housekeeping from 8:00 A.M. to midnight for any of their service needs. Housekeeping keeps one

49.

person on duty during this time to respond to guest calls. Each room averages 0.7 call per day to housekeeping (Poisson distributed), and a guest request requires an average of 30 minutes (exponentially distributed) for the staff person to respond to and take care of. Determine the portion of time the staff person is busy and how long a guest must wait for his or her request to be addressed. Does the housekeeping system seem adequate?

Jim Carter builds custom furniture, primarily cabinets, bookcases, small tables, and chairs. He works on only one piece of furniture for a customer at a time. It takes him an average of 5 weeks (exponentially distributed) to build

- a piece of furniture. An average of 14 customers approach Jim to
- 50.** order pieces of furniture each year (Poisson distributed); however, Jim will take only a maximum of 8 advance orders. Determine the average time a customer must wait to receive a furniture order once it is placed and how busy Jim is. What is the probability that a customer will be able to place an order with Jim?

Judith Lewis is a doctoral student at State University, and she also works full-time as an academic tutor for 10 scholarshiped student athletes. She took the job, hoping it would leave her free time between tutoring to devote to her own studies. An athlete visits her for tutoring an average of every

16 hours, and she spends an average of 1.5 hours (exponentially distributed) with him or her. She is able to tutor only one athlete at a time, and athletes

51. study while they are waiting.

- 1.** Determine how long a player must wait to see her and the percentage of time Judith is busy. Does the job seem to meet Judith's expectations, and does the system seem adequate to meet the athletes' needs?
- 2.** If the results in (a) indicate that the tutoring arrangement is ineffective, suggest an adjustment that could improve it for both the athletes and Judith.

52. TSA security agents at the security gate at the Tri-Cities Regional Airport are able to check passengers and process them through the security gate at a rate of 50 per hour (Poisson distributed). Passengers arrive at the gate throughout the day at a rate of 40 per hour (Poisson distributed). Determine the average waiting time and waiting line.

[Page 609]

In [Problem 52](#) the passenger traffic arriving at the airport security gate varies significantly during the day. At times close to flight takeoffs, the traffic is very heavy, while at other times there is little or no passenger traffic through the

53. security gate. At the times leading up to flight takeoffs, passengers arrive at a rate of 110 per hour (Poisson distributed). Suggest a waiting line system that can accommodate this level of passenger traffic.

The inland port at Front Royal, Virginia, is a transportation hub that primarily transfers shipping containers from trucks coming from eastern seaports to rail cars for shipment to inland destinations. Containers arrive at the inland

54. ports at a rate of eight per hour. It takes 28 minutes (exponentially distributed) for a crane to unload a container from a truck, place it on a flatbed railcar, and secure it. Suggest a waiting line system that will effectively handle this level of

container traffic at the inland port.

The Old Colony theme park has a new ride, the Double Cyclone. The ride holds 30 people in double roller coaster cars, and it takes 3.8 minutes to complete the ride circuit. It also takes the ride attendants another 3 minutes

55. (with virtually no variation) to load and unload passengers.

Passengers arrive at the ride during peak park hours at a rate of 4 per minute (Poisson distributed).

Determine the length of the waiting line for the ride.

Case Problems

THE COLLEGE OF BUSINESS COPY CENTER

The copy center at the college of business at State University has become an increasingly contentious item among the college administrators. The department heads have complained to the associate dean about the long lines and waiting times for their secretaries at the copy center.

They claim that it is a waste of scarce resources for the secretaries to stand in line talking when they could be doing more productive work in the office. Alternatively, Handford Burris, the associate dean, says the limited operating budget will not allow the college to purchase a new copier, or several copiers, to relieve the problem. This standoff has been going on for several years.

To make her case for improved copying facilities, Lauren Moore, the department head

for management science, assigned students a class project to gather some information about the copy center. The students were to record the arrivals at the center and the length of time it took to do a copy job once the secretary actually reached a copy machine. In addition, the students were to describe how the copy center system worked.

When the students completed the project, they turned in a report to Professor Moore. The report described the copy center as containing two

machines. When secretaries arrived for a copy job, they joined a queue, which looked more like milling around to the students. But they acknowledged that the secretaries knew when it was their turn, and, in effect, they formed a single queue for the first available copy machine. Also, because copy jobs are assigned tasks, secretaries always stayed to do the job, no matter how long the line was or how long they had to wait. They never left the queue.

From the data the students

gathered, Professor Moore is able to determine that secretaries arrive every 8 minutes for a copy job and that the arrival rate is Poisson distributed. Furthermore, she was able to determine that the average time it takes to complete a job is 12 minutes and that this is exponentially distributed.

Using her own personnel records and some data from the university's personnel office, Dr. Moore determines that a secretary's average salary is \$8.50 per hour. From her

academic calendar she adds up the actual days in the year when the college and departmental offices are open and finds there are 247. However, as she adds up working days, it occurs to her that during the summer months, the workload is much lighter, and the copy center would also probably get less traffic. The summer includes about 70 days, during which she expects the copy center traffic would be about half of what it is during the normal year, but she speculates that

the average time of a copying job would remain about the same.

Professor Moore next calls a local office supply firm to check the prices on copiers. A new copier of the type in the copy center now would cost \$36,000. It would also require \$8,000 per year for maintenance and would have a normal useful life of 6 years.

Do you think Dr. Moore will be able to convince the associate dean that a new copier machine will be cost-effective?

Case Problem

NORTHWOODS BACKPACKERS

Bob and Carol Packer operate a successful outdoor-wear store in Vermont called Northwoods Backpackers. They stock mostly cold weather outdoor items such as hiking and backpacking clothes, gear, and accessories. They established an excellent reputation throughout New England for quality products and service. Eventually, Bob

and Carol noticed that more and more of their sales were to customers who did not live in the immediate vicinity but were calling in orders on the telephone. As a result, the Packers decided to distribute a catalog and establish a phone-order service. The order department consisted of five operators working 8 hours per day from 10:00 A.M. to 6:00 P.M., Monday through Friday. For a few years the mail-order service was only moderately successful; the Packers just about broke even on their

investment. However, during the holiday season of the third year of the catalog-order service, they were overwhelmed with phone orders. Although they made a substantial profit, they were concerned about the large number of lost sales they estimated they had incurred. Based on information provided by the telephone company regarding call volume and complaints from customers, the Packers estimated that they lost sales of approximately \$100,000. Also, they felt they

had lost a substantial number of old and potentially new customers because of the poor service of the catalog order department.

Prior to the next holiday season, the Packers explored several alternatives for improving the catalog-order service. The current system includes the five original operators with computer terminals who work 8-hour days, 5 days per week. The Packers hired a consultant to study this system, and she reported that the time for an

operator to take a customer order is exponentially distributed, with a mean of 3.6 minutes. Calls are expected to arrive at the telephone center during the 6-week holiday season, according to a Poisson distribution, with a mean rate of 175 calls per hour. When all operators are busy, callers are put on hold, listening to music until an operator can answer. Waiting calls are answered on a FIFO basis. Based on her experience with other catalog telephone-order operations and data from Northwoods

Backpackers, the consultant has determined that if Northwoods Backpackers can reduce customer call-waiting time to approximately 0.5 minute or less, the company will save \$135,000 in lost sales during the coming holiday season.

Therefore, the Packers have adopted this level of call service as their goal. However, in addition to simply avoiding lost sales, the Packers believe it is important to reduce waiting time in order to maintain their reputation for good customer

service. Thus, they would like for about 70% of their callers to receive immediate service.

The Packers can maintain the same number of workstations and computer terminals they currently have and increase their service to 16 hours per day with two operator shifts running from 8:00 A.M. to midnight. The Packers believe when customers become aware of their extended hours, the calls will spread out uniformly, resulting in a new call average arrival rate of 87.5 calls per hour (still Poisson distributed).

This schedule change would cost Northwoods Backpackers approximately \$11,500 for the 6-week holiday season.

Another alternative for reducing customer waiting times is to offer weekend service. However, the Packers believe that if they offer weekend service, it must coincide with whatever service they offer during the week. In other words, if they have phone-order service 8 hours per day during the week, they must have the same service during the weekend; the same

is true with 16-hours-per-day service. They feel that if weekend hours differ from week-day hours, it will confuse customers. If 8-hour service is offered 7 days per week, the new call arrival rate will be reduced to 125 calls per hour, at a cost of \$3,600. If they offer 16-hour service, the mean call arrival rate will be reduced to 62.5 calls per hour, at a cost of \$7,200.

Still another possibility is to add more operator stations. Each station includes a desk, an operator, a phone, and a

computer terminal. An additional station that is in operation 5 days per week, 8 hours per day, will cost \$2,900 for the holiday season. For a 16-hour day the cost per new station is \$4,700. For 7-day service the cost of an additional station for 8-hours-per-day service is \$3,800; for 16-hours-per-day service, the cost is \$6,300.

The facility Northwoods Backpackers uses to house its operators can accommodate a maximum of 10 stations. Additional operators in excess

of 10 would require the Packers to lease, remodel, and wire a new facility, which is a capital expenditure they do not want to undertake this holiday season. Alternatively, the Packers do not want to reduce their current number of operator stations.

Determine what order service configuration the Packers should use to achieve their goals and explain your recommendation.

Chapter 14.

Simulation

[Page 612]

Simulation represents a major divergence from the topics presented in the previous chapters of this text. Previous topics usually dealt with mathematical models and formulas that could be applied to certain types of problems. The solution approaches to these problems were, for the most part, analytical. However,

not all real-world problems can be solved by applying a specific type of technique and then performing the calculations. Some problem situations are too complex to be represented by the concise techniques presented so far in this text. In such cases, **simulation** is an alternative form of analysis.

Analogue simulation is a form of simulation that is familiar to most people. In analogue simulation, an original physical system is replaced by an analogous physical system that is easier

to manipulate. Much of the experimentation in staffed spaceflight was conducted using physical simulation that re-created the conditions of space. For example, conditions of weightlessness were simulated using rooms filled with water. Other examples include wind tunnels that simulate the conditions of flight and treadmills that simulate automobile tire wear in a laboratory instead of on the road.

Analogue

simulation replaces
a physical system
with an analogous
physical system that
is easier to
manipulate.

This chapter is concerned with
an alternative type of
simulation, *computer
mathematical simulation*. In this
form of simulation, systems are
replicated with mathematical
models, which are analyzed

using a computer. This form of simulation has become very popular and has been applied to a wide variety of business problems. One reason for its popularity is that it offers a means of analyzing very complex systems that cannot be analyzed by using the other management science techniques in this text.

However, because such complex systems are beyond the scope of this text, we will not present actual simulation models; instead, we will present simplified simulation models of

systems that can also be analyzed analytically. We will begin with one of the simplest forms of simulation models, which encompasses the Monte Carlo process for simulating random variables.

In computer mathematical simulation, a system is replicated with a mathematical model that is analyzed by using the computer.

The Monte Carlo Process

One characteristic of some systems that makes them difficult to solve analytically is that they consist of random variables represented by probability distributions. Thus, a large proportion of the applications of simulations are for probabilistic models.

The term **Monte Carlo** has become synonymous with probabilistic simulation in

recent years. However, the Monte Carlo technique can be more narrowly defined as a technique for selecting numbers *randomly* from a probability distribution (i.e., "sampling") for use in a *trial* (computer) run of a simulation. The Monte Carlo technique is not a type of simulation model but rather a mathematical process used within a simulation.

Monte Carlo is a
technique for
selecting numbers

*randomly from a
probability
distribution.*

The name *Monte Carlo* is appropriate because the basic principle behind the process is the same as in the operation of a gambling casino in Monaco. In Monaco such devices as roulette wheels, dice, and playing cards are used. These devices produce numbered results at random from well-

defined populations. For example, a 7 resulting from thrown dice is a random value from a population of 11 possible numbers (i.e., 2 through 12). This same process is employed, in principle, in the Monte Carlo process used in simulation models.

The Monte Carlo process is analogous to gambling devices.

The Use of Random Numbers

The Monte Carlo process of selecting random numbers according to a probability distribution will be demonstrated using the following example. The manager of Computer-World, a store that sells computers and related equipment, is attempting to determine how many laptop PCs the store should order each week. A primary consideration in this

decision is the average number of laptop computers that the store will sell each week and the average weekly revenue generated from the sale of laptop PCs. A laptop sells for \$4,300. The number of laptops demanded each week is a random variable (which we will define as x) that ranges from 0 to 4. From past sales records, the manager has determined the frequency of demand for laptop PCs for the past 100 weeks. From this frequency distribution, a probability distribution of demand can be

developed, as shown in [Table 14.1](#).

[Page 613]

Table 14.1. Probability distribution of demand for laptop PCs

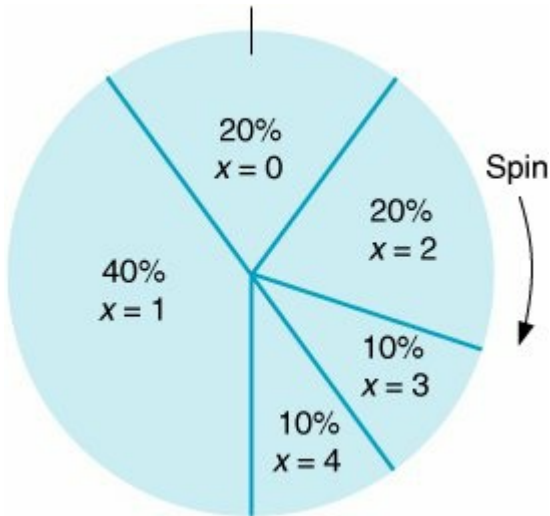
PCs Demanded per Week	Frequency of Demand	Probability of Demand, $P(x)$
0	20	.20
1	40	.40
2	20	.20

3	10	.10
4	10	.10
	100	1.00

The purpose of the Monte Carlo process is to generate the random variable, demand, by sampling from the probability distribution, $P(x)$. The demand per week can be randomly generated according to the

probability distribution by spinning a wheel that is partitioned into segments corresponding to the probabilities, as shown in [Figure 14.1](#).

Figure 14.1. A roulette wheel for demand



In the Monte Carlo process, values for a random variable are generated by

*sampling from a
probability
distribution.*

Because the surface area on the roulette wheel is partitioned according to the probability of each weekly demand value, the wheel replicates the probability distribution for demand if the values of demand occur in a random manner. To simulate demand for 1 week, the

manager spins the wheel; the segment at which the wheel stops indicates demand for 1 week. Over a period of weeks (i.e., many spins of the wheel), the frequency with which demand values occur will approximate the probability distribution, $P(x)$. This method of generating values of a variable, x , by randomly selecting from the probability distribution the wheel is the Monte Carlo process.

By spinning the wheel, the manager artificially reconstructs the purchase of

PCs during a week. In this reconstruction, a long period of *real time* (i.e., a number of weeks) is represented by a short period of **simulated time** (i.e., several spins of the wheel).

A long period of real time is represented by a short period of simulated time.

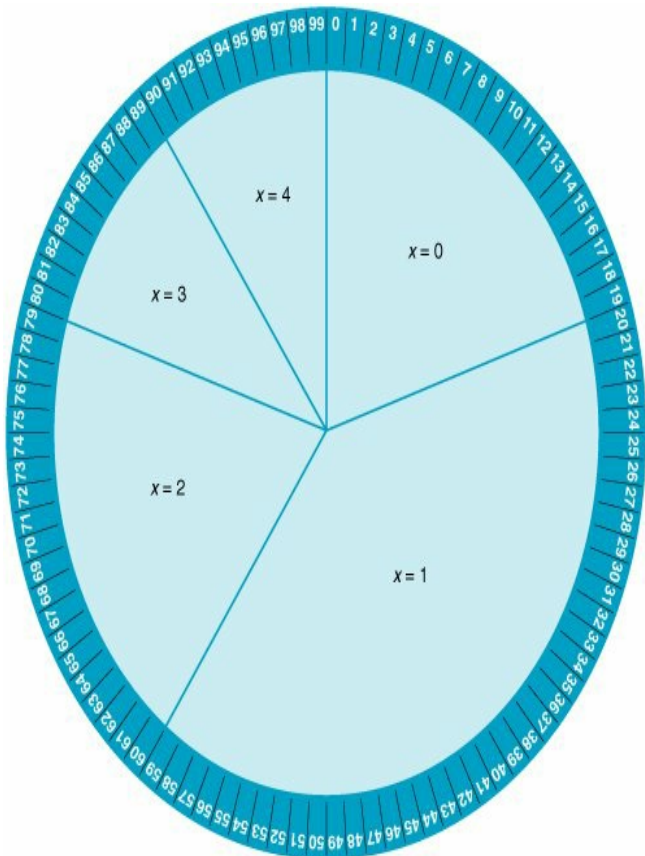
Now let us slightly reconstruct

the roulette wheel. In addition to partitioning the wheel into segments corresponding to the probability of demand, we will put numbers along the outer rim, as on a real roulette wheel. This reconstructed roulette wheel is shown in [Figure 14.2](#).

Figure 14.2.

Numbered roulette wheel

(This item is displayed on page 614 in the print version)



There are 100 numbers from 0 to 99 on the outer rim of the wheel, and they have been partitioned according to the probability of each demand value. For example, 20 numbers from 0 to 19 (i.e., 20% of the total 100 numbers) correspond to a demand of no (0) PCs. Now we can determine the value of demand by seeing which number the wheel stops at as well as by looking at the segment of the wheel.

When the manager spins this new wheel, the actual demand for PCs will be determined by a number. For example, if the number 71 comes up on a spin, the demand is 2 PCs per week; the number 30 indicates a demand of 1. Because the manager does not know which number will come up prior to the spin and there is an equal chance of any of the 100 numbers occurring, the numbers occur at random; that is, they are **random numbers**.

Obviously, it is not generally practical to generate weekly demand for PCs by spinning a wheel. Alternatively, the process of spinning a wheel can be replicated by using random numbers alone.

First, we will transfer the ranges of random numbers for each demand value from the roulette wheel to a table, as in [Table 14.2](#). Next, instead of spinning the wheel to get a random number, we will select a random number from [Table 14.3](#), which is referred to as a **[random number table](#)**.

(These random numbers have been generated by computer so that they are all *equally likely to occur*, just as if we had spun a wheel. The development of random numbers is discussed in more detail later in this chapter.) As an example, let us select the number 39, the first entry in [Table 14.3](#). Looking again at [Table 14.2](#), we can see that the random number 39 falls in the range 2059, which corresponds to a weekly demand of 1 laptop PC.

Table 14.2.

Generating demand from random numbers

(This item is displayed on page 614 in the print version)

Demand, x	Ranges of Random Numbers, r	
0	0-19	
1	20-59	← $r = 39$
2	60-79	
3	80-89	
4	90-99	

Table 14.3. Random number table

39 65	19 90	20 26	58 24	95 06
76 45	69 64	36 31	97 14	70 99
45	61	62	97	00

73 71	65 97	31 56	47 83	30 62
23 70	60 12	34 19	75 51	38 20
90	11	19	33	46

72 18	51 67	98 40	23 05	59 78
47 33	47 97	07 17	09 51	11 52
84	19	66	80	49

75 12	17 95	24 33	69 81	93 22
25 69	21 78	45 77	84 09	70 45
17	58	48	29	80

37 17	63 52	01 31	35 07	28 99
79 88	06 34	60 10	79 71	52 01
74	30	27	53	41

02 48	85 53	56 27	21 78	72 61
08 16	83 29	09 24	55 09	88 73
94	95	43	82	61

87 89	37 79	48 13	41 92	09 18
15 70	49 12	93 55	45 71	25 58
07	38	96	51	94

98 18	89 09	00 06	14 36	54 45
71 70	39 59	41 41	59 25	17 24
15	24	20	47	89

10 83	76 62	58 76	59 53	66 04
58 07	16 48	17 14	11 52	18 72
04	68	86	21	87

47 08	71 82	27 55	28 04	95 23
56 37	13 50	10 24	67 53	00 84
31	41	92	44	47

93 90	34 18	74 13	68 95	36 63
31 03	04 52	39 35	23 92	70 35
07	35	22	35	33

21 05	11 20	76 51	13 79	98 16
11 47	99 45	94 84	93 37	04 41
99	18	86	55	67

95 89	27 37	79 57	09 61	56 20
94 06	83 28	95 13	87 25	11 32
97	71	91	21	44

97 18	10 65	77 31	20 44	26 99
31 55	81 92	61 95	90 32	76 75

73	59	46	64	63
69 08	59 71	48 38	73 37	60 82
88 86	74 17	75 93	32 04	29 20
13	32	29	05	25
41 26	87 63	81 83	77 45	79 88
10 25	93 95	83 04	85 50	01 97
03	17	49	51	30
91 47	08 61	92 79	29 18	14 85
14 63	74 51	43 89	94 51	11 47
62	69	79	23	23
80 94	08 52	48 40	72 65	50 03
54 18	85 08	35 94	71 08	42 99
47	40	22	86	36

67 06	89 85	64 71	89 37	61 65
77 63	84 46	06 21	20 70	70 22
99	06	66	01	12

59 72	42 29	06 94	81 30	81 33
24 13	72 23	76 10	15 39	17 16
75	19	08	14	33

63 62	79 53	94 61	20 21	84 95
06 34	36 02	09 43	14 68	48 46
41	95	62	86	45

78 47	79 93	34 85	85 43	14 93
23 53	96 38	52 05	01 72	87 81
90	63	09	73	40

87 68	97 48	53 16	59 97	24 62
62 15	72 66	71 13	50 99	20 42
43	48	81	52	31

47 60	26 97	88 46	72 68	75 70
92 10	05 73	38 03	49 29	16 08
77	51	58	31	24

56 88	06 87	65 88	88 02	85 81
87 59	37 78	69 58	84 27	56 39
41	48	39	83	38

22 17	87 02	68 69	11 29	49 34
68 65	22 57	80 95	01 95	35 36
84	51	44	80	47

19 36	39 77	79 57	89 74	08 58
27 59	32 77	92 36	39 82	94 34
46	09	59	15	74

16 77	28 06	22 45	87 80	09 71
23 02	24 25	44 84	61 65	91 74

77	93	11	31	25
78 43	97 67	30 45	59 73	53 33
76 71	63 99	67 93	19 85	65 97
61	61	82	23	21

03 28	69 30	53 58	66 56	45 56
28 26	16 09	47 70	45 65	20 19
08	05	93	79	47

04 31	33 73	26 72	53 77	60 61
17 21	99 19	39 27	57 68	97 22
56	87	67	93	61

61 06	87 14	43 00	45 60	98 99
98 03	77 43	65 98	33 01	46 50
91	96	50	07	47

23 68	99 53	52 70	56 65	90 92
35 26	93 61	05 48	05 61	10 70
00	28	34	86	80

15 39	93 86	15 33	22 87	86 96
25 70	52 77	59 05	26 07	98 29
99	65	28	47	06

58 71	18 46	85 13	49 18	74 16
96 30	23 34	99 24	09 79	32 23
24	27	44	49	02

93 22	07 10	87 03	08 13	55 34
53 64	63 76	04 79	13 85	57 72
39	35	88	51	69

78 76	92 38	52 06	82 63	69 66
58 54	70 96	79 79	18 27	92 19
74	92	45	44	09

61 81	00 57	46 72	55 66	08 99
31 96	25 60	60 18	12 62	55 64
82	59	77	11	57

42 88	24 98	47 21	27 80	10 92
07 10	65 63	61 88	30 21	35 36
05	21	32	60	12

77 94	28 10	12 73	49 99	96 88
30 05	99 00	73 99	57 94	57 17
39	27	12	82	91

*Random numbers are
equally likely to
occur.*

By repeating this process of selecting random numbers from [Table 14.3](#) (starting anywhere in the table and moving in any direction but not repeating the same sequence) and then determining weekly demand from the random number, we can simulate demand for a period of time. For example, [Table 14.4](#) shows demand for a period of 15 consecutive weeks.

Table 14.4. Randomly generated demand for 15 weeks

Week	r	Demand, x	Revenue
1	39	1	\$ 4,300
2	73	2	8,600
3	72	2	8,600
4	75	2	8,600
5	37	1	4,300
6	02	0	0

7	87	3	12,900
8	98	4	17,200
9	10	0	0
10	47	1	4,300
11	93	4	17,200
12	21	1	4,300
13	95	4	17,200
14	97	4	17,200

15

69

2

8,600

 $\Sigma=31$

\$133,300

From [Table 14.4](#) the manager can compute the estimated average weekly demand and revenue:

$$\text{estimated average demand} = \frac{31}{15} = 2.07 \text{ laptop PCs per week}$$

$$\text{estimated average revenue} = \frac{\$133,300}{15} = \$8,886.67$$

The manager can then use this

information to help determine the number of PCs to order each week.

Although this example is convenient for illustrating how simulation works, the average demand could have more appropriately been calculated *analytically* using the formula for expected value. The *expected value* or average for weekly demand can be computed analytically from the probability distribution, $P(x)$:

$$E(x) = \sum_{i=1}^n P(x_i)x_i$$

where

x_i = demand value i

$P(x_i)$ = probability of demand

n = the number of different
demand values

Therefore,

$$E(x) = (.20)(0) + (.40)(1) + (.20)(2) + (.10)(3) + (.10)(4)$$

= 1.5 PCs per week

Simulation results will not equal analytical results unless enough trials of the simulation have been conducted to reach steady state.

The analytical result of 1.5 PCs is close to the simulated result of 2.07 PCs, but clearly there is some difference. The margin of difference (0.57 PCs) between the simulated value and the analytical value is a result of the number of periods over which the simulation was conducted. The results of any simulation study are subject to the number of times the simulation occurred (i.e., the number of *trials*). Thus, the more periods for which the

simulation is conducted, the more accurate the result. For example, if demand were simulated for 1,000 weeks, in all likelihood an average value exactly equal to the analytical value (1.5 laptop PCs per week) would result.

Time Out: For John Von Neumann

The mathematics of the Monte Carlo method have been known for years; the British mathematician Lord Kelvin used the technique in a paper in 1901. However, it was formally identified and given this name by the Hungarian mathematician John Von Neumann while working on the Los Alamos atomic bomb project during World War II. During this project, physicists confronted a problem in determining how far neutrons would travel through various materials (i.e., neutron diffusion in fissile material). The Monte Carlo process was suggested to Von Neumann by a colleague at Los Alamos, Stanislas Ulam, as a means to solve this problem that is, by selecting random

numbers to represent the random actions of neutrons. However, the Monte Carlo method as used in simulation did not gain widespread popularity until the development of the modern electronic computer after the war. Interestingly, this remarkable man, John Von Neumann, is credited with being the key figure in the development of the computer.

Once a simulation has been repeated enough times that it reaches an average result that remains constant, this result is analogous to the *steady-state* result, a concept we discussed previously in our presentation of queuing. For this example, 1.5 PCs is the long-run average or steady-state result, but we have seen that the simulation might have to be repeated more than 15 times (i.e., weeks) before this result is reached.

Comparing our simulated result with the analytical (expected value) result for this example points out one of the problems that can occur with simulation. It is often difficult to validate the results of a simulation model that is, to make sure that the true steady-state average result has been reached. In this case we were able to compare the simulated result with the expected value (which is the true steady-state result), and we found there was a slight difference. We logically deduced that the 15 trials of

the simulation were not sufficient to determine the steady-state average. However, simulation most often is employed whenever analytical analysis is not possible (this is one of the reasons that simulation is generally useful). In these cases, there is no analytical standard of comparison, and validation of the results becomes more difficult. We will discuss this problem of validation in more detail later in the chapter.

It is often difficult to

*validate that the
results of a
simulation truly
replicate reality.*

Computer Simulation with Excel Spreadsheets

The simulation we performed manually for the ComputerWorld example was not too difficult. However, if we had performed the simulation for 1,000 weeks, it would have taken several hours. On the other hand, this simulation could be done on a computer in several seconds. Also, our simulation example was not very complex. As simulation

models get progressively more complex, it becomes virtually impossible to perform them manually, thus making the computer a necessity.

Simulations are normally done on the computer.

Although we will not develop a simulation model in a computer language for this example, we will demonstrate how a

computerized simulation model is developed by using Excel spreadsheets.

The first step in developing a simulation model is to generate a random number, r . Numerous subroutines that are available on practically every computer system generate random numbers. Most are quite easy to use and require the insertion of only a few statements in a program. These random numbers are generated by mathematical processes as opposed to a physical process, such as spinning a roulette

wheel. For this reason, they are referred to as **pseudorandom numbers**. It should be apparent from the previous discussion that random numbers play a very important part in a probabilistic simulation. Some of the random numbers we used came from **Table 14.3**, a table of random numbers. However, random numbers do not come just from tables, and their generation is not as simple as one might initially think. If random numbers are not truly random, the validity of

simulation results can be significantly affected.

[Page 618]

*Random numbers generated by a mathematical process instead of a physical process are **pseudorandom numbers.***

The random numbers in [Table](#)

14.3 were generated by using a *numerical technique*. Thus, they are not true random numbers but *pseudorandom numbers*. True random numbers can be produced only by a physical process, such as spinning a roulette wheel over and over. However, a physical process, such as spinning a roulette wheel, cannot be conveniently employed in a computerized simulation model. Thus, there is a need for a numerical method that artificially creates random numbers.

Random numbers are typically generated on the computer by using a numerical technique.

To truly reflect the system being simulated, the artificially created random numbers must have the following characteristics:

- 1.** The random numbers must be uniformly distributed.

This means that each random number in the interval of random numbers (i.e., 0 to 1 or 0 to 100) has an equal chance of being selected. If this condition is not met, then the simulation results will be biased by the random numbers that have a *more likely* chance of being selected.

- 2.** The numerical technique for generating random numbers should be efficient. This means that the random numbers

should not degenerate into constant values or recycle too frequently.

- 3.** The sequence of random numbers should not reflect any pattern. For example, the sequence of numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, and so on, although uniform, is not random.

A table of random numbers must be uniform, efficiently

*generated, and
absent of patterns.*

Random numbers between 0 and 1 can be generated in Excel by entering the formula **=RAND()** in a cell. The random numbers generated by this formula include all the necessary characteristics for randomness and uniformity that we discussed earlier.

[Exhibit 14.1](#) is an Excel spreadsheet with 100 random

numbers generated by entering the formula **=RAND()** in cell A3 and copying to the cells in the range **A3:J12**. Recall that we can copy things in a range of cells in two ways. You can first cover cells **A3:J12** with the cursor and then type the formula **=RAND()** into cell A3. Then you press the "Ctrl" and "Enter" keys simultaneously. Alternatively, you can type **=RAND()** in cell A3, copy this cell (using the right mouse button), then cover cells **A3:J12** with the cursor, and (again with the right mouse

button) paste this formula in these cells.

Exhibit 14.1.

[\[View full size image\]](#)

[illegible]

If you attempt to replicate this spreadsheet, you will generate different random numbers from those shown in [Exhibit 14.1](#).

Every time you generate random numbers, they will be different. In fact, any time you recalculate anything on your spreadsheet, the random numbers will change. You can see this by pressing the F9 key and observing that all the random numbers change.

However, sometimes it is useful in a simulation model to be able to use the same set (or stream) of random numbers

over and over. You can freeze the random numbers you are using on your spreadsheet by first covering the cells with random numbers in them with the cursor for example, cells **A3:J12** in [Exhibit 14.1](#). Next, you copy these cells (using the right mouse button); then you click on the "Edit" menu at the top of your spreadsheet and select "Paste Special" from this menu. Next, you select the "Values" option and click on "OK." This procedure pastes a copy of the numbers in these cells over the same cells with

=**RAND**() formulas in them, thus freezing the numbers in place.

[Page 619]

Notice one more thing from [Exhibit 14.1](#): The random numbers are all between 0 and 1, whereas the random numbers in [Table 14.3](#) are whole numbers between 0 and 100. We used whole random numbers previously for illustrative purposes; however, computer programs such as Excel generally provide random

numbers between 0 and 1.

Now we are ready to duplicate our example simulation model for the ComputerWorld store by using Excel. The spreadsheet in [Exhibit 14.2](#) includes the simulation model originally developed in [Table 14.4](#).

Exhibit 14.2.

[\[View full size image\]](#)

Enter this formula in G6 and copy it to G7:G20.

The screenshot shows a Microsoft Excel spreadsheet titled "Microsoft Excel - Exhibit 14.7.xls". The worksheet contains a simulation model for "ComputerWorld Simulation Example".

Simulation Data:

Week	RN	Demand	Revenue
1	0.3835	1	4300
2	0.5534	1	4300
3	0.1494	0	0
4	0.2172	1	4300
5	0.3131	1	4300
6	0.4194	1	4300
7	0.9014	4	17200
8	0.4203	1	4300
9	0.3734	1	4300
10	0.4932	1	4300
11	0.5215	1	4300
12	0.9811	3	12900
13	0.8191	3	12900
14	0.9894	3	12900
15	0.2815	1	4300
Total		23	98900.00

Summary Statistics:

Metric	Value
Average Demand	= 1.57
Average Revenue	= \$592.33

A callout box points to cell H6, containing the formula: Enter =4300*G6 in H6 and copy it to H7:H20.

Enter =4300*G6
in H6 and copy it
to H7:H20.

= AVERAGE(G6:G20)

Generate random numbers for cells F6:F20 with the formula =RAND() in F6 and copy them to F7:F20.

Note that the probability distribution for the weekly demand for laptops has been entered in cells **A6:C10**. Also notice that we have entered a new set of cumulative probability values in column B. We generated these cumulative probabilities by first entering 0 in cell B6, then entering the formula **=A6+B6** in cell B7, and copying this formula to cells **B8:B10**. This cumulative probability creates a range of random numbers for each demand value. For example, any random number less than

0.20 will result in a demand value of 0, and any random number greater than 0.20 but less than 0.60 will result in a demand value of 1, and so on. (Notice that there is no value of 1.00 in cell B11; the last demand value, 4, will be selected for any random number equal to or greater than .90.)

Random numbers are generated in cells **F6:F20** by entering the formula **=RAND()** in cell F6 and copying it to the range of cells in **F7:F20**.

Now we need to be able to generate demand values for each of these random numbers in column F. We accomplish this by first covering the cumulative probabilities and the demand values in cells **B6:C10** with the cursor. Then we give this range of cells the name "Lookup." This can be done by typing "Lookup" directly on the formula bar in place of B6 or by clicking on the "Insert" button at the top of the spreadsheet and selecting "Name" and "Define" and then entering the

name "Lookup." This has the effect of creating a table called "Lookup" with the ranges of random numbers and associated demand values in it. Next, we enter the formula **=VLOOKUP(F6,Lookup,2)** in cell G6 and copy it to the cells in the range **G7:G20**. This formula will compare the random numbers in column F with the cumulative probabilities in **B6:B10** and generate the correct demand value from cells **C6:C10**.

Once the demand values have been generated in column G,

we can determine the weekly revenue values by entering the formula **=4300*G6** in H6 and copying it to cells **H7:H20**.

Average weekly demand is computed in cell C13 by using the formula **=AVERAGE(G6:G20)**, and the average weekly revenue is computed by entering a similar formula in cell C14.

Notice that the average weekly demand value of 1.53 in [Exhibit 14.2](#) is different from the simulation result (2.07) we obtained from [Table 14.4](#). This

is because we used a different stream of random numbers. As mentioned previously, to acquire an average closer to the true steady-state value, the simulation probably needs to include more repetitions than 15 weeks. As an example, [Exhibit 14.3](#) simulates demand for 100 weeks. The window has been "frozen" at row 16 and scrolled up to show the first 10 weeks and the last 6 weeks on the screen in [Exhibit 14.3](#).

Exhibit 14.3.

[\[View full size image\]](#)

Microsoft Excel - Exhibit14.3.xls

File Edit View Insert Format Tools Data Window Help

Form a question for help

Formulas: =LOOKUP(F96,Lookup2)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	ComputerWorld Simulation Example (100 Weeks)																
2																	
3	Probability of Weekly Demand				Simulation												
4																	
5	P(v)	Cumulative	Demand		Week	RN	Demand	Revenue									
6	0.20	0	0		1	0.3830	1	4300									
7	0.40	0.20	1		2	0.5524	1	4300									
8	0.20	0.60	2		3	0.1404	0	0									
9	0.10	0.70	3		4	0.3172	1	4300									
10	0.10	0.80	4		5	0.2121	1	4300									
11	1.00				6	0.4164	1	4300									
12					7	0.9014	4	17200									
13	Average Demand = 1.49				8	0.4203	1	4300									
14	Average Revenue = 6407.00				9	0.3734	1	4300									
15					10	0.4932	1	4300									
16					95	0.2370	1	4300									
161					96	0.1215	2	8600									
162					97	0.0869	2	8600									
163					98	0.8262	0	0									
164					99	0.9200	0	0									
165					100	0.3653	2	8600									
166					Total		149	6407.00									
167																	

Spreadsheet "frozen" at row 16 to show first 10 weeks and last 6.

Decision Making with Simulation

In our previous example, the manager of ComputerWorld acquired some useful information about the weekly demand and revenue for laptops that would be helpful in making a decision about how many laptops would be needed each week to meet demand. However, this example did not lead directly to a decision. Next, we will expand our ComputerWorld store example

so that a possible decision will result.

[Page 621]

From the simulation in [Exhibit 14.3](#) the manager of the store knows that the average weekly demand for laptop PCs will be approximately 1.49; however, the manager cannot order 1.49 laptops each week. Because fractional laptops are not possible, either one or two must be ordered. Thus, the manager wants to repeat the earlier simulation with two

possible order sizes, 1 and 2. The manager also wants to include some additional information in the model that will affect the decision.

If too few laptops are on hand to meet demand during the week, then not only will there be a loss of revenue, but there will also be a shortage cost of \$500 per unit incurred because the customer will be unhappy. However, each laptop still in stock at the end of each week that has not been sold will incur an inventory or storage cost of \$50. Thus, it costs the

store money to have either too few or too many laptops on hand each week. Given this scenario, the manager wants to order either one or two laptops, depending on which order size will result in the greatest average weekly revenue.

[Exhibit 14.4](#) shows the Excel spreadsheet for this revised example. The simulation is for 100 weeks. The columns labeled "1," "2," and "4" for "Week," "RN," and "Demand" were constructed similarly to the model in [Exhibit 14.3](#). The array of cells **B6:C10** was

given the name "Lookup," and the formula
=VLOOKUP(F6,Lookup,2)
was entered in cell H6 and copied to cells **H7:H105**.

Exhibit 14.4.

[\[View full size image\]](#)

The simulation in [Exhibit 14.4](#) is for an order size of one laptop each week. The "Inventory" column (3) keeps track of the amount of inventory available each week the one laptop that comes in on order plus any laptops carried over from the previous week. The cumulative inventory is computed each week by entering the formula **=1+MAX(G6H6,0)** in cell G7 and copying it to cells **G8:G105**. This formula adds

the one laptop on order to either the number left over from the previous week (**G6-H6**) or 0, if there were not enough laptops on hand to meet demand. It does not allow for negative inventory levels, called [back orders](#). In other words, if a sale cannot be made due to a shortage, then it is gone. The inventory values in column 3 are eventually multiplied by the inventory cost of \$50 per unit in column 8, using the formula =**G6*50**.

If there is a shortage, it is recorded in column 5, labeled "Shortage." The shortage is computed by entering the formula **=MIN(G6-H6,0)** in cell I6 and copying it to cells **I7:I105**. Shortage costs are computed in column 7 by multiplying the shortage values in column 5 by \$500 by entering the formula **=I6*500** in cell K6 and copying it to cells **K7:K105**.

Weekly revenues are computed in column 6 by entering the formula **=4300* MIN(H6,G6)** in cell J6 and copying it to cells

J7:J105. In other words, the revenue is determined by either the inventory level in column 3 or the demand in column 4, whichever is smaller.

Total weekly revenue is computed in column 9 by subtracting shortage costs and inventory costs from revenue by entering the formula **=J6-K6-L6** in cell M6 and copying it to cells **M7:M105**.

The average weekly demand, 1.50, is shown in cell C13. The average weekly revenue, \$3,875, is computed in cell

C14.

Next, we must repeat this same simulation for an order size of two laptops each week. The spreadsheet for an order size of two is shown in [Exhibit 14.5](#).

Notice that the only actual difference is the use of a new formula to compute the weekly inventory level in column 3.

This formula in cell G7, reflecting two laptops ordered each week, is shown on the formula bar at the top of the spreadsheet.

[\[View full size image\]](#)

ComputerWorld Simulation Example												
Probability of Weekly Demand												
P(D)	Cumulative	Demand	Week	RN	Inventory	Demand	Shortage	Revenue	Cost	Inventory	Total	
0.20	0	0	1	0.3630	2	1	0	4300	0	100	4200	
0.40	0.20	1	2	0.5534	3	1	0	4300	0	150	4150	
0.20	0.60	2	3	0.1484	4	0	0	0	0	200	-200	
0.10	0.80	3	4	0.3172	6	1	0	4300	0	300	4000	
0.10	0.90	4	5	0.3131	2	1	0	4300	0	350	3950	
1.00			6	0.4194	8	1	0	4300	0	400	3900	
			7	0.9014	9	4	0	17200	0	450	16750	
			8	0.4203	7	1	0	4300	0	350	3950	
			9	0.3734	8	1	0	4300	0	400	3900	
			10	0.4832	9	1	0	4300	0	450	3950	
			11	0.5215	10	1	0	4300	0	500	3800	
			95	0.0161	44	0	0	0	0	2200	-2200	
			96	0.1269	45	0	0	0	0	2300	-2300	
			97	0.2281	48	1	0	4300	0	2400	1900	
			98	0.7927	49	2	0	8600	0	2450	8150	
			99	0.1976	48	0	0	0	0	2450	-2450	
			100	0.4241	51	1	0	4300	0	2550	1750	
			Total		3245	150	0	845000	0	16250	492750	

This second simulation, in [Exhibit 14.5](#), results in an average weekly demand of 1.50 laptops and an average weekly total revenue of \$4,927.50. This is higher than the total weekly revenue of \$3,875 achieved in the first simulation run in [Exhibit 14.4](#), even though the store would incur significantly higher inventory costs. Thus, the correct decision based on weekly revenue would be to order two laptops per week.

However, there are probably additional aspects of this problem the manager would want to consider in the decision-making process, such as the increasingly high inventory levels as the simulation progresses. For example, there may not be enough storage space to accommodate this much inventory. Such questions as this and others can also be analyzed with simulation. In fact, one of the main attributes of simulation is its usefulness as a model for experimenting,

called **what-if? analysis**.

[Page 623]

This example briefly demonstrates how simulation can be used to make a decision (i.e., to "optimize"). In this example we experimented with two order sizes and determined the one that resulted in the greater revenue. The same basic modeling principles can be used to solve larger problems with hundreds of possible order sizes and a probability distribution for

demand with many more values plus variable lead times (i.e., the time it takes to receive an order), the ability to back order, and other complicating factors. These factors make the simulation model larger and more complex, but such models are frequently developed and used in business.

Simulation of a Queuing System

To demonstrate the simulation of a queuing system, we will use an example. Burlingham Mills produces denim, and one of the main steps in the production process is dyeing the cotton yarn that is subsequently woven into denim cloth. The yarn is dyed in a large concrete vat like a narrow swimming pool. The yarn is strung over a series of rollers

so that it passes through the vat, up over a set of rollers, back down into the vat, back up and over another set of rollers, and so on. Yarn is dyed in batches that arrive at the dyeing vat in 1-, 2-, 3-, or 4-day intervals, according to the probability distribution shown in [Table 14.5](#). Once a batch of yarn arrives at the dyeing facility, it takes either 0.5, 1.0, or 2.0 days to complete the dyeing process, according to the probability distribution shown in [Table 14.6](#).

Table 14.5. Distribution of a

intervals		
Arrival Interval(days), x	Probability, $P(x)$	Cumulative Probability
1.0	.20	.20
2.0	.40	.60
3.0	.30	.90
4.0	.10	1.00

Table 14.6. Distribution of service times

Service Time (days), y	Probability, $P(y)$	Cumulative Probability	Random Number Range, r_2
0.5	.20	.20	120
1.0	.50	.70	2170
2.0	.30	1.00	7199, 00

[Table 14.5](#) defines the interarrival time, or how often batches arrive at the dyeing facility. For example, there is a .20 probability of a batch arriving *1 day* after the previous batch. [Table 14.6](#) defines the service time for a batch. Notice that cumulative probabilities have been included in [Tables 14.5](#) and [14.6](#). The cumulative probability provides a means for determining the ranges of random numbers associated with each probability, as we demonstrated with our Excel

example in the previous section. For example, in [Table 14.5](#) the first random number range for r_1 is from 1 to 20, which corresponds to the cumulative probability of .20. The second range of random numbers is from 21 to 60, which corresponds to a cumulative probability of .60. Although the cumulative probability goes up to 1.00, [Table 14.3](#) contains only random number values from 0 to 99. Thus, the number 0 is used in place of 100 in the last random number range of each

table.

Developing the cumulative probability distribution helps to determine random number ranges.

[Page 624]

[Table 14.7](#) illustrates the simulation of 10 batch arrivals at the dyeing vat. The manual

simulation process illustrated in [Table 14.7](#) can be interpreted as follows:

- 1.** Batch 1 arrives at time 0, which is recorded on an *arrival clock*. Because there are no batches in the system, batch 1 approaches the dyeing facility immediately, also at time 0. The waiting time is 0.
- 2.** Next, a random number, $r_2 = 65$, is selected from the second column in [Table 14.3](#). Observing [Table 14.6](#), we see that the random

number 65 results in a service time, y , of 1 day. After leaving the dyeing vat, the batch departs at time 1 day, having been in the queuing system a total of 1 day.

- 3.** The next random number, $r_1 = 71$, is selected from [Table 14.3](#), which specifies that batch 2 arrives 3 days after batch 1, or at time 3.0, as shown on the arrival clock. Because batch 1 departed the service facility at time 1.0, batch 2

can be served immediately and thus incurs no waiting time.

4. The next random number, $r_2 = 18$, is selected from [Table 14.3](#), which indicates that batch 2 will spend 0.5 days being served and will depart at time 3.5.

Table 14.7. Simulation at the Dyeing Facility

Batch	r_1	Arrival Interval, x	Arrival Clock	Enter Facility Clock	Waiting Time
-------	-------	--------------------------	---------------	----------------------	--------------

1			0.0	0.0	0.0
2	71	3.0	3.0	3.0	0.0
3	12	1.0	4.0	4.0	0.0
4	48	2.0	6.0	6.0	0.0
5	18	1.0	7.0	8.0	1.0
6	08	1.0	8.0	10.0	2.0
7	05	1.0	9.0	12.0	3.0
8	18	1.0	10.0	14.0	4.0

9	26	2.0	12.0	14.5	2.5	0.0
10	94	4.0	16.0	16.0	0.0	0.0
					12.5	0.0

This process of selecting random numbers and generating arrival intervals and service times continues until 10 batch arrivals have been simulated, as shown in [Table 14.7](#).

Once the simulation is

complete, we can compute operating characteristics from the simulation results, as follows:

$$\text{average waiting time} = \frac{12.5 \text{ days}}{10 \text{ batches}} = 1.25 \text{ days per batch}$$

$$\text{average time in the system} = \frac{24.5 \text{ days}}{10 \text{ batches}} = 2.45 \text{ days per batch}$$

However, as in our previous example, these results must be viewed with skepticism. Ten trials of the system do not ensure steady-state results. In general, we can expect a considerable difference between the true average

values and the values estimated from only 10 random draws. One reason is that we cannot be sure that the random numbers we selected in this example replicated the actual probability distributions because we used so few random numbers. The stream of random numbers that was used might have had a preponderance of high or low values, thus biasing the final model results. For example, of nine arrivals, five had interarrival times of 1 day. This corresponds to a probability of

.55 (i.e., $5/9$); however, the actual probability of an arrival interval of 1 day is .20 (from [Table 14.5](#)). This excessive number of short interarrival times (caused by the sequence of random numbers) has probably artificially inflated the operating statistics of the system.

[Page 625]

As the number of random trials is increased, the probabilities in the simulation will more closely conform to the actual

probability distributions. That is, if we simulated the queuing system for 1,000 arrivals, we could more reasonably expect that 20% of the arrivals would have an interarrival time of 1 day.

An additional factor that can affect simulation results is the starting conditions. If we start our queuing system with no batches in the system, we must simulate a length of time before the system replicates normal operating conditions. In this example, it is logical to start simulating at the time the

vat starts operating in the morning, especially if we simulate an entire working day. Some queuing systems, however, start with items already in the system. For example, the dyeing facility might logically start each day with partially completed batches from the previous day. In this case, it is necessary to begin the simulation with batches already in the system.

A factor that can affect simulation results is the starting

conditions.

By adding a second random variable to a simulation model, such as the one just shown, we increase the complexity and therefore the manual operations. To simulate the example in [Table 14.7](#) manually for 1,000 trials would require several hours. It would be far preferable to perform this type of simulation on a computer. A number of mathematical

computations would be required to determine the various column values of [Table 14.7](#), as we will demonstrate in the Excel spreadsheet simulation of this example in the next section.

Computer Simulation of the Queuing Example with Excel

The simulation of the dyeing process at Burlingham Mills shown in [Table 14.7](#) can also be done in Excel. [Exhibit 14.6](#)

shows the spreadsheet simulation model for this example.

Exhibit 14.6.

[\[View full size image\]](#)

This formula is entered in D15 and copied to D16:D23.

Microsoft Excel - Burlington Mills Queuing Simulation Example

File Edit Format Tools Data Window Help

Formulas: =MAX(E15,J14) in F15

Batch	Req'd	Arrival Time	Arrival Clock	Enter Factory Clock	Waiting Time	Req'd	Service Time	Departure Clock	Time in System
1			0.0	0.0	0.0	0.0006	1.0	1.0	1.0
2	0.0048		3.0	3.0	0.0	0.0006	1.0	4.0	1.0
3	0.0041		6.0	6.0	0.0	0.0009	2.0	8.0	2.0
4	0.0219		1.0	7.0	0.0	0.0019	2.0	19.0	3.0
5	0.2629		2.0	9.0	18.0	0.0009	1.0	11.0	2.0
6	0.0006		1.0	10.0	11.0	0.0019	2.0	13.0	3.0
7	0.2007		2.0	12.0	13.0	0.0021	2.0	15.0	3.0
8	0.0007		3.0	15.0	16.0	0.0009	1.0	16.0	1.0
9	0.1704		1.0	16.0	16.0	0.0019	2.0	18.0	2.0
10	0.0385		2.0	18.0	18.0	0.0019	0.0	18.0	0.0
Total					4.00				18.0

Average waiting time = 0.4 days, Average time in the system = 1.8 days

Clock time is generated by entering =MAX(E15,J14) in F15 and copying it to F16:F23.

Copy = VLOOKUP(H14,LOOKUP2,2) to I14:I23.

Arrival times are generated by entering =E14+D15 in E15 and copying it to E16:E23.

=AVERAGE(G14:G23)

The stream of random numbers in column C is generated by the formula =**RAND()**, used in the ComputerWorld examples in [Exhibits 14.2](#) through [14.5](#). (Notice that for this computer simulation, we have changed our random numbers from whole numbers to numbers between 0.0 and 1.0.) The arrival times are generated from the cumulative probability distribution of arrival intervals in cells **B6:C9**. This array of cells is renamed "Lookup1" because there are two probability distributions in the

model. The formula
=VLOOKUP(C15, Lookup1,2)
is entered in cell D15 and
copied to cells **D16:D23** to
generate the arrival times in
column D. The arrival clock
times are computed by entering
the formula **=E14+D15** in cell
E15 and copying it to cells
E16:E23.



Management Science

Application: Simulating the Israeli Army Recruitment Process

Israel has compulsory army service starting at age 18. Candidates are examined while they are still in high school, a year before their recruitment. Prior to 1995 the examinations were conducted at six recruitment offices around the country. Each office included five major testing stations, where candidates were required to go through 2 days of examinations. The first day a candidate went through an admissions station, a premed station, and then a psychometric exam. Several weeks later the candidate

would be scheduled for a second visit that included a personal interview and standard medical exam. More than 70% of the candidates did not finish the examination process during the 2 scheduled days and had to return for additional visits to finish all the tests. The recruitment offices were overcrowded, and staff worked at 100% capacity. Candidates spent over 40% of their time in the recruitment offices waiting in lines, and the average station waiting time was 30 minutes. This problem of overcrowding and delays was expected to get worse because of Israel's population growth from immigration (about 20%) during the 1990s. Possible solutions being considered included building an additional recruitment office and adding more personnel and staff.

A team was assigned to examine the

recruitment process and overcrowding problem, with goals of finding a more cost-effective solution than building new facilities and adding personnel, and providing candidates with faster service, specifically reducing the number of extra visits. Preliminary analysis indicated that areas of improvement included the sequencing of the stations and the scheduling of the psychometric exam. The team created a simulation model to examine the various queuing situations that existed and the candidate flows through the system. The simulation model was used to analyze bottleneck situations and to evaluate possible alternatives, including changing the arrival patterns at stations, changing the way the psychometric test was administered, and looking at different possible routes through the process. After a

year, a new recruiting configuration was developed, using the simulation model. A new computer system was installed that eliminated the use of personal files and thus saved wasted time for files to be physically transported between stations. A computerized routing system using bar codes assigned to candidates upon admission routed candidates to the stations with the shortest waiting times. The psychometric exam was computerized, thus eliminating the need to wait until examining rooms filled or emptied and to collect paper test results. Candidates' arrival times were evenly distributed during the morning.

The new system was implemented in 1995 in the Haifa office with successful results almost identical to those predicted by the simulation model. Ninety-nine percent of all

candidates were able to complete all their tests in 1 day, in an average of 4 hours. Average station waiting time was reduced to 5 minutes. With the same workforce and facilities, throughput more than doubled. The new system was subsequently implemented in the five other recruiting offices with similar results. The two biggest offices in Tel Aviv were eventually merged, thus eliminating an office. This savings alone covered the cost of the entire project. Savings included \$3.3 million for the closed offices plus \$350,000 annually for reduced costs, including personnel and office expenses.



Source: O. Shtrichman, R. Ben-Haim, and M. Pollatschek, "Using Simulation to Increase Efficiency in an Army Recruitment Office," *Interfaces* 31, no. 4 (July/August 2001): 6170.

A batch of yarn can enter the dyeing facility as soon as it arrives (in column E) if there is not another batch being dyed or as soon as any batches being dyed or waiting to be dyed have departed the facility (column J). This clock time is computed by entering the formula **=MAX(E15, J14)** in cell F15 and copying it to cells **F16:F23**. The waiting time is computed with the formula **=F14-E14**, copied in cells **G14:G23**.

A second set of random numbers is generated in

column H by using the **RAND()** function. The service times are generated in column I from the cumulative probability distribution in cells **H6:I8**, using the "Lookup" function again. In this case the array of cells in **H6:I8** is named "Lookup2," and the service times in column I are generated by copying the formula **=VLOOKUP(H14, Lookup2,2)** in cells **I14:I23**. The departure times in column J are determined by using the formula **=F14+I14** copied in cells **J14:J23**, and the "Time in

System" values are computed by using the formula **=J14-E14**, copied in cells **K14:K23**.

The operating statistic, average waiting time, is computed by using the formula

=AVERAGE(G14:G23) in cell G26, and the average time in the system is computed with a similar formula in cell L26.

Notice that both the average waiting time of 0.4 days and the average time in the system of 1.9 days are significantly lower than the simulation conducted in [Table 14.7](#), as we speculated they might be.

Continuous Probability Distributions

In the first example in this chapter, ComputerWorld's store manager considered a probability distribution of discrete demand values. In the queuing example, the probability distributions were for discrete interarrival times and service times. However, applications of simulation models reflecting continuous distributions are more common

than those of models employing discrete distributions.

We have concentrated on examples with discrete distributions because with a discrete distribution, the ranges of random numbers can be explicitly determined and are thus easier to illustrate. When random numbers are being selected according to a continuous probability distribution, a continuous function must be used. For example, consider the following continuous probability function, $f(x)$, for time (minutes), x :

$$f(x) = \frac{x}{8}, 0 \leq x \leq 4$$

The area under the curve, $f(x)$, represents the probability of the occurrence of the random variable x . Therefore, the area under the curve must equal 1.0 because the sum of all probabilities of the occurrence of a random variable must equal 1.0. By computing the area under the curve from 0 to any value of the random variable x , we can determine the cumulative probability of that value of x , as follows:

$$F(x) = \int_0^x \frac{x}{8} dx = \frac{1}{8} \int_0^x x dx = \frac{1}{8} \left(\frac{1}{2} x^2 \right) \Bigg|_0^x$$

$$F(x) = \frac{x^2}{16}$$

Cumulative probabilities are analogous to the discrete ranges of random numbers we used in previous examples. Thus, we let this function, $F(x)$, equal the random number, r ,

$$r = \frac{x^2}{16}$$

and solve for x ,

$$x = 4\sqrt{r}$$

By generating a random number, r , and substituting it into this function, we determine a value for x , "time." (However, for a continuous function, the range of random numbers must be between zero and one to correspond to probabilities between 0.0 and 1.00.) For example, if $r = .25$, then

$$x = 4\sqrt{.25} = 2 \text{ min.}$$

The purpose of briefly presenting this example is to demonstrate the difference between discrete and continuous functions. This continuous function is relatively simple; as functions become more complex, it becomes more difficult to develop the equation for determining the random variable x from r . Even this simple example required some calculus, and developing more complex models would require a higher level of mathematics.

Simulation of a Machine Breakdown and Maintenance System

In this example we will demonstrate the use of a continuous probability distribution. The Bigelow Manufacturing Company produces a product on a number of machines. The elapsed time between breakdowns of the machines is defined by the following continuous probability distribution:

$$f(x) = \frac{x}{8}, 0 \leq x \leq 4 \text{ weeks}$$

where

x = weeks between machine breakdowns

A continuous probability distribution of the time between machine breakdowns.

As indicated in the previous

section on continuous probability distributions, the equation for generating x , given the random number r_1 , is

$$x = 4\sqrt{r_1}$$

When a machine breaks down, it must be repaired, and it takes either 1, 2, or 3 days for the repair to be completed, according to the discrete probability distribution shown in [Table 14.8](#). Every time a machine breaks down, the cost to the company is an estimated \$2,000 per day in lost

production until the machine is repaired.

Table 14.8. Probability distribution of machine repair time

Machine Repair Time, y (days)	Probability of Repair Time, $P(y)$	Cumulative Probability	Random Number Range, r_2
1	.15	.15	0.00.15
2	.55	.70	.16.70
3	.30	1.00	.711.00

The company would like to know whether it should implement a machine maintenance program at a cost of \$20,000 per year that would reduce the frequency of breakdowns and thus the time for repair. The maintenance program would result in the following continuous probability function for time between breakdowns:

$$f(x) = x/18, 0 \leq x \leq 6 \text{ weeks}$$

where

x = weeks between machine breakdowns

[Page 629]

The equation for generating x , given the random number r_1 , for this probability distribution is

$$x = 6\sqrt{r_1}$$

The reduced repair time resulting from the maintenance program is defined by the discrete probability distribution shown in [Table 14.9](#).

Table 14.9. Revised probability distribution of machine repair time with the maintenance program

Machine Repair Time, y (days)	Probability of Repair Time, $P(y)$	Cumulative Probability	Random Number Range, r_2
1	.40	.40	0.00.40
2	.50	.90	.41.90
3	.10	1.00	.911.00

To solve this problem, we must first simulate the existing system to determine an estimate of the average annual repair costs. Then we must simulate the system with the maintenance program installed to see what the average annual repair costs will be with the maintenance program. We will then compare the average annual repair cost with and without the maintenance program and compute the difference, which will be the average annual savings in repair costs with the

maintenance program. If this savings is more than the annual cost of the maintenance program (\$20,000), we will recommend that it be implemented; if it is less, we will recommend that it not be implemented.

First, we will manually simulate the existing breakdown and repair system without the maintenance program, to see how the simulation model is developed. [Table 14.10](#) illustrates the simulation of machine breakdowns and repair for 20 breakdowns, which occur

over a period of approximately 1 year (i.e., 52 weeks).

Table 14.10. Simulation breakdowns and repair

Breakdowns	r_1	Time Between Breakdowns, x (weeks)	r_2	Repair Time, y (days)
1	.45	2.68	.19	2
2	.90	3.80	.65	2
3	.84	3.67	.51	2
4	.17	1.65	.17	2

5	.74	3.44	.63	2
6	.94	3.88	.85	3
7	.07	1.06	.37	2
8	.15	1.55	.89	3
9	.04	0.80	.76	3
10	.31	2.23	.71	3
11	.07	1.06	.34	2
12	.99	3.98	.11	1

13	.97	3.94	.27	2
14	.73	3.42	.10	1
15	.13	1.44	.59	2
16	.03	0.70	.87	3
17	.62	3.15	.08	1
18	.47	2.74	.08	1
19	.99	3.98	.89	3
20	.75	3.46	.42	2

[Page 630]

The simulation in [Table 14.10](#) results in a total annual repair cost of \$84,000. However, this is for only 1 year, and thus it is probably not very accurate.

The next step in our simulation analysis is to simulate the machine breakdown and repair system with the maintenance program installed. We will use

the revised continuous probability distribution for time between breakdowns and the revised discrete probability distribution for repair time shown in [Table 14.9](#). [Table 14.11](#) illustrates the manual simulation of machine breakdowns and repair for 1 year.

Table 14.11. Simulation breakdowns and repair maintenance prog

Breakdowns	Time Between Breakdowns, r_1 x (weeks)	Repair Time, r_2 y (days)
------------	---	-------------------------------------

1	.45	4.03	.19	1
2	.90	5.69	.65	2
3	.84	5.50	.51	2
4	.17	2.47	.17	1
5	.74	5.16	.63	2
6	.94	5.82	.85	2
7	.07	1.59	.37	1

8	.15	2.32	.89	2
9	.04	1.20	.76	2
10	.31	3.34	.71	2
11	.07	1.59	.34	1
12	.99	5.97	.11	1
13	.97	5.91	.27	1
14	.73	5.12	.10	1

[Table 14.11](#) shows that the annual repair cost with the maintenance program totals \$42,000. Recall that in the manual simulation shown in [Table 14.10](#), the annual repair cost was \$84,000 for the system without the maintenance program. The difference between the two annual repair costs is $\$84,000 - \$42,000 = \$42,000$. This figure represents the savings in average annual repair cost with the maintenance program.

Because the maintenance program will cost \$20,000 per year, it would seem that the recommended decision would be to implement the maintenance program and generate an expected annual savings of \$22,000 per year (i.e., $\$42,000 - \$20,000 = \$22,000$).

However, let us now concern ourselves with the potential difficulties caused by the fact that we simulated each system (the existing one and the system with the maintenance program) only *once*. Because

the time between breakdowns and the repair times are probabilistic, the simulation results could exhibit significant variation. The only way to be sure of the accuracy of our results is to simulate each system many times and compute an average result. Performing these many simulations manually would obviously require a great deal of time and effort. However, Excel can be used to accomplish the required simulation analysis.

Manual simulation is limited because of the amount of real time required to simulate even one trial.

Computer Simulation of the Machine Breakdown Example Using Excel

[Exhibit 14.7](#) shows the Excel

spreadsheet model of the simulation of our original machine breakdown example simulated manually in [Table 14.10](#). The Excel simulation is for 100 breakdowns. The random numbers in **C14:C113** are generated using the **RAND()** function, which was used in our previous Excel examples. The "Time Between Breakdowns" values in column D are developed using the formula for the continuous cumulative probability function, **=4*SQRT(C14)**, typed in cell D14 and copied in cells

D15:D113.

[Page 631]

Exhibit 14.7.

[\[View full size image\]](#)

Click on Window to freeze panes at row 24.

From Table 14.8

Microsoft Excel - Table14.8.xls

File Edit Insert Format Tools Data Window Help

Formulas: =SUM(B1:B15)

Table 14.8: Digital Manufacturing Company Machine Breakdown Simulation Example

Pro	Cumulative	Repair Time, y
0.15	0	1
0.55	0.15	2
0.70	0.70	3
1.00		

Average time between breakdowns = 2.73 hours
 Average repair time = 2.16 days
 Average annual cost = \$33,202.26

Breakdowns	Pro	Time Between Breakdowns, y	Cumulative Time	RBO	Repair Time, y	Cost, \$
1	0.3335	2.18	2.18	0.2204	3	4000
2	0.7702	3.52	5.71	0.0860	2	6000
3	0.1023	1.76	7.46	0.4716	2	4000
4	0.5863	3.02	10.48	0.7241	3	6000
5	0.6297	3.06	14.34	0.7533	3	6000
6	0.7038	3.49	17.82	0.0403	1	3000
7	0.2002	2.04	19.86	0.4009	2	4000
8	0.2198	2.08	21.95	0.1305	1	3000
9	0.0618	0.98	22.94	0.0260	3	6000
10	0.2178	1.87	24.81	0.5126	3	4000
11	0.4698	3.32	28.13	0.7874	3	6000
12	0.3270	1.91	30.04	0.0847	1	3000
13	0.2402	1.08	31.24	0.6870	2	4000
14	0.4273	3.17	34.41	0.5567	2	4000
15	0.7895	3.58	38.00	0.0370	1	3000
16	0.8630	3.67	42.83	0.0370	1	3000

Spreadsheet frozen at row 24 to show first 10 breakdowns and last 6.

Copy =E14+D15 to E15:E113.

Copy =VLOOKUP(F14, LOOKUP,2) to G14:G113.

The "Cumulative Time" in column E is computed by

copying the formula
=E14+D15 in cells **E15:E113**.
The second stream of random
numbers in column F is
generated using the **RAND()**
function. The "Repair Time"
values in column G are
generated from the cumulative
probability distribution in the
array of cells **B6:C8**. As in our
previous examples, we name
this array "Lookup" and copy
the formula
=VLOOKUP(F14,Lookup,2) in
cells **G14:G113**. The cost
values in column H are
computed by entering the

formula =**2000*G14** in cell
H14 and copying it to cells
H15:H113.

[Page 632]

The "Average Annual Cost" in
cell H7 is computed with the
formula
=**SUM(H14:H113)/(E113/52)**
For this, the original problem,
the annual cost is \$82,397.35,
which is not too different from
the manual simulation in [Table
14.10](#).

[Exhibit 14.8](#) shows the Excel

spreadsheet simulation for the modified breakdown system with the new maintenance program, which was simulated manually in [Table 14.11](#). The two differences in this simulation model are the cumulative probability distribution formulas for the time between breakdowns and the reduced repair time distributions from [Table 14.9](#) in cells **A6:C8**.

Exhibit 14.8.

(This item is displayed on page

631 in the print version)

[\[View full size image\]](#)

Revised formula for time between breakdowns

Probability distribution or repair time from Table 14.9

Figure Manufacturing Company Machine Breakdown Simulation Example

Prob	Cumulative	Repair Time, y
0.40	0	1
0.80	0.40	2
0.10	0.90	3
0.50	1.00	

Average time between breakdowns = 4.00 hours	
Average repair time = 1.75 days	
Average annual cost = \$44,534.54	

Breakdowns	Prob	Time between Breakdowns, x	Cumulative Time	Prob	Repair Time, y	Cost, \$
1	0.4000	0.29	3.29	0.2204	1	2000
2	0.7700	0.38	6.57	0.0000	3	6000
3	0.7820	2.60	11.20	0.4215	2	4000
4	0.8000	4.02	15.72	0.7241	2	4000
5	0.8297	6.75	21.81	0.7833	2	4000
6	0.7600	0.23	26.74	0.0403	1	2000
7	0.2602	3.08	29.89	0.4959	2	4000
8	0.2718	3.15	32.82	0.5000	1	2000
9	0.8810	1.49	34.41	0.0000	2	4000
10	0.2178	2.80	37.21	0.5126	2	4000
11	0.8000	4.00	38.71	0.7074	2	4000
12	0.2270	2.06	388.37	0.0847	1	2000
13	0.2402	2.90	383.36	0.8870	2	4000
14	0.8373	4.75	388.11	0.5967	2	4000
15	0.7095	0.30	403.44	0.0270	1	2000
16	0.9470	0.11	403.95	0.0270	1	2000

The average annual cost for this model, shown in cell H8, is \$44,504.74. This annual cost is only slightly higher than the \$42,000 obtained from the manual simulation in [Table 14.11](#). Thus, as before, the decision should be to implement the new maintenance system.

Statistical Analysis of Simulation Results

In general, the outcomes of a simulation model are statistical measures such as averages, as in the examples presented in this chapter. In our ComputerWorld example, we generated average revenue as a measure of the system we simulated; in the Burlingham Mills queuing example, we generated the average waiting time for batches to be dyed;

and in the Bigelow Manufacturing Company machine breakdown example, the system was measured in terms of average repair costs. However, we also discussed the care that must be taken in accepting the accuracy of these statistical results because they were frequently based on relatively few observations (i.e., simulation replications). Thus, as part of the simulation process, these statistical results are typically subjected to additional statistical analysis to determine their degree of

accuracy.

One of the most frequently used tools for the analysis of the statistical validity of simulations results is confidence limits. Confidence limits can be developed within Excel for the averages resulting from simulation models in several different ways. Recall that the statistical formulas for 95% confidence limits are

$$\text{upper confidence limit} = \bar{x} + (1.96)(s / \sqrt{n})$$

$$\text{lower confidence limit} = \bar{x} - (1.96)(s / \sqrt{n})$$

where $\bar{x}$ is the mean and s is

the sample standard deviation from a sample of size n from any population. Although we cannot be sure that the sample mean will exactly equal the population mean, we can be 95% confident that the true population mean will be between the upper confidence limit (UCL) and lower confidence limit (LCL) computed using these formulas.

[Exhibit 14.9](#) shows the Excel spreadsheet for our machine breakdown example (from [Exhibit 14.8](#)), with the upper and lower confidence limits for

average repair cost in cells L13 and L14. Cell L11 contains the average repair cost (for each incidence of a breakdown), computed by using the formula **=AVERAGE(H14:H113)**. Cell L12 contains the sample standard deviation, computed by using the formula **=STDEV(H14:H113)**. The upper confidence limit is computed in cell L13 by using the formula shown on the formula bar at the top of the spreadsheet, and the lower control limit is computed similarly. Thus, we can be 95%

confident that the true average repair cost for the population is between \$3,248.50 and \$3,751.50.

[Page 633]

Exhibit 14.9.

[\[View full size image\]](#)

"Tools" menu. (If this option is not available on your "Tools" menu, select the "Add-ins" option from the "Tools" menu and then select the "Analysis ToolPak" option.) Select the "Data Analysis" option from the "Tools" menu at the top of the spreadsheet, and then from the resulting menu select "Descriptive Statistics." This will result in a dialog box like the one shown in [Exhibit 14.10](#). This box, completed as shown, results in the summary statistics for repair costs shown in cells **J8:K23** in [Exhibit](#)

[14.11](#). These summary statistics include the mean, standard deviation, and confidence limits we computed in [Exhibit 14.9](#), plus several other statistics.

Exhibit 14.10.

[\[View full size image\]](#)

Cost (\$) values in column H

This copies the label in H13 onto the statistical summary report; if the first row is a data point, don't check this.

Specifies the location of the statistical summary report on the spreadsheet.

Descriptive Statistics

Input

Input Range:

Grouped By: ☒ Columns ☐ Rows

☒ Labels in first row

Output options

☒ Output Range:

☐ New Worksheet Ply:

☐ New Workbook

☒ Summary statistics

☒ Confidence Level for Mean: %

☐ Kth Largest:

☐ Kth Smallest:

OK Cancel Help

Exhibit 14.11.

[\[View full size image\]](#)

Microsoft Excel - Exhibit14.11.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Help & guidance for help

Formulas

fx = SUMPRODUCT(B3:B5,C3:E5)

Worksheet: Sheet1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1	Bigeye Manufacturing Company Machine Breakdown Simulation Example														
2															
3															
4															
5		Run	Cumulative	Repair											
6		Time, y		Time, y											
7		0.40	0	1			Average time between breakdowns =	4.00	weeks						
8		0.50	0.40	2			Average repair time =	1.75	days						
9		0.10	0.90	3			Average annual cost =	44594.34							
10		1.00													
11															
12															
13															
14															
15															
16															
17															
18															
19															
20															
21															
22															
23															
24															
25															
26															
27															
28															
29															
30															
31															
32															
33															
34															
35															
36															
37															
38															
39															
40															
41															
42															
43															
44															
45															
46															
47															
48															
49															
50															
51															
52															
53															
54															
55															
56															
57															
58															
59															
60															
61															
62															
63															
64															
65															
66															
67															
68															
69															
70															
71															
72															
73															
74															
75															
76															
77															
78															
79															
80															
81															
82															
83															
84															
85															
86															
87															
88															
89															
90															
91															
92															
93															
94															
95															
96															
97															
98															
99															
100															
101															
102															
103															
104															
105															
106															
107															
108															
109															
110															
111															
112															
113															
114															

Statistical summary report

Crystal Ball

So far in this chapter we have used simulation examples that included mostly discrete probability distributions that we set up on an Excel spreadsheet. These are the easiest types of probability distributions to work with in spreadsheets. However, many realistic problems contain more complex probability distributions, like the normal distribution, which are not discrete but are continuous, or

they include discrete probability distributions that are more difficult to work with than the simple ones we have used. However, there are several simulation add-ins for Excel that provide the user with the capability to perform simulation analysis, using a variety of different probability distributions in a spreadsheet environment. One of these add-ins is Crystal Ball, published by Decisioneering; it is available for download from the CD that accompanies this text. Crystal Ball is a risk

analysis and forecasting program that uses Monte Carlo simulation to forecast a statistical range of results possible for a given situation. In this section we will provide an overview of how to apply Crystal Ball to a simple example for profit analysis that we first introduced in [Chapter 1](#).

Simulation of a Profit Analysis Model

In [Chapter 1](#) we used a simple

example for the Western Clothing Company to demonstrate break-even and profit analysis. In that example, Western Clothing Company produced denim jeans. The price (p) for jeans was \$23, the variable cost (c_v) was \$8 per pair of jeans, and the fixed cost (c_f) was \$10,000. Given these parameters, we formulated a profit (Z) function as follows:

$$Z = vp - c_f - vc_v$$

Our objective in that analysis

was to determine the break-even volume, v , that would result in no profit or loss. This was accomplished by setting $Z = 0$ and solving the profit function for v , as follows:

$$v = \frac{c_f}{p - c_v}$$

[Page 635]

Substituting the values for p , c_f , and c_v into this formula resulted in the break-even volume:

$$v = \frac{10,000}{23 - 8}$$

= 666.7 pairs of jeans

To demonstrate the use of Crystal Ball, we will modify that example. First, we will assume that volume is actually volume *demand*ed and that it is a random variable defined by a normal probability distribution, with a mean value of 1,050 pairs of jeans and a standard deviation of 410.

Furthermore, we will assume that the price is not fixed but is also uncertain and defined by a

uniform probability distribution (from \$20 to \$26) and that variable cost is not a constant value but defined by a triangular probability distribution. Instead of seeking to determine the break-even volume, we will simulate the profit model, given probabilistic demand, price, and variable costs to determine average profit and the probability that Western Clothing will break even.

The first thing we need to do is access Crystal Ball, which you can download from the CD that

comes with this text.

[Exhibit 14.12](#) shows the Excel spreadsheet for our example.[\[1\]](#)

We have described the parameters of each probability distribution in our profit model next to its corresponding cell. For example, cell C4 contains the probability distribution for demand. What we want to do is generate demand values in this cell according to the probability distribution for demand (i.e., Monte Carlo simulation). We also want to do this in cell C5 for price and in cell C7 for

variable cost. This is the same process we used in our earlier ComputerWorld example to generate demand values from a discrete probability distribution, using random numbers. Notice that cell C9 contains our formula for profit, **=C4*C5-C6-(C4*C7)**. This is the only cell formula in our spreadsheet.

[1] This simulation example is located on the CD accompanying this text. It is in a file titled "[Exhibit 14.12](#)," located in the Crystal Ball folder on the accompanying text CD and created when Crystal Ball is loaded from the CD.

Exhibit 14.12.

[\[View full size image\]](#)

2. Click on "Define" to access
"Define Assumption."

The screenshot shows a Microsoft Excel spreadsheet titled "Exhibit14.12.xls". The spreadsheet is titled "Simulation of Profit Analysis for Western Clothing Company". It contains several input variables and their corresponding probability distributions:

Variable	Value	Distribution
volume (v)	1,050	normal distribution (mean = 1,050, standard deviation = 410)
price (p)	23	uniform distribution (minimum = 20, maximum = 26)
fixed cost (c _f)	10000	
variable cost (c _v)	8	triangular distribution (min = 6.75, most likely = 8, max = 9.12)
Profit (Z)	5750	

The profit formula is shown in cell C7: $\text{Profit (Z)} = C4 * C5 - C6 - (C4 * C7)$.

1. Click on cell C4 to define normal distribution parameters.

To set up the normal probability

distribution for demand, we first enter the mean value 1,050 in cell C4. Cells require some initial value to start with. Next we click on "Define" from the top of the spreadsheet, as shown in [Exhibit 14.12](#), which will result in the menu shown in [Exhibit 14.13](#). We select "Define Assumption" from this menu, which will result in the Distribution Gallery window shown in [Exhibit 14.14](#).

Exhibit 14.13.

Click here to access
the Distribution
Gallery window.

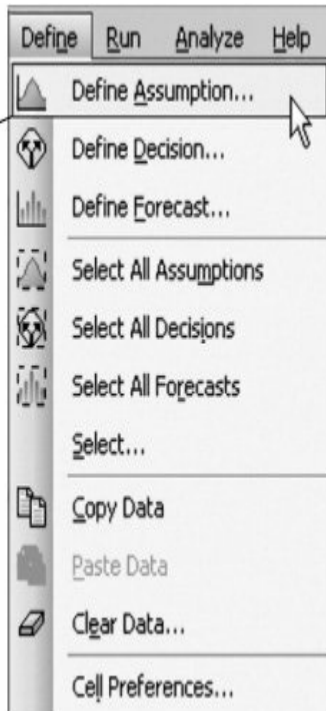
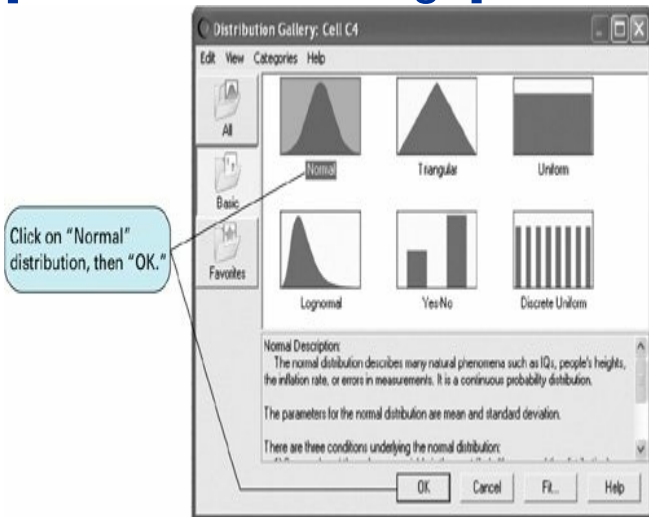


Exhibit 14.14.

[\[View full size image\]](#)



The Distribution Gallery window includes several different probability distributions we can use. Because we have indicated that demand is defined by a normal distribution, we click on this box and then on "OK." This will result in the window for the normal distribution shown in [Exhibit 14.15](#).

Exhibit 14.15.

(This item is displayed on page 637 in the print version)

[\[View full size image\]](#)

Define Assumption: Cell C4

Edt View Parameters Preferences Help

Name: volume (v) =

Normal Distribution

Probability

0 300 600 900 1,200 1,500 1,800 2,100

Mean 1,050 Std. Dev. 410

OK Cancel Enter Gallery Correlate... Help

1. Click on Windows Tab key to enter parameters.

2. Click "Enter" to configure distribution in window.

3. Click on "OK" to return to the spreadsheet.

The "Name" value in the box at the top of the window in [Exhibit 14.15](#) was automatically pulled from the spreadsheet, where it is the heading "volume (v) ="; however, a new or different name could be typed in. Next, we click on "Mean" or use the Tab key to toggle down to the "Mean" display in the lower-left-hand corner of this window. Because we entered the mean value of 1,050 in cell C4 on our spreadsheet, this value will already be shown in this window. Next, we click on "Std Dev" or use the Tab key to

move to the "Std Dev" window and enter the standard deviation of 410. Then we click on the Enter button, which will configure the normal distribution figure in the window, and then we click on "OK."

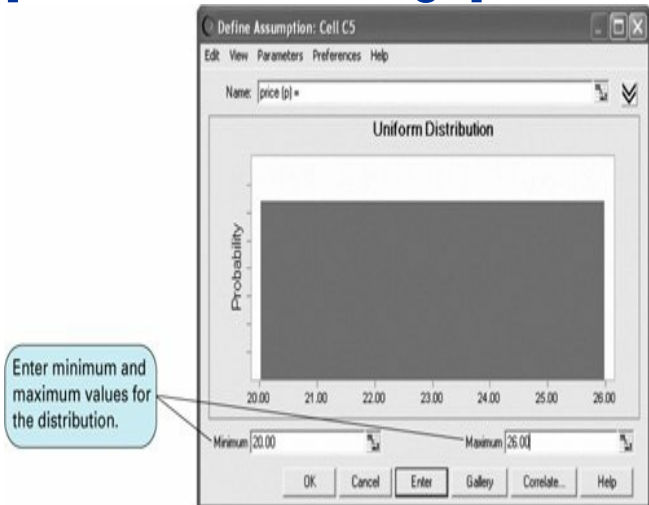
[Page 637]

We will repeat this same process to enter the parameters for the uniform distribution for price in cell C5. First, we enter the value for price, 23, in cell C5. Next (with

cell C5 activated), we click on "Define" at the top of the spreadsheet and then select "Define Assumption" from the menu, as shown earlier in [Exhibit 14.13](#). The Distribution Gallery window will again appear, and this time you should click on "Uniform Distribution" and then "OK." This will result in the "Uniform Distribution" window shown in [Exhibit 14.16](#).

Exhibit 14.16.

[\[View full size image\]](#)



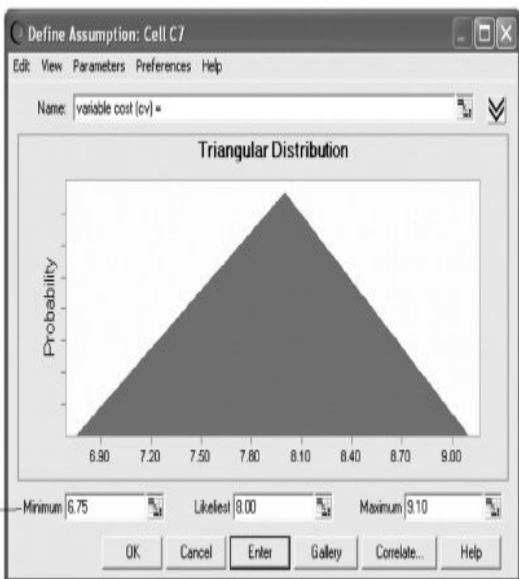
As before, the "Name" value, "price (p)," was pulled from the original spreadsheet in [Exhibit 14.12](#). Next, we click on "Minimum" or use the Tab key to move to the "Minimum" display at the bottom of the window and enter 20, the lower limit of the uniform distribution specified in the problem statement. Next, we activate the "Maximum" display window and enter 26. Then we click on the "Enter" button to configure the distribution graph in the window. Finally, we click on "OK" to exit this window.

We repeat the same process to enter the triangular distribution parameters in cell C7. A triangular probability distribution is defined by three estimated values: a minimum, a most likely, and a maximum. It is a very useful approximation when enough data points do not exist to allow for the construction of a distribution, but the user can estimate what the endpoints and the midpoint of the distribution might be. Clicking on "Define Assumption" from the cell menu and then selecting the

triangular distribution from the distribution gallery results in the window shown in [Exhibit 14.17](#).

Exhibit 14.17.

[\[View full size image\]](#)



We enter the "Minimum" value of 6.75, the "Likeliest" value of

8.00, and the "Maximum" value of 9.10. Clicking on "Enter" will configure the graph of the triangular distribution shown in the window. We click on "OK" to exit this window and return to the spreadsheet.

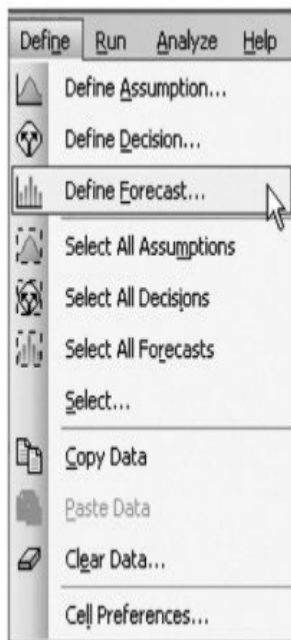
Next, we click on cell C9 on our original spreadsheet. Recall that this is the cell in which we entered our profit formula in [Exhibit 14.12](#). The profit value of 5,750, computed from the other cell values entered on the original spreadsheet, will be shown in cell C9. We click on the "Define" menu at the

top of the spreadsheet and select "Define Forecast," as shown in [Exhibit 14.18](#). This will result in the window shown in [Exhibit 14.19](#). The heading "Profit(Z) =" will already be entered from the spreadsheet. We click on the "Units" display and enter "dollars." We then click on "OK" to exit this window. This completes the process of entering our simulation parameters and data. [Exhibit 14.20](#) shows the spreadsheet with changes resulting from the parameter inputs. The next step is to run

the simulation.

Exhibit 14.18.

(This item is displayed on page
639 in the print version)



Click on "Define Forecast."

Exhibit 14.19.

(This item is displayed on page 639 in the print version)

[\[View full size image\]](#)

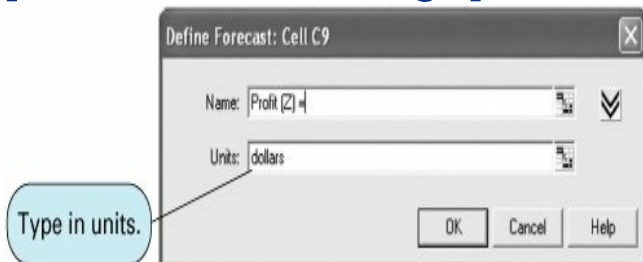


Exhibit 14.20.

(This item is displayed on page 639 in the print version)

[\[View full size image\]](#)

Click on "Run" to begin the simulation.

Microsoft Excel - File Edit View Insert Format Tools Data Window Help

File Edit View Insert Format Tools Data Window Help

Formulas: =C4*C5-C6-(C4*C7)

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Simulation of Profit Analysis for Western Clothing Company																
2																	
3																	
4		volume (v) =	1,000	normal distribution (mean = 1,000, standard deviation = 410)													
5		price (p) =	25	uniform distribution (minimum = 20, maximum = 30)													
6		fixed cost (c_f) =	10,000														
7		variable cost (c_v) =	8	triangular distribution (min = 6.75, most likely = 8, max = 9.10)													
8																	
9		Profit (Z) =	6,750														
10																	

Cursor is on cell C9.

The mechanics of the simulation are similar to those of our previous Excel spreadsheet models. Using random numbers, we want to generate a value for demand in cell C4, then a value for price in C5, and then a value for variable cost in C7. These three values are then substituted into the profit formula in cell C9 to compute a profit value. This represents one repetition, or *trial*, of the simulation. The simulation is run for many trials in order to develop a distribution for profit.

To run the simulation, we access the "Run" menu at the top of the spreadsheet. This will result in the menu shown in [Exhibit 14.21](#). We click on "Run Preferences," which will activate the window shown in [Exhibit 14.22](#). We then enter the number of simulations for the simulation run. For this example we will run the simulation for 5,000 trials. Next, we click on "Sampling" at the top of this window to activate the window shown in

[Exhibit 14.23](#). In this window we must enter the seed value for a sequence of random numbers for the simulation, which is always 999. We click on "OK" and then go back to the "Run" menu. From the "Run" menu ([Exhibit 14.21](#)), we click on "Start Simulation," which will run the simulation. [Exhibit 14.24](#) shows the simulation window with the simulation completed for 5,000 trials and the frequency distribution for this simulation.

Exhibit 14.21.

(This item is displayed on page 640 in the print version)

2. Click here to start the simulation.

1. Click here to establish the number of trials and the seed number.

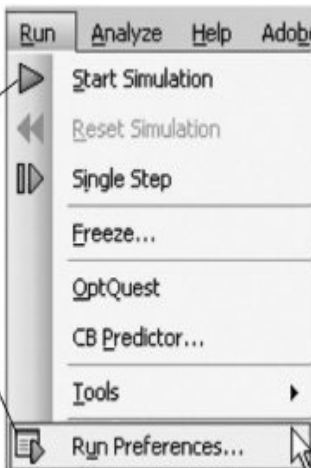


Exhibit 14.22.

(This item is displayed on page 640 in the print version)

[\[View full size image\]](#)

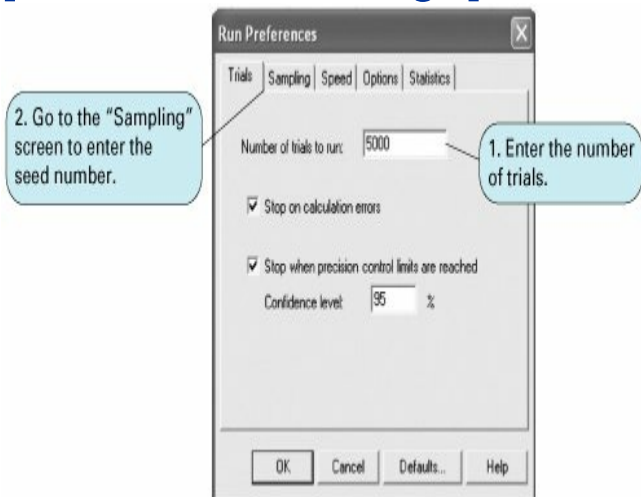


Exhibit 14.23.

(This item is displayed on page
640 in the print version)

[\[View full size image\]](#)

Click here to repeat
the same simulation.

Run Preferences

Trial Sampling Speed Options Statistics

Random number generation

☒ Use same sequence of random numbers

Initial seed value: 999

Sampling method

☒ Monte Carlo (more random)

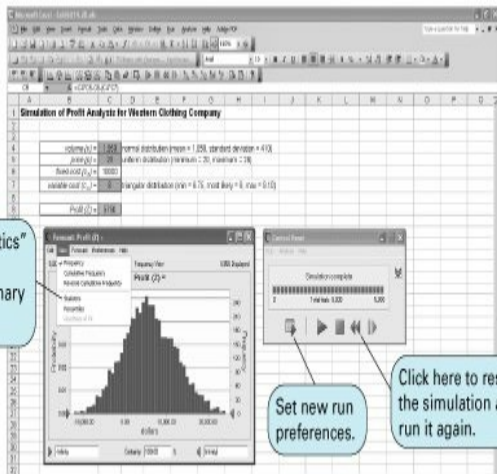
☐ Latin Hypercube (more even)

Sample size: 500

OK Cancel Defaults... Help

Exhibit 14.24.

[\[View full size image\]](#)



A statistical summary report for this simulation can be obtained by clicking on "View" at the top of the forecast window and then selecting "Statistics" from the drop-down menu. This results in the window shown in [Exhibit 14.25](#). You can return to the forecast window by selecting "Frequency" from the "View" menu at the top of the statistics window.

In our original example formulated in [Chapter 1](#), we wanted to determine the break-even volume. In this revised example, Western Clothing

Company wants to know the average profit and the probability that it will break even from this simulation analysis. The mean profit (from the Statistics window in [Exhibit 14.25](#)) is \$5,833.78. The probability of breaking even is determined by clicking on the arrow on the left side of the horizontal axis of the window shown in [Exhibit 14.26](#) and "grabbing" it and moving it to "0.00," or by clicking on the lower limit, currently set at "Infinity"; we change this to 0 and press the Enter key. This

will shift the lower limit to zero, the break-even point. The frequency chart that shows the location of the new lower limit and the "Certainty" of zero profit is shown as 81.73% at the bottom of the window as shown in [Exhibit 14.26](#). Thus, there is a .8173 probability that the company will break even.

Exhibit 14.25.

(This item is displayed on page 642 in the print version)

[\[View full size image\]](#)

Click on "View"
to return to the
Frequency window.



Forecast: Profit (Z) -

Edit View Forecast Preferences Help

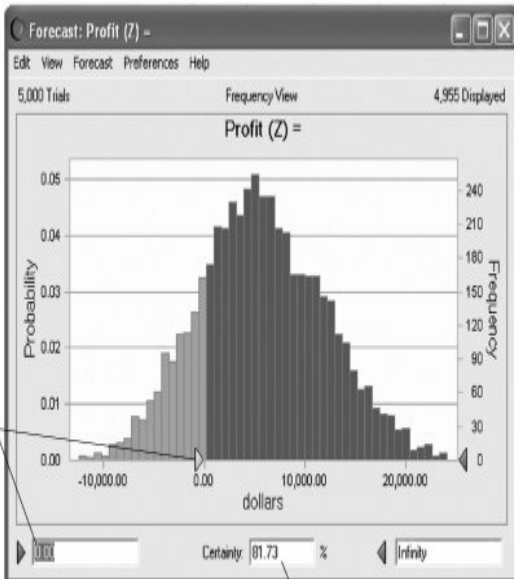
5,000 Trials Statistics View 4,955 Displayed

	Statistic	Forecast values
▶	Trials	5,000
	Mean	5,833.78
	Median	5,487.15
	Mode	...
	Standard Deviation	6,536.98
	Variance	42,732,071.56
	Skewness	0.25457
	Kurtosis	3.27
	Coef. of Variability	1.12
	Minimum	-18,815.19
	Maximum	34,295.49
	Mean Std. Error	92.45

Exhibit 14.26.

(This item is displayed on page
642 in the print version)

[\[View full size image\]](#)



Move arrow to "0.00" or "set lower limit equal to 0.00."

Probability (.8173) that the company will break even

We have demonstrated using Crystal Ball with a straightforward example that was not very complex or detailed. Crystal Ball has the capability to perform much more sophisticated simulation analyses than what we have shown in this section. However, the demonstration of these capabilities and other features of Crystal Ball would require more space and in-depth coverage than is possible here. However, although using Crystal Ball to simulate more complex situations requires a

greater degree of knowledge than we have provided, this basic introduction to and demonstration of Crystal Ball provide a good starting point to understanding the basic features of Crystal Ball and its use for simulation analysis.

Verification of the Simulation Model

Even though we may be able to verify the statistical results of a simulation model, we still may not know whether the model actually replicates what is going on in the real world. The user of simulation generally wants to be certain that the model is internally correct and that all the operations performed in the simulation are logical and mathematically

correct. An old adage often associated with computer simulation is "garbage in, garbage out." To gain some assurances about the validity of simulation results, there are several testing procedures that the user of a simulation model can apply.

Simulation models must be validated to make sure they are accurately replicating the system being simulated.

First, the simulation model can be run for short periods of time or for only a few simulation trials. This allows the user to compare the results with manually derived solutions (as we did in the examples in this chapter) to check for discrepancies. Another means of testing is to divide the model into parts and simulate each part separately. This reduces the complexity of seeking out errors in the model. Similarly, the mathematical relationships

in the simulation model can be simplified so that they can more easily be tested to see if the model is operating correctly.

Sometimes manual simulation of several trials is a good way to validate a simulation.

To determine whether the model reliably represents the system being simulated, the

simulation results can sometimes be compared with actual real-world data. Several statistical tests are available for performing this type of analysis. However, when a model is developed to simulate a *new* or *unique* system, there is no realistic way to ensure that the results are valid.

An additional problem in determining whether a simulation model is a valid representation of the system under analysis relates to starting conditions. Should the simulation be started with the

system empty (e.g., should we start by simulating a queuing system with no customers in line), or should the simulation be started as close as possible to normal operating conditions? Another problem, as we have already seen, is the determination of how long the simulation should be run to reach true steady-state conditions, if indeed a steady state exists.

In general, a standard, foolproof procedure for validation is simply not possible. In many cases, the

user of a simulation model must rely on the expertise and experience of whoever develops the model.

Areas of Simulation Application

Simulation is one of the most useful of all management science techniques. The reason for this popularity is that simulation can be applied to a number of problems that are too difficult to model and solve analytically. Some analysts feel that complex systems should be studied via simulation whether or not they can be analyzed analytically because simulation

provides such an easy vehicle for experimenting on the system. As a result, simulation has been applied to a wide range of problems. Surveys conducted during the 1990s indicated that a large majority of major corporations use simulation in such functional areas as production, corporate planning, engineering, financial analysis, research and development, marketing, information systems, and personnel. Following are descriptions of some of the most common applications of

simulation.

Queuing

A major application of simulation has been in the analysis of queuing systems. As indicated in [Chapter 13](#), the assumptions required to solve the operating characteristic formulas are relatively restrictive. For the more complex queuing systems (which result from a relaxation of these assumptions), it is not possible to develop analytical

formulas, and simulation is often the only available means of analysis.

[Page 644]

Inventory Control

Most people are aware that product demand is an essential component in determining the amount of inventory a commercial enterprise should keep. Most of the mathematical formulas used to analyze inventory systems make the

assumption that this demand is certain (i.e., not a random variable). In practice, however, demand is rarely known with certainty. Simulation is one of the few means for analyzing inventory systems in which demand is a random variable, reflecting demand uncertainty. Inventory control is discussed in [Chapter 16](#).

Production and Manufacturing

Simulation is often applied to

production problems, such as production scheduling, production sequencing, assembly line balancing (of work-in-process inventory), plant layout, and plant location analysis. It is surprising how often various production processes can be viewed as queuing systems that can only be analyzed using simulation. Because machine breakdowns typically occur according to some probability distributions, maintenance problems are also frequently analyzed using simulation.

Finance

Capital budgeting problems require estimates of cash flows, which are often a result of many random variables.

Simulation has been used to generate values of the various contributing factors to derive estimates of cash flows.

Simulation has also been used to determine the inputs into rate of return calculations in which the inputs are random variables, such as market size, selling price, growth rate, and market share.

Marketing

Marketing problems typically include numerous random variables, such as market size and type, and consumer preferences. Simulation can be used to ascertain how a particular market might react to the introduction of a product or to an advertising campaign for an existing product.

Another area in marketing where simulation is applied is the analysis of distribution channels to determine the most efficient distribution system.

Public Service Operations

The operations of police departments, fire departments, post offices, hospitals, court systems, airports, and other public systems have all been analyzed by using simulation. Typically, such operations are so complex and contain so many random variables that no technique except simulation can be employed for analysis.

Environmental and Resource Analysis

Some of the more recent innovative applications of simulation have been directed at problems in the environment. Simulation models have been developed to ascertain the impact on the environment of projects such as nuclear power plants, reservoirs, highways, and dams. In many cases, these models include measures to analyze the financial feasibility of such projects. Other models have been developed to simulate pollution conditions. In the area of resource

analysis, numerous models have been developed in recent years to simulate energy systems and the feasibility of alternative energy sources.

[Page 645]

Management Science

Application: Simulating a 10-km Race in Boulder, Colorado

The Bolder Boulder, a popular 10-kilometer race held each Memorial Day in Colorado, attracts many of the world's best runners among its 20,000 participants. The race starts at the Bank of Boulder at the northeastern corner of the city, winds through the city streets, and ends at the University of Colorado's football stadium in the center of the city. As the race grew in size (from 2,200 participants in 1979 to 20,000 in 1985), its quality suffered from overcrowding problems, especially at the finish line, where runners are

individually tagged as they finish. Large waiting lines built up at the finish line, causing many complaints from the participants.

To correct this problem, race management implemented an interval-start system in 1986, wherein 24 groups of up to 1,000 runners each were started at 1-minute intervals. While this solution alleviated the problem of street crowding, it did not solve the queuing problem at the finish line.

A simulation model of the race was then developed to evaluate several possible solutions specifically, increasing the number of finish line chutes from the 8 used previously to either 12 or 15. The model was also used to identify a set of block-start intervals that would eliminate finish line queuing problems with either

chute scenario. Recommendations based on the simulation model were for a 12-chute finish line configuration and specific block-start intervals. The race conducted using the recommendations from the simulation model was flawless. The actual race behavior was almost identical to the simulation results. No overcrowding or queuing problems occurred at the finish line. The simulation model was used to fine-tune the 1986 and 1987 races, which were also conducted with virtually no problems.



Source: R. Farina, et al., "The Computer Runs the Bolder Boulder: A Simulation of a Major Running Race," *Interfaces* 19, no. 2 (March/April 1989): 4855.

Summary

Simulation has become an increasingly important management science technique in recent years. Various surveys have shown simulation to be one of the techniques most widely applied to real-world problems. Evidence of this popularity is the number of specialized simulation languages that have been developed by the computer industry and academia to deal

with complex problem areas.

[Page 646]

The popularity of simulation is due in large part to the flexibility it allows in analyzing systems, compared to more confining analytical techniques. In other words, the problem does not have to fit the model (or technique) the simulation model can be constructed to fit the problem. A primary benefit of simulation analysis is that it enables us to experiment with the model. For example, in our

queuing example we could expand the model to represent more service facilities, more queues, and different arrival and service times; and we could observe their effects on the results. In many analytical cases, such experimentation is limited by the availability of an applicable formula. That is, by changing various parts of the problem, we may create a problem for which we have no specific analytical formula. Simulation, however, is not subject to such limitations. Simulation is limited only by

one's ability to develop a computer program.

Simulation is one of the most important and widely used management science techniques.

Simulation provides a laboratory for experimentation on a real system.

Simulation is a management science technique that does not usually result in an optimal solution. Generally, a simulation model reflects the *operation of a system*, and the results of the model are in the form of operating statistics, such as averages. However, optimal solutions can sometimes be obtained for simulation models by employing [search techniques](#).

Simulation does not

*usually provide a recommended decision as does an optimization model; it provides **operating characteristics**.*

However, in spite of its versatility, simulation has limitations and must be used with caution. One limitation is that simulation models are typically unstructured and must be developed for a system or

problem that is also unstructured. Unlike some of the structured techniques presented in this text, they cannot simply be applied to a specific type of problem. As a result, developing simulation models often requires imagination and intuitiveness that are not required by some of the more straightforward solution techniques we have presented. In addition, the validation of simulation models is an area of serious concern. It is often impossible realistically to validate simulation results,

to know if they accurately reflect the system under analysis. This problem has become an area of such concern that "output analysis" of simulation results is developing into a new field of study. Another limiting factor in simulation is the cost in money and time of model building. Because simulation models are developed for unstructured systems, they often take large amounts of staff time, computer time, and money to develop and run. For many business companies, these

costs can be prohibitive.

*Simulation has
certain limitations.*

References

Hammersly, J. M., and Handscomb, D. C. *Monte Carlo Methods*. New York: John Wiley & Sons, 1964.

Markowitz, H. M., Karr, H. W., and Hausner, B. *SIMSCRIPT: A Simulation Programming Language*. Upper Saddle River, NJ: Prentice Hall, 1963.

Meier, R. C., Newell, W. T., and Pazer, H. L. *Simulation in Business and Economics*. Upper Saddle River, NJ: Prentice Hall, 1969.

Mize, J., and Cox, G. *Essentials*

of Simulation. Upper Saddle River, NJ: Prentice Hall, 1968.

Naylor, T. H., Balintfy, J. L., Burdinck, D. S., and Chu, K. *Computer Simulation Techniques*. New York: John Wiley & Sons, 1966.

Pritsker, A. A. B. *Modeling and Analysis Using Q-GERT Networks*, 2nd ed. New York: John Wiley & Sons, 1977.

. *The GASP IV Simulation Language*. New York: John Wiley & Sons, 1974.

Schriber, T. J. *Simulation Using GPSS*. New York: John Wiley & Sons, 1974.

Tocher, K. D. "Review of Computer Simulation."
Operational Research Quarterly 16 (June 1965): 189-217.

Van Horne, R. L. "Validation of Simulation Results."
Management Science 17 (January 1971): 247-57.

Example Problem Solution

The following example problem demonstrates a manual simulation using discrete probability distributions.

Problem Statement

Members of the Willow Creek Emergency Rescue Squad know from past experience that they will receive between zero and

six emergency calls each night,
according to the following
discrete probability
distribution:

Calls	Probability
0	.05
1	.12
2	.15
3	.25
4	.22

5	.15
6	.06
	1.00

The rescue squad classifies each emergency call into one of three categories: minor, regular, or major emergency. The probability that a particular call will be each type of emergency is as follows:

Emergency Type	Probability
Minor	.30
Regular	.56
Major	.14
	1.00

The type of emergency call determines the size of the crew sent in response. A minor

emergency requires a two-person crew, a regular call requires a three-person crew, and a major emergency requires a five-person crew.

Simulate the emergency calls received by the rescue squad for 10 nights, compute the average number of each type of emergency call each night, and determine the maximum number of crew members that might be needed on any given night.

Solution

Develop Random Number Probability Distribution

Calls	Probability
-------	-------------

0	.05
---	-----

1	.12
---	-----

2	.15
---	-----

Step 1.

3 .25

4 .22

5 .15

6 .06

1.00

Emergency Type

Probabilit

Minor	.30
Regular	.56
Major	.14
	1.00

Set Up a Tabular Simulation

Use the second column
[Table 14.3](#):

Night	r_1	Number of Calls	r_2	Expected Calls
1	65	4	71	
2	65	4	71	
3	65	4	71	
4	65	4	71	
5	65	4	71	
6	65	4	71	
7	65	4	71	
8	65	4	71	
9	65	4	71	
10	65	4	71	
11	65	4	71	
12	65	4	71	
13	65	4	71	
14	65	4	71	
15	65	4	71	
16	65	4	71	
17	65	4	71	
18	65	4	71	
19	65	4	71	
20	65	4	71	
21	65	4	71	
22	65	4	71	
23	65	4	71	
24	65	4	71	
25	65	4	71	
26	65	4	71	
27	65	4	71	
28	65	4	71	
29	65	4	71	
30	65	4	71	
31	65	4	71	
32	65	4	71	
33	65	4	71	
34	65	4	71	
35	65	4	71	
36	65	4	71	
37	65	4	71	
38	65	4	71	
39	65	4	71	
40	65	4	71	
41	65	4	71	
42	65	4	71	
43	65	4	71	
44	65	4	71	
45	65	4	71	
46	65	4	71	
47	65	4	71	
48	65	4	71	
49	65	4	71	
50	65	4	71	
51	65	4	71	
52	65	4	71	
53	65	4	71	
54	65	4	71	
55	65	4	71	
56	65	4	71	
57	65	4	71	
58	65	4	71	
59	65	4	71	
60	65	4	71	
61	65	4	71	
62	65	4	71	
63	65	4	71	
64	65	4	71	
65	65	4	71	
66	65	4	71	
67	65	4	71	
68	65	4	71	
69	65	4	71	
70	65	4	71	
71	65	4	71	
72	65	4	71	
73	65	4	71	
74	65	4	71	
75	65	4	71	
76	65	4	71	
77	65	4	71	
78	65	4	71	
79	65	4	71	
80	65	4	71	
81	65	4	71	
82	65	4	71	
83	65	4	71	
84	65	4	71	
85	65	4	71	
86	65	4	71	
87	65	4	71	
88	65	4	71	
89	65	4	71	
90	65	4	71	
91	65	4	71	
92	65	4	71	
93	65	4	71	
94	65	4	71	
95	65	4	71	
96	65	4	71	
97	65	4	71	
98	65	4	71	
99	65	4	71	
100	65	4	71	

18

12

17

2 48 3 89

18

83

3 08 1 90

4 05 0

5 89 5 18

08

26

Step 2.

47

94

6 06 1 72

7	62	4	47
---	----	---	----

			68
--	--	--	----

			60
--	--	--	----

			88
--	--	--	----

8	17	1	36
---	----	---	----

9	77	4	43
---	----	---	----

28

31

06

10

68

4

39

71

22

Step 3. Compute Results

$$\text{average number of minor emergency calls per night} = \frac{10}{10} = 1.0$$

$$\text{average number of regular emergency calls per night} = \frac{13}{10} = 1.3$$

$$\text{average number of major emergency calls per night} = \frac{4}{10} = 0.40$$

If all the calls came in at the same time, the maximum number of squad members required during any 1 night would be 14.

Problems

The Hoylake Rescue Squad receives an emergency call every 1, 2, 3, 4, 5, or 6 hours, according to the following probability distribution. The squad is on duty 24 hours per day, 7 days per week:

Time Between Emergency Calls (hr.)	Probability
1	.05
2	.10

1.

3	.30
4	.30
5	.20
6	.05
	1.00

1. Simulate the emergency calls for 1000 hours (assuming that this will require a "running," or "on-call," person on an hourly clock), using the random numbers.
2. Compute the average time between calls and compare this value with the expected time between calls from the probability distribution.

distribution. Why are the results d

- 3.** How many calls were made during period? Can you logically assume average number of calls per 3-day how could you simulate to determine average?

The time between arrivals of cars at the Service Station is defined by the following distribution:

Time Between Arrivals (min.)	Probability
1	.15
2	.30

2.	3	.40
	4	.15
		1.00

- 1.** Simulate the arrival of cars at the for 20 arrivals and compute the a between arrivals.
- 2.** Simulate the arrival of cars at the for 1 hour, using a different strear numbers from those used in (a) a average time between arrivals.
- 3.** Compare the results obtained in (

The Dynaco Manufacturing Company produces a product in a process consisting of operating machines. The probability distribution of machines that will break down in a week

[Page 650]

**Machine
Breakdowns per
Week**

Probab

0

.10

1

.10

2

.20

3

.25

3.

4	.30
---	-----

5	.05
---	-----

	1.00
--	------

- 1.** Simulate the machine breakdown: 20 weeks.
 - 2.** Compute the average number of will break down per week.
- 4.** Solve [Problem 19](#) at the end of [Chapte](#) simulation.

- 5.** Simulate the decision situation described in [Problem 16\(a\)](#) at the end of [Chapter 12](#) for 20 trials and recommend the best decision.

Every time a machine breaks down at the Manufacturing Company ([Problem 3](#)), 6 hours are required to fix it, according to the following probability distribution:

Repair Time (hr.)	Probability
1	.30
2	.50
3	.20
	1.00

-
1. Simulate the repair time for 20 weeks and compute the average weekly repair time.
 2. If the random numbers that are used for breakdowns per week are also used for repair time per breakdown, will the results be affected in any way? Explain.
 3. If it costs \$50 per hour to repair a machine when it breaks down (including lost production), determine the average weekly breakdown cost.
 4. The Dynaco Company is considering a preventive maintenance program that would use the probabilities of machine breakdown shown in the following table:
- 6.
-

Machine Breakdowns per Week	Probability
-----------------------------	-------------

Week

0

.20

1

.30

2

.20

3

.15

4

.10

5

.05

1.00

The weekly cost of the preventive maintenance program is \$150. Using simulation, determine if the company should institute the preventive maintenance program.

Sound Warehouse in Georgetown sells CD players (with speakers), which it orders from Fuji in Japan. Because of shipping and handling charges, an order must be for five CD players. Because it takes two weeks to receive an order, the warehouse places an order every time the present inventory falls to five CD players. It costs \$100 to place an order. If a customer asks for a CD player and the warehouse is out of stock, it costs the warehouse \$400 in lost sales. If a customer asks for a CD player and the warehouse has a CD player in stock, it costs the warehouse \$40 to keep each CD player stored in the warehouse. If a customer asks for a CD player when it is requested, the customer will not wait until one comes in from a competitor. The following probability distribution of demand for CD players has been determined.

Demand per Month	Probability
0	.04
1	.08
2	.28
3	.40
4	.16
5	.02

6

.02

1.00

The time required to receive an order o
has the following probability distributio

**Time to Receive
an Order (mo.)**

Probability

1

.60

2

.30

3

.10

The warehouse has five CD players in stock and always received at the beginning of the month. Sound Warehouse's ordering and sales are on a monthly basis, using the first column of random numbers from [Table 14.3](#). Compute the average monthly inventory.

First American Bank is trying to determine how many drive-through teller windows should install one or two drive-through teller windows. The following probability distributions for the number of drive-through teller windows and service times have been collected from historical data:

**Time Between
Automobile**

Probability

Arrivals (min.)	
1	.20
2	.60
3	.10
4	.10
	1.00
Service Time (min.)	Probability

8.	2	.10
	3	.40
	4	.20
	5	.20
	6	.10
		1.00

Assume that in the two-server system, a ship will join the shorter queue. When the queue lengths are equal, there is a 50/50 chance that a ship will enter the queue for either window.

1. Simulate both the one- and two-server systems. Compute the average queue length and percentage utilization for each system.
2. Discuss your results in (a) and suggest to which they could be used to make a decision about which system to employ.

The time between arrivals of oil tankers dock at Prudhoe Bay is given by the following probability distribution:

Time Between Ship Arrivals (days)	Probability
--	--------------------

1 .05

2 .10

3 .20

4 .30

5 .20

6 .10

7 .05

1.00

9.

The time required to fill a tanker with oil for sea is given by the following probability distribution

Time to Fill and Prepare (days)	Probability
3	.10
4	.20
5	.40
6	.30
	1.00

-
1. Simulate the movement of tankers to the single loading dock for the first 100 tankers. Compute the average time between arrivals, average waiting time to load, and average number of tankers waiting to be loaded.
 2. Discuss any hesitation you might have about using your results for decision making.

The Saki automobile dealer in the Minneapolis area orders the Saki sport compact, which gets 30 miles per gallon of gasoline, from the manufacturer in Japan. However, the dealer never knows how many months it will take to receive an order once placed. It can take 1, 2, or 3 months, with the following probabilities:

Months to Receive an

Order	Probab
1	.50
2	.30
3	.20
	1.00

[Page 653]

The demand per month is given by the
10. distribution:

Demand per Month (cars)	Probab
1	.10
2	.30
3	.40
4	.20
	1.00

The dealer orders when the number of gets down to a certain level. To determ appropriate level of cars to use as an ir

to order, the dealer needs to know how many cars will be demanded during the time required to receive an order. Simulate the demand for 30 orders and find the average number of cars demanded during the time required to receive an order. At what level of stock should the dealer place an order?

State University is playing Tech in their football game on Saturday. A sportswriter has followed the team all season and accumulated the following data. State runs four basic plays: a sweep, a pass, a run, and an off tackle; Tech uses three basic defenses: a wide tackle, an Oklahoma, and a blitz. The number of yards State will gain for each play against each defense is shown in the following table:

		Tech Defense	
State Play		Wide Tackle	Oklahoma

Sweep	3	5
Pass	12	4
Draw	2	1
Off tackle	7	3

The probability that State will run each is shown in the following table:

Play	Probability
------	-------------

11.

Sweep	.10
-------	-----

Pass	.20
Draw	.20
Off tackle	.50

The probability that Tech will use each c
follows:

Defense	Probability
Wide tackle	.30
Oklahoma	.50

The sportswriter estimates that State will win during the game. The sportswriter believes that if State gains 300 or more yards, it will win; however, if State holds State to fewer than 300 yards, it will lose. The sportswriter uses a computer simulation to determine which team the sportswriter will predict to win the game.

[Page 654]

Each semester, the students in the college at State University must have their college advisor approved by the college adviser. The students must arrive at the hallway outside the adviser's office according to the following distribution:

**Time Between Arrivals
(min.)**

Probab

4

.20

5

.30

6

.40

7

10

1.00

12.

The time required by the adviser to exa
approve a schedule corresponds to the

probability distribution:

Schedule Approval (min.)	Probab
6	.30
7	.50
8	.20
	1.00

Simulate this course approval system f
Compute the average queue length and
time a student must wait. Discuss thes

A city is served by two newspapers the *Daily News*. Each Sunday readers purchase newspapers at a stand. The following table shows the probabilities of a customer's buying a newspaper in a week, given the newspaper bought the previous Sunday:

13.

	Next Sunday	
	Tribune	Daily News
This Sunday		
Tribune	.65	.35
Daily News	.45	.55

Simulate a customer's purchase of newspapers over several weeks to determine the steady-state probabilities of a customer buying each newspaper in the long run.

Loebuck Grocery orders milk from a dairy on a daily basis. The manager of the store has determined the following probability distribution for the number of cases of milk ordered (in cases):

14.

Demand (cases)	Probability
15	.20
16	.25
17	.40
18	.15
	1.00

The milk costs the grocery \$10 per case. The carrying cost is \$0.50 per case per week, and the shortage cost is \$1 per case.

Simulate the ordering system for Loeb for 20 weeks. Use a weekly order size of 100 and compute the average weekly profit size. Explain how the complete simulation determining order size would be developed for this problem.

[Page 655]

The Paymore Rental Car Agency rents cars in a small town. It wants to determine how many cars it should maintain. Based on market projections and historical data, the manager has determined probability distributions for the number of rentals per day and rental duration (in days only) as shown in the following tables:

Number of Customers/Day	Probability
-------------------------	-------------

1

2

3

15.

Rental Duration (days)	Probab
-------------------------------	---------------

1

.10

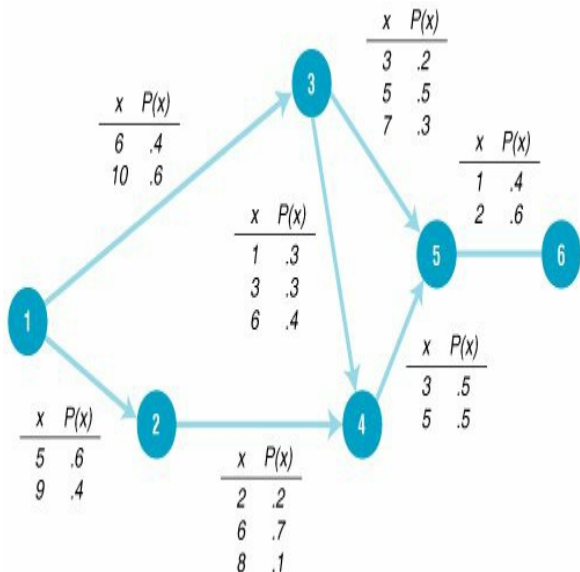
2

.30

3	.40
4	.10
5	.10
	1.00

Design a simulation experiment for the simulate using a fleet of four rental cars. Compute the probability that the agency car available upon demand. Should the its fleet? Explain how a simulation experiment designed to determine the optimal fleet Paymore Agency.

A CPM/PERT project network has probability times (x) as shown on each branch of example, activity 13 has a .40 probability completed in 6 weeks and a .60 probability completed in 10 weeks:



Simulate the project network 10 times the critical path each time. Compute the critical path time and the frequency at which each activity is critical. How does this simulation analysis compare with the critical path method compare with regular analysis?

A robbery has just been committed at Market in the downtown area of the city. The store owner was able to activate the alarm, but the thief fled on foot. Police officers arrived a few minutes later and asked the owner, "How long ago did the robbery occur?"

leave?" "He left only a few minutes ago," the store owner responded. "He's probably 10 blocks away now," one of the officers said to the other. "He said the store owner. "He was so stone cold."

17. I bet even if he has run 10 blocks, he's still within a few blocks of here! He's probably just a few circles!"

Perform a simulation experiment that will test the store owner's hypothesis. Assume that at each city block there is an equal chance that a person will go in any one of the four possible directions: north, south, east, or west. Simulate for five trials and indicate in how many of the trials the person is within 10 blocks of the store.

Compcomm, Inc., is an international communications and information technology company that has seen the value of its common stock appreciate significantly in recent years. A stock analyst would like to use a simulation to predict the stock prices of Compcomm for an extended period. Based on historical data, the analyst has developed the following probability

distribution for the movement of Comp
prices per day:

Stock Price Movement	Probab
Increase	.45
Same	.30
Decrease	.25
	1.00

The analyst has also developed the foll
distributions for the amount of the incre
decreases in the stock price per day:

18.	Probabil	
	Stock Price Change	Increase
	1/8	.40
	1/4	.17
	3/8	.12
	1/2	.10
	5/8	.08
	3/4	.07

7/8

.04

1

.02

1.00

The price of the stock is currently 62.

Develop a Monte Carlo simulation model of the stock price of Compcomm stock and simulate 1000 days. Indicate the new stock price at the end of 1000 days. How would this model be expanded to a complete simulation of 1 year's stock price movement?

The emergency room of the community Farmburg has one receptionist, one doctor, and one nurse. The emergency room opens at 10:00 a.m. Patients begin to arrive some time later at the emergency room according to the following probability distribution:

Time Between Arrivals (min.)	Probability
5	.06
10	.10
15	.23
20	.29

25	.18
30	.14
	1.00

The attention needed by a patient who emergency room is defined by the follo distribution:

Patient Needs to See	Probability
-------------------------------------	--------------------

Doctor alone	.50
--------------	-----

19.

Nurse alone	.20
-------------	-----

Both	.30
------	-----

	1.00
--	------

If a patient needs to see both the doctor and nurse, he or she cannot see one before the other is, the patient must wait to see both to get the treatment.

The length of the patient's visit (in minutes) is determined by the following probability distributions:

Doctor Probability	Nurse Probability
--------------------	-------------------

10	.22	5	.08
15	.31	10	.24
20	.25	15	.51
25	.12	20	.17
30	.10		1.00
	1.00		

Simulate the arrival of 20 patients to the room and compute the probability that wait and the average waiting time. Based on simulation, does it appear that this system provides adequate patient care?

The Western Outfitters Store specializes in jeans. The variable cost of the jeans varies due to several factors, including the cost of the distributor, labor costs, handling, and so on. Price also is a random variable that varies according to competitors' prices.

[Page 658]

Sales volume also varies each month. The probability distributions for volume, price, and variable cost per month are as follows:

Sales Volume	Probability
---------------------	--------------------

300 .12

400 .18

500 .20

600 .23

700 .17

800 .10

1.00

	Price	Probability
	\$22	.07
20.	23	.16
	24	.24
	25	.25
	26	.18
	27	.10
		1.00

Variable Cost	Probability
\$ 8	.17
9	.32
10	.29
11	.14
12	.08
	1.00

Fixed costs are \$9,000 per month for 1

Simulate 20 months of store sales and probability that the store will at least break the average profit (or loss).

Randolph College and Salem College are 10 miles of each other, and the students at each college frequently date each other. The students at Randolph College are debating how good their dates at Salem College are. The Randolph students have asked several hundred of their fellow students to rate their dates from 1 to 5 (1 is excellent and 5 is poor) according to physical attractiveness, intelligence, and personality. The following are the resulting probability distributions for traits for students at Salem College:

Physical Attractiveness Probability

1

.27

2

.35

3

.14

4

.09

5

.15

1.00

Intelligence**Probability**

21.

1

.10

2

.16

3

.45

4

.17

5

.12

1.00

Personality	Probability
1	.15
2	.30
3	.33
4	.07
5	.15
	1.00

Simulate 20 dates and compute an average rating of the Salem students.

- 22.** In [Problem 21](#) discuss how you might assess the accuracy of the average rating for Salem students based on only 20 simulated data.

Burlingham Mills produces denim cloth for jeans manufacturers. It is negotiating a contract with Troy Clothing Company to provide denim on a weekly basis. Burlingham has established its available production capacity for this contract between 0 and 600 yards, according to the following probability distribution:

$$f(x) = \frac{x}{180,000}, \quad 0 \leq x \leq 600 \text{ yd.}$$

Troy Clothing's weekly demand for denim

according to the following probability di

	Demand (yd.)	Probability
--	---------------------	--------------------

	0	.03
--	---	-----

23.

100	.12
-----	-----

200	.20
-----	-----

300	.35
-----	-----

400	.20
-----	-----

500	.10
-----	-----

Simulate Troy Clothing's cloth orders to determine the average weekly capacity. Also determine the probability that Burl has sufficient capacity to meet demand.

[Page 660]

A baseball game consists of plays that are as follows.

Play	Description
------	-------------

An out when

No advance

can advance
strikeouts,
flies, and th

Groundout

Each runner
one base.

Possible double play

Double play
runner on f
fewer than
lead runner
forced is ou
out advance
there is no
or there are
play is treat
advance."

Long fly

A runner or
score.

Very long fly

Runners on
third base &
base.

Walk

Includes a f

Infield single

All runners &
base.

Outfield single

A runner or
advances o
runner on s
base score

Long single

All runners &
maximum c

Double

Runners can
maximum c

Long double

All runners s

Triple

Home run

Note: Singles also include a factor for e
the batter to reach first base.

Distributions for these plays for two tea
Sox (visitors) and the Yankees (home)

Team: White Sox

Play	Probability
No advance	.03
Groundout	.39
24. Possible double play	.06
Long fly	.09
Very long fly	.08
Walk	.06
Infield single	.02

Outfield single	.10
-----------------	-----

Long single	.03
-------------	-----

Double	.04
--------	-----

Long double	.05
-------------	-----

Triple	.02
--------	-----

Home run	.03
----------	-----

	1.00
--	------

Team: Yankees

Play	Probability
------	-------------

No advance	.04
------------	-----

Groundout	.38
-----------	-----

Possible double play	.04
----------------------	-----

Long fly	.10
----------	-----

Very long fly	.06
---------------	-----

Walk	.07
------	-----

Infield single	.04
Outfield single	.10
Long single	.04
Double	.05
Long double	.03
Triple	.01
Home run	.04
	1.00

Simulate a nine-inning baseball game using the preceding information.[\[2\]](#)

[2] This problem was adapted from R. E. Trueman, "A Simulation Model of Baseball: With Particular Application to the Analysis of the Game," in R. E. Machol, S. P. Ladany, and D. G. L. (eds.), *Management Science in Sports* (New York: North-Holland Co., 1976), 114.

[Page 661]

Tracy McCoy is shopping for a new car. She has identified a particular sports utility vehicle. She has heard that it has high maintenance costs. She decided to develop a simulation model to estimate maintenance costs for the life of the vehicle. She estimates that the projected life of the owner (before it is sold) is uniformly distributed with a minimum of 2.0 years and a maximum of 10.0 years. Furthermore, she believes that the mile

25. the car each year can be defined by a triangular distribution with a minimum value of 3,000 miles, a maximum value of 14,500 miles, and a most likely value of 9,000 miles. She has determined from automobile association data that the cost per mile driven for the vehicle she is interested in is normally distributed, with a mean of \$0.03 per mile and a standard deviation of \$0.02 per mile. Using Crystal Ball, develop a simulation model (with 1,000 trials) and determine the average main maintenance cost over the life of the car with Tracy and the probability the cost will be less than \$3,000.

In [Problem 20](#), assume that the sales volume at Western Outfitters Store is normally distributed with a mean of 600 pairs of jeans and a standard deviation of 200; the price is uniformly distributed, with a minimum of \$22 and a maximum of \$28; and the cost of the jeans is defined by a triangular distribution with a minimum value of \$6, a maximum of \$11, and a most likely value of \$9. Develop a simulation model (with 1,000 trials) using Crystal Ball (with 1,000 trials) and determine the average profit and the probability of a profit of less than \$100.

average profit and the probability that Outfitters will break even.

In [Problem 21](#), assume that the students at Salem College have redefined the probability distributions for their dates at Salem College as follows: attractiveness is uniformly distributed from 1 to 5; intelligence is defined by a triangular distribution with a minimum rating of 1, a maximum of 5, and a most likely rating of 2; and personality is defined by a triangular distribution with a minimum of 1, a maximum of 5, and a most likely rating of 3. Develop a simulation by using Crystal Ball and determine the average rating (for 1,000 trials). Also compute the probability that the rating will be "better" than 3.0.

In [Problem 23](#), assume that production at Burlingham Mills for the Troy Clothing Company contract is normally distributed, with a mean of 100 yards per month and a standard deviation of 10 yards, and that Troy Clothing's demand is normally distributed, with a mean of 120 yards per month and a standard deviation of 15 yards.

28.

distributed between 0 and 500 yards. Use a simulation model by using Crystal Ball to determine the average monthly shortage or surplus of cloth (for 1,000 trials). Also determine the probability that Burlingham will always have sufficient capacity.

Erin Jones has \$100,000 and, to diversify, she wants to invest equal amounts of \$50,000 each in two mutual funds selected from a list of four mutual funds. She wants to invest for a year. She has used historical data from the five-year period to determine the mean return and standard deviation (normally distributed) for each fund, as follows:

Fund	Return (r)
	μ

1. Internet	.20
2. Index	.12
3. Entertainment	.16
29. 4. Growth	.14

The possible combinations of two investments are (1,2), (1,3), (1,4), (2,3), (2,4), and (3,4).

1. Use Crystal Ball to simulate each of the six investment combinations to determine the expected return in 3 years. (Note the formula for the future value, FV , of a present investment, P , with return, r , for t years: $FV = P(1 + r)^t$.)

future is $FV_n = P_r(1 + r)^n$.) Indicate which investment combination has the highest expected return.

[Page 662]

2. Erin wants to reduce her risk as much as possible. She knows that if she invests her money in a CD at the bank, she is guaranteed to receive \$20,000 after 3 years. Using the charts for the simulation runs in Chapter 16, determine which combination of investments would result in the greatest probability of receiving a return of \$120,000 or more.

In [Chapter 16](#), the formula for the optimal quantity of an item, Q , given its demand, C_o , and the cost of holding, or carrying, inventory, C_c , is as follows:

$$Q = \sqrt{\frac{2C_o D}{C_c}}$$

The total inventory cost formula is

30.
$$TC = \frac{C_o D}{Q} + C_c \frac{Q}{2}$$

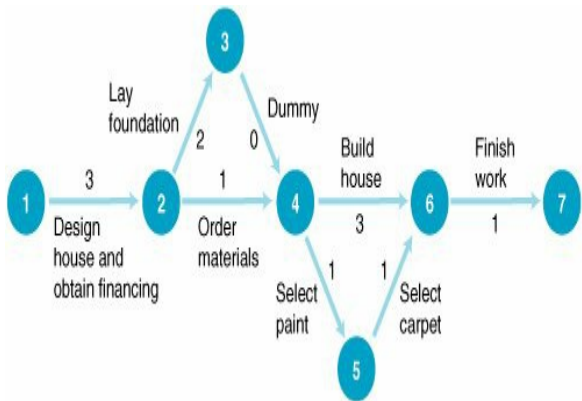
Order cost, C_o , and carrying cost, C_c , are values that the company is often able to determine with certainty because they are internal costs. Demand, D , is usually not known with certainty because it is external to the company. The order quantity formula given here, however, assumes demand as if it were certain. To consider the uncertainty in demand, it must be simulated.

Using Crystal Ball, simulate the preceding problem to find Q and TC to determine their average values. Assume demand per item, with $C_o = \$150$, $C_c = \$0.75$, and D that is normally distributed with a mean of 10,000 and a standard deviation of 4,000.

The Management Science Association (MSA) has arranged to hold its annual conference at the Grand Hotel in Orlando next year. Based on his experience, MSA believes the number of rooms it will have members attending the conference is normally distributed, with a mean of 800 and a standard deviation of 270. The MSA can reserve rooms now (at least a year prior to the conference) for \$80; any rooms not reserved now, the cost will be the hotel's regular room rate of \$120. The MSA guarantees the room rate of \$80 to its members who reserve fewer than the number of rooms reserved. If MSA reserves more rooms than the number of members, MSA must pay the hotel for the extra rooms at the \$120 room rate. If MSA does not reserve enough rooms, it must pay the extra cost that the hotel charges for each room.

- 31.**
 - 1.** Using Crystal Ball, determine whether MSA should reserve 600, 700, 800, 900, or 1000 rooms in advance to realize the largest expected profit.
 - 2.** Can you determine a more exact number of rooms to reserve to maximize the expected profit?

In [Chapter 8](#), [Figure 8.6](#) shows a simple network for building a house, as follows:



[Page 663]

There are four paths through this network:

Path A: 123467

Path B: 1234567

Path C: 12467

Path D: 124567

The time parameters (in weeks) defining the probability distribution for each activity follows:

32.

Activity	Time Parameters	
	Minimum	Likeliest
12	1	3
23	1	2

24	.5	1
34	0	0
45	1	2
46	1	3
56	1	2
67	1	2

1. Using Crystal Ball, simulate each p network and identify the longest p critical path).

2. Observing the simulation run frequency for path A, determine the probability that the total time will exceed the critical path time. What does this tell you about the simulation results for this network versus an analytical result?

Case Problem

JET COPIES

James Banks was standing in line next to Robin Cole at Klecko's Copy Center, waiting to use one of the copy machines. "Gee, Robin, I hate this," he said. "We have to drive all the way over here from Southgate and then wait in line to use these copy machines. I hate wasting time like this."

"I know what you mean," said Robin. "And look who's here. A lot of these students are from Southgate Apartments or one of the other apartments near us. It seems as though it would be more logical if Klecko's would move its operation over to us, instead of all of us coming over here."

James looked around and noticed what Robin was talking about. Robin and he were students at State University, and most of the customers at Klecko's were also students. As Robin suggested, a lot of the

people waiting were State students who lived at Southgate Apartments, where James also lived with Ernie Moore. This gave James an idea, which he shared with Ernie and their friend Terri Jones when he got home later that evening.

"Look, you guys, I've got an idea to make some money," James started. "Let's open a copy business! All we have to do is buy a copier, put it in Terri's duplex next door, and sell copies. I know we can get customers because I've just

seen them all at Klecko's. If we provide a copy service right here in the Southgate complex, we'll make a killing."

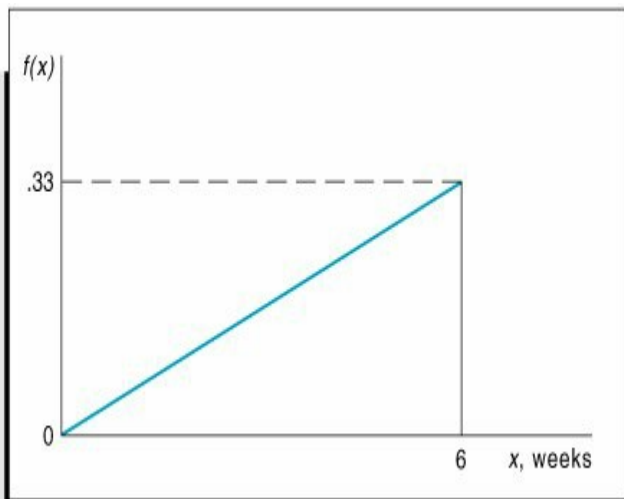
Terri and Ernie liked the idea, so the three decided to go into the copying business. They would call it JET Copies, named for James, Ernie, and Terri. Their first step was to purchase a copier. They bought one like the one used in the college of business office at State for \$18,000. (Terri's parents provided a loan.) The company that sold them the copier touted the copier's reliability,

but after they bought it, Ernie talked with someone in the dean's office at State, who told him that the University's copier broke down frequently and when it did, it often took between 1 and 4 days to get it repaired. When Ernie told this to Terri and James, they became worried. If the copier broke down frequently and was not in use for long periods while they waited for a repair person to come fix it, they could lose a lot of revenue. As a result, James, Ernie, and Terri thought they might need to

purchase a smaller backup copier for \$8,000 to use when the main copier broke down. However, before they approached Terri's parents for another loan, they wanted to have an estimate of just how much money they might lose if they did not have a backup copier. To get this estimate, they decided to develop a simulation model because they were studying simulation in one of their classes at State.

To develop a simulation model, they first needed to know how frequently the copier might break down specifically, the time between breakdowns. No one could provide them with an exact probability distribution, but from talking to staff members in the college of business, James estimated that the time between breakdowns was probably between 0 and 6 weeks, with the probability increasing the longer the copier went without breaking down. Thus, the probability distribution of breakdowns

generally looked like the following:



Next, they needed to know how long it would take to get the copier repaired when it broke

down. They had a service contract with the dealer that "guaranteed" prompt repair service. However, Terri gathered some data from the college of business from which she developed the following probability distribution of repair times:

Repair Time (days)	Probability
1	.20
2	.45

3

.25

4

.10

1.00

Finally, they needed to estimate how much business they would lose while the copier was waiting for repair. The three of them had only a vague idea of how much business they would do but finally estimated that they

would sell between 2,000 and 8,000 copies per day at \$0.10 per copy. However, they had no idea about what kind of probability distribution to use for this range of values.

Therefore, they decided to use a uniform probability distribution between 2,000 and 8,000 copies to estimate the number of copies they would sell per day.

James, Ernie, and Terri decided that if their loss of revenue due to machine downtime during 1 year was \$12,000 or more, they should purchase a backup

copier. Thus, they needed to simulate the breakdown and repair process for a number of years to obtain an average annual loss of revenue.

However, before programming the simulation model, they decided to conduct a manual simulation of this process for 1 year to see if the model was working correctly. Perform this manual simulation for JET Copies and determine the loss of revenue for 1 year.

Case Problem

BENEFITCOST ANALYSIS OF THE SPRADLIN BLUFF RIVER PROJECT

The U.S. Army Corps of Engineers has historically constructed dams on various rivers in the southeastern United States. Its primary instrument for evaluating and selecting among many projects under consideration is benefitcost analysis. The Corps estimates both the annual

benefits deriving from a project in several different categories and the annual costs and then divides the total benefits by the total costs to develop a benefitcost ratio. This ratio is then used by the Corps and Congress to compare numerous projects under consideration and select those for funding. A benefitcost ratio greater than 1.0 indicates that the benefits are greater than the costs; and the higher a project's benefitcost ratio, the more likely it is to be selected over projects with lower ratios.

The Corps is evaluating a project to construct a dam over the Spradlin Bluff River in southwest Georgia. The Corps has identified six traditional areas in which benefits will accrue: flood control, hydroelectric power, improved navigation, recreation, fish and wildlife, and area commercial redevelopment. The Corps has made three estimates (in dollars) for each benefit: a minimum possible value, a most likely value, and a

maximum benefit value. These benefit estimates are as follows:

Benefit	Estimate		
	Minimum	Most Likely	Maximum
Flood control	\$1,695,200	\$2,347,800	\$3,000,000
Hydroelectric power	8,068,250	11,845,000	14,000,000
Navigation	50,400	64,000	78,400

Recreation	6,404,000	9,774,000	14,
Fish and wildlife	104,300	255,000	.
Area redevelopment	0	1,630,000	2,

There are two categories of costs associated with a construction project of this type: the total capital cost, annualized over 100 years (at a rate of interest specified by the government), and annual

operation and maintenance costs. The cost estimates for this project are as follows:

Cost	Estimate	
	<i>Minimum</i>	<i>Most Likely</i>
Annualized capital cost	\$12,890,750	\$14,150,500
Operation and maintenance	3,483,500	4,890,000

Using Crystal Ball, determine a simulated mean benefitcost ratio and standard deviation.

What is the probability that this project will have a benefitcost ratio greater than 1.0?

Chapter 15.

Forecasting

[Page 667]

A **forecast** is a prediction of what will occur in the future. Meteorologists forecast the weather, sportscasters predict the winners of football games, and managers of business firms attempt to predict how much of their product will be demanded in the future. In fact, managers are constantly trying to predict the future, making decisions in

the present that will ensure the continued success of their firms. Often a manager will use judgment, opinion, or past experiences to forecast what will occur in the future.

However, a number of mathematical methods are also available to aid managers in making decisions. In this chapter, we present two of the traditional forecasting methods: time series analysis and regression. Although no technique will result in a totally accurate forecast (i.e., it is impossible to predict the future

exactly), these forecasting methods can provide reliable guidelines for decision making.

Forecasting Components

A variety of forecasting methods exist, and their applicability is dependent on the *time frame* of the forecast (i.e., how far in the future we are forecasting), the *existence of patterns* in the forecast (i.e., seasonal trends, peak periods), and the *number of variables* to which the forecast is related. We will discuss each of these factors separately.

In general, forecasts can be classified according to three time frames: short range, medium range, and long range.

Short-range forecasts

typically encompass the immediate future and are concerned with the daily operations of a business firm, such as daily demand or resource requirements. A short-range forecast rarely goes beyond a couple months into the future. A **medium-range forecast** typically encompasses anywhere from 1 or 2 months to 1 year. A

forecast of this length is generally more closely related to a yearly production plan and will reflect such items as peaks and valleys in demand and the necessity to secure additional resources for the upcoming year. A **long-range forecast** typically encompasses a period longer than 1 or 2 years. Long-range forecasts are related to management's attempt to plan new products for changing markets, build new facilities, or secure long-term financing. In general, the further into the future one seeks to predict, the

more difficult forecasting becomes.

These classifications should be viewed as generalizations. The line of demarcation between medium- and long-range forecasts is often quite arbitrary and not always distinct. For some firms a medium-range forecast could be several years, and for other firms a long-range forecast could be in terms of months.

*A **trend** is a gradual, long-term, up-or-*

down movement of demand.

Forecasts often exhibit patterns, or trends. A **trend** is a long-term movement of the item being forecast. For example, the demand for personal computers has shown an upward trend during the past decade, without any long downward movement in the market. Trends are the easiest patterns of demand behavior to

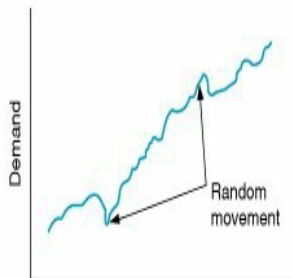
detect and are often the starting point for developing a forecast. [Figure 15.1\(a\)](#) illustrates a demand trend in which there is a general upward movement or increase. Notice that [Figure 15.1\(a\)](#) also includes several random movements up and down. **[Random variations](#)** are movements that are not predictable and follow no pattern (and thus are virtually unpredictable).

Figure 15.1. Forms of

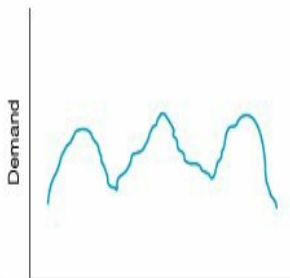
**forecast movement:
(a) trend, (b) cycle,
(c) seasonal pattern,
and (d) trend with
seasonal pattern**

(This item is displayed on page
668 in the print version)

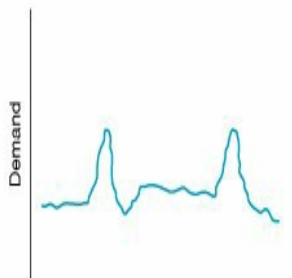
[\[View full size image\]](#)



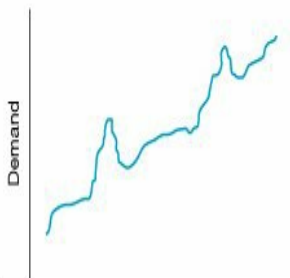
Time
(a)



Time
(b)



Time
(c)



Time
(d)

A **cycle** is an undulating movement in demand, up and down, that repeats itself over a lengthy time span (i.e., more than 1 year). For example, new housing starts and thus construction-related products tend to follow cycles in the economy. Automobile sales tend to follow cycles in the same fashion. The demand for winter sports equipment increases every 4 years, before and after the Winter Olympics. **Figure 15.1(b)** shows the

general behavior of a demand cycle.

*A **cycle** is an up-and-down repetitive movement in demand.*

*A **seasonal pattern** is an up-and-down, repetitive movement within a trend occurring periodically.*

A **seasonal pattern** is an oscillating movement in demand that occurs periodically (in the short run) and is repetitive. Seasonality is often weather related. For example, every winter the demand for snowblowers and skis increases dramatically, and retail sales in general increase during the Christmas season. However, a seasonal pattern can occur on a daily or weekly basis. For example, some restaurants are busier at lunch than at dinner,

and shopping mall stores and theaters tend to have higher demand on weekends. [Figure 15.1\(c\)](#) illustrates a seasonal pattern in which the same demand behavior is repeated each period at the same time.

[Page 668]

Of course, demand behavior will frequently display several of these characteristics simultaneously. Although housing starts display cyclical behavior, there has been an upward trend in new house

construction over the years. As we noted, demand for skis is seasonal; however, there has been a general upward trend in the demand for winter sports equipment during the past 2 decades. [Figure 15.1\(d\)](#) displays the combination of two demand patterns, a trend with a seasonal pattern.

There are instances in which demand behavior exhibits no pattern. These are referred to as *irregular movements*, or variations. For example, a local flood might cause a momentary increase in carpet demand, or

negative publicity resulting from a lawsuit might cause product demand to drop for a period of time. Although this behavior is causal, and thus not totally random, it still does not follow a pattern that can be reflected in a forecast.

Forecasting Methods

The factors discussed previously determine to a certain extent the type of forecasting method that can or should be used. In this chapter

we discuss the basic types of forecasting: *time series*, *regression methods*, and *qualitative methods*. **Time series** is a category of statistical techniques that uses historical data to predict future behavior. **Regression** (or causal) methods attempt to develop a mathematical relationship (in the form of a regression model) between the item being forecast and factors that cause it to behave the way it does. Most of the remainder of this chapter is about time series and regression

forecasting methods. In this section we focus our discussion on qualitative forecasting.

Types of forecasting methods are time series, regression, and qualitative.

[Page 669]

Qualitative methods use management judgment, expertise, and opinion to make

forecasts. Often called "the jury of executive opinion," they are the most common type of forecasting method for the long-term strategic planning process. There are normally individuals or groups within an organization whose judgments and opinions regarding the future are as valid or more valid than those of outside experts or other structured approaches. Top managers are the key group involved in the development of forecasts for strategic plans. They are generally most familiar with

their firms' own capabilities and resources and the markets for their products.

The sales force is a direct point of contact with the consumer.

This contact provides an awareness of consumer expectations in the future that others may not possess.

Engineering personnel have an innate understanding of the technological aspects of the type of products that might be feasible and likely in the future.

Consumer, or market, research

is an organized approach that uses surveys and other research techniques to determine what products and services customers want and will purchase, and to identify new markets and sources of customers. Consumer and market research is normally conducted by the marketing department within an organization, by industry organizations and groups, and by private marketing or consulting firms. Although market research can provide accurate and useful forecasts of

product demand, it must be skillfully and correctly conducted, and it can be expensive.

The **Delphi method** is a procedure for acquiring informed judgments and opinions from knowledgeable individuals, using a series of questionnaires to develop a consensus forecast about what will occur in the future. It was developed at the RAND Corporation shortly after World War II to forecast the impact of a hypothetical nuclear attack on the United States. Although the

Delphi method has been used for a variety of applications, forecasting has been one of its primary uses. It has been especially useful for forecasting technological change and advances.

Technological forecasting has become increasingly crucial for successful competition in today's global business environment. New enhanced computer technology, new production methods, and advanced machinery and equipment are constantly being made available to companies.

These advances enable them to introduce more new products into the marketplace faster than ever before. The companies that succeed do so by getting a technological jump on their competitors through accurate prediction of future technology and its capabilities. What new products and services will be technologically feasible, when they can be introduced, and what their demand will be are questions about the future for which answers cannot be predicted from historical data. Instead,

the informed opinion and judgment of experts are necessary to make these types of single, long-term forecasts.

Time Series Methods

Time series methods are statistical techniques that make use of historical data accumulated over a period of time. Time series methods assume that what has occurred in the past will continue to occur in the future. As the name *time series* suggests, these methods relate the forecast to only one factor *time*. Time series methods tend to be most useful for short-range

forecasting, although they can be used for longer-range forecasting. We will discuss two types of time series methods: the *moving average* and *exponential smoothing*.

Moving Average

A time series forecast can be as simple as using demand in the current period to predict demand in the next period. For example, if demand is 100 units this week, the forecast for next week's demand would be

100 units; if demand turned out to be 90 units instead, then the following week's demand would be 90 units, and so forth. This is sometimes referred to as *naïve* forecasting. However, this type of forecasting method does not take into account any type of historical demand behavior; it relies only on demand in the current period. As such, it reacts directly to the normal, random up-and-down movements in demand.

Alternatively, the **moving average** method uses several values during the recent past to develop a forecast. This tends to *dampen*, or *smooth out*, the random increases and decreases of a forecast that uses only one period. As such, the simple moving average is particularly useful for forecasting items that are relatively stable and do not display any pronounced behavior, such as a trend or seasonal pattern.

The moving average

*method is good for
stable demand with
no pronounced
behavioral patterns.*

Moving averages are computed for specific periods, such as 3 months or 5 months, depending on how much the forecaster desires to smooth the data. The longer the moving average period, the smoother it will be. The formula for computing the simple moving average is as

follows:

$$MA_n = \frac{\sum_{i=1}^n D_i}{n}$$

where

n = number of periods in the moving average

D_i = data in period i

To demonstrate the *moving average* forecasting method, we will use an example. The Instant Paper Clip Supply Company sells and delivers

office supplies to various companies, schools, and agencies within a 30-mile radius of its warehouse. The office supply business is extremely competitive, and the ability to deliver orders promptly is an important factor in getting new customers and keeping old ones. (Offices typically order not when their inventory of supplies is getting low but when they completely run out. As a result, they need their orders immediately.) The manager of the company wants to be certain that enough

drivers and delivery vehicles are available so that orders can be delivered promptly.

Therefore, the manager wants to be able to forecast the number of orders that will occur during the next month (i.e., to forecast the demand for deliveries).

From records of delivery orders, the manager has accumulated data for the past 10 months. These data are shown in [Table 15.1](#).

Table 15.1. Orders for 10-

month period

Month	Orders Delivered per Month
January	120
February	90
March	100
April	75
May	110
June	50
July	75

August	130
September	110
October	90

The moving average forecast is computed by dividing the sum of the values of the forecast variable, orders per month for a sequence of months, by the number of months in the sequence. Frequently, a moving

average is calculated for three or five time periods. The forecast resulting from either the 3- or the 5-month moving average is typically for the next month in the sequence, which in this case is November. The moving average is computed from the demand for orders for the last 3 months in the sequence, according to the following formula:

$$MA_3 = \frac{\sum_{i=1}^3 D_i}{3} = \frac{90 + 110 + 130}{3} = 110 \text{ orders}$$

The 5-month moving average is computed from the last 5 months of demand data, as follows:

$$MA_5 = \frac{\sum_{i=1}^5 D_i}{5} = \frac{90 + 110 + 130 + 75 + 50}{5} = 91 \text{ orders}$$

The 3- and 5-month moving averages for all the months of demand data are shown in [Table 15.2](#). Notice that we have computed forecasts for all the

months. Actually, only the forecast for November, based on the most recent monthly demand, would be used by the manager. However, the earlier forecasts for prior months allow us to compare the forecast with actual demand to see how accurate the forecasting method is (i.e., how well it does).

Table 15.2. 3- and 5-month averages

Month	Orders per Month	3-Month Moving Average	5-Month Moving Average
--------------	---------------------------------	---------------------------------------	---------------------------------------

January	120		
February	90		
March	100		
April	75	103.3	
May	110	88.3	
June	50	95.0	99.0
July	75	78.3	85.0
August	130	78.3	82.0

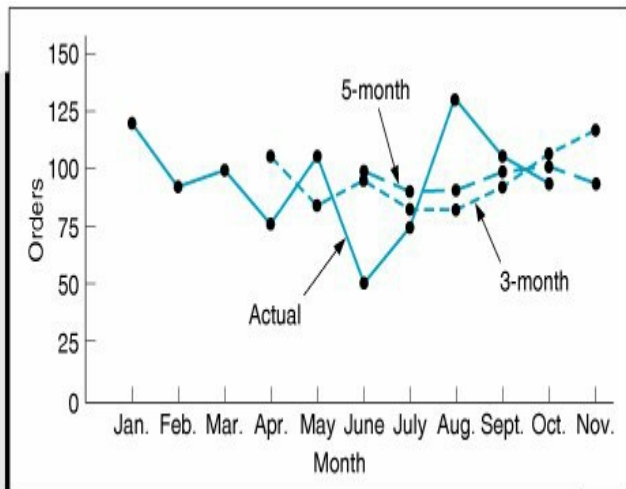
September	110	85.0	88.0
October	90	105.0	95.0
November		110.0	91.0

Both moving average forecasts in [Table 15.2](#) tend to smooth out the variability occurring in the actual data. This smoothing effect can be observed in [Figure 15.2](#), in which the 3-month and 5-month averages have been superimposed on a graph of the original data. The

extremes in the actual orders per month have been reduced. This is beneficial if these extremes simply reflect random fluctuations in orders per month because our moving average forecast will not be strongly influenced by them.

Figure 15.2. 3- and 5-month moving averages

(This item is displayed on page 672 in the print version)



Notice that the 5-month moving average in [Figure 15.2](#) smooths out fluctuations to a greater extent than the 3-

month moving average. However, the 3-month average more closely reflects the most recent data available to the office supply manager. (The 5-month average forecast considers data all the way back to June; the 3-month average does so only to August.) In general, forecasts computed using the longer-period moving average are slower to react to recent changes in demand than those made using shorter-period moving averages. The extra periods of data dampen the speed with which the

forecast responds. Establishing the appropriate number of periods to use in a moving average forecast often requires some amount of trial-and-error experimentation.

*Longer-period
moving averages
react more slowly to
recent demand
changes than do
shorter-period
moving averages.*

One additional point to mention relates to the period of the forecast. Sometimes a forecaster will need to forecast for a short planning horizon rather than just a single period. For both the 3- and 5-month moving average forecasts computed in [Table 15.2](#), the final forecast is for one period only in the future. Because the moving average includes multiple periods of data for each forecast, and it is most often used to forecast in a stable demand environment, it would be appropriate to use the

forecast for multiple periods in the future. It could then be updated as actual demand data became available.

[Page 672]

The major disadvantage of the moving average method is that it does not react well to variations that occur for a reason, such as trends and seasonal effects (although this method does reflect trends to a moderate extent). Those factors that cause changes are generally ignored. It is basically

a "mechanical" method, which reflects historical data in a consistent fashion. However, the moving average method does have the advantage of being easy to use, quick, and relatively inexpensive, although moving averages for a substantial number of periods for many different items can result in the accumulation and storage of a large amount of data. In general, this method can provide a good forecast for the short run, but an attempt should not be made to push the forecast too far into the distant

future.

Weighted Moving Average

The moving average method can be adjusted to reflect more closely more recent fluctuations in the data and seasonal effects. This adjusted method is referred to as a **weighted moving average** method. In this method, weights are assigned to the most recent data according to the following formula:

$$WMA_n = \sum_{i=1}^n W_i D_i$$

where

W_i = the weight for period i , between 0% and 100%

$$\sum W_i = 1.00$$

*In a **weighted moving average**, weights are assigned*

to the most recent data.

For example, if the Instant Paper Clip Supply Company wants to compute a 3-month weighted moving average with a weight of 50% for the October data, a weight of 33% for the September data, and a weight of 17% for August, it is computed as

$$WMA_3 = \sum_{i=1}^3 W_i D_i = (.50)(90) + (.33)(110) + (.17)(130) = 103.4 \text{ orders}$$

Notice that the forecast includes a fractional part, .4. Because .4 order would be impossible, this appears to be unrealistic. In general, the fractional parts need to be included in the computation to achieve mathematical accuracy, but when the final forecast is achieved, it must be rounded up or down.

Management Science

Application: Product

Forecasting at Nabisco

Nabisco Biscuit Company is the largest domestic operating company within Nabisco, a *Fortune* 500 company with \$9 billion in annual sales. Nabisco Biscuit Company is the largest cookie and cracker manufacturing company in the United States, with annual sales of \$3.5 billion. More than 30 company-owned and contract bakeries produce hundreds of different cookie, cracker, and snack products, including Oreo, Chips Ahoy!, Ritz, and Snack-Wells. These products are shipped from the bakeries to more than 100 distribution center warehouses throughout the United States, and

then from these distribution centers, Nabisco delivers orders to more than 100,000 final destinations, mostly retail grocery and convenience stores. Orders placed with a distribution center are delivered to a store within a short lead-time period, typically 24 to 48 hours. An accurate weekly forecast of product demand from the distribution centers is particularly critical to Nabisco because of the short lead time for orders, the wide geographic dispersion of the distribution centers, the large number of Nabisco products, and the limited product shelf life of bakery products. The forecasting process is complicated by product promotional events at the stores and the continuous introduction of new products.

Nabisco uses what is referred to as a *top-down* approach to forecasting. It

starts with a national production forecast for each product for a 4-week sales period. This national forecast considers information from finance and marketing and from a national product promotional schedule. It is provided to the distribution centers on a weekly basis to help create weekly forecasts of product demand, using statistical forecasting methods based on sales history. One of the forecast techniques it uses is a moving average. This demand forecast is combined with a statistical forecast based on individual account (e.g., store) promotional events and activities. Nabisco then uses this combined set of forecasts to determine the appropriate shipments from the bakeries to the distribution centers.

Forecasts for new products present

Nabisco with a different challenge. Most of the forecasting techniques used for existing products are developed using several years of historical data, which are not available for new products. At Nabisco a planning forecast is used to develop initial production planning and scheduling and to set initial inventory levels for a new product. However, when the product is introduced, which is generally a gradual process across different geographic regions, actual sales can be different than planned, depending on consumer acceptance of the product. If consumer acceptance is greater than planned, the company can expect shortages and lost sales. If consumer demand is lower than planned and the new product fails, the company can be stuck with excess inventory that must be destroyed. Thus, it is

important that Nabisco be able to adjust its forecasts quickly during the product introductory period so that they will be as accurate as possible. Nabisco uses a forecast model for new products developed using the exponential smoothing technique, with adjustments for trend and seasonality (one of the forecast techniques presented in this chapter).



Sources: S. Amrute, "Forecasting New Products with Limited History:

Nabisco's Experience," *Journal of Business Forecasting* 17, no. 3 (Fall 1998): 711; M. Barash and D. Mitchell, "Account Based Forecasting at Nabisco Biscuit Company," *Journal of Business Forecasting* 17, no. 2 (Summer 1998): 36.

Also notice that this forecast is slightly lower than our previously computed 3-month average forecast of 110 orders, reflecting the lower number of orders in October (the most recent month in the sequence).

Determining the precise weights to use for each period of data frequently requires some trial-and-error experimentation, as does determining the exact number of periods to include in the moving average. If the most recent months are weighted too heavily, the forecast might overreact to a random fluctuation in orders; if they are weighted too lightly, the forecast might underreact to an actual change in the pattern of orders.

Exponential Smoothing

The **exponential smoothing** forecast method is an averaging method that weights the most recent past data more strongly than more distant past data. Thus, the forecast will react more strongly to immediate changes in the data. This is very useful if the recent changes in the data are the results of an *actual* change (e.g., a seasonal pattern)

instead of just random fluctuations (for which a simple moving average forecast would suffice).

Exponential smoothing is an averaging method that reacts more strongly to recent changes in demand than to more distant past data.

We will consider two forms of exponential smoothing: *simple exponential smoothing* and *adjusted exponential smoothing* (adjusted for trends, seasonal patterns, etc.). We will discuss the simple exponential smoothing case first, followed by the adjusted form.

To demonstrate simple exponential smoothing, we will return to the Instant Paper Clip Supply Company example. The simple exponential smoothing forecast is computed by using the formula

$$F_{t+1} = \alpha D_t + (1 - \alpha)F_t$$

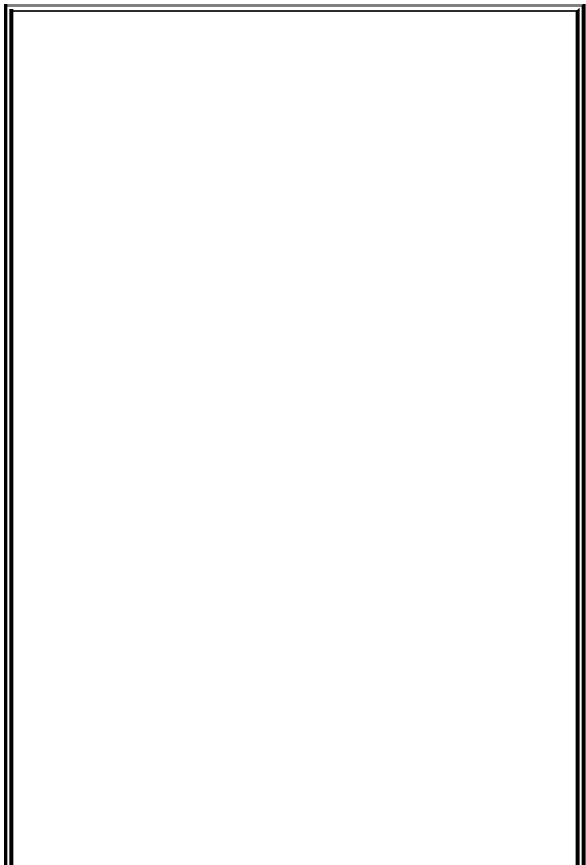
where

F_{t+1} = the forecast for the next period

D_t = actual demand in the present period

F_t = the previously determined forecast for the present period

α = a weighting factor referred to as the *smoothing constant*



Management Science

Application: Forecasting

Customer Demand at Taco Bell

Taco Bell is an international fast-food business with 6,500 locations worldwide, with annual sales of approximately \$4.6 billion.

Labor costs at Taco Bell are approximately 30% of every sales dollar, and these costs are very difficult to manage because labor is so closely linked to sales demand. It is not possible for Taco Bell to produce its food products in advance and store them in inventory like a manufacturing company; instead, it must prepare its food products on

demand that is, when they are ordered. Demand is highly variable during both the day and the week, with 52% of sales occurring during the 3-hour lunch period from 11:00 A.M. to 2:00 P.M. Determining how many employees to schedule at given times during the day is very difficult. This problem is exacerbated by the fact that quality service is determined by the speed of service.

To develop an effective labor-management system, Taco Bell needed an accurate forecasting model to predict customer demand that is, arrivals and sales. Taco Bell determined that the best forecasting method was a 6-week moving average for each 15-minute interval of every day of the week. For example, customer demand at a particular restaurant for all Thursdays from 11:00 A.M. to 11:15 A.M. for a

6-week period constitutes the time series database to forecast customer transactions for the next Thursday in the future. On a weekly basis, the forecasts are compared to statistical forecasting control limits that are continuously updated, and the length of the moving average is adjusted when the forecasts move out of control. Taco Bell achieved labor savings of over \$40 million from 1993 to 1996 by using the new labor-management system, of which this forecasting model is an integral part.



Source: J. Hueter and W. Swart, "An Integrated Labor-Management System for Taco Bell," *Interfaces* 28, no. 1 (January/February 1998): 7591.

The smoothing constant, α , is between zero and one. It reflects the weight given to the most recent demand data. For example, if $\alpha = .20$,

$$F_{t+1} = .20D_t + .80F_t$$

which means that our forecast for the next period is based on 20% of recent demand (D_t) and 80% of past demand (in the form of the forecast F_t because F_t is derived from previous demands and forecasts). If we go to one extreme and let $\alpha = 0.0$, then

$$F_{t+1} = 0D_t + 1F_t \\ = F_t$$

and the forecast for the next period is the same as for this period. In other words, *the forecast does not reflect the most recent demand at all.*

On the other hand, if $\alpha = 1.0$, then

The closer α is to one, the greater the reaction to the most recent demand.

$$\begin{aligned}
 F_{t+1} &= 1D_t + 0F_t \\
 &= 1D_t
 \end{aligned}$$

and we have considered only the most recent occurrence in our data (demand in the present period) and nothing else. Thus, we can conclude that the higher α is, the more sensitive the forecast will be to changes in recent demand. Alternatively, the closer α is to zero, the greater will be the dampening or smoothing effect. As α approaches zero, the forecast will react and adjust

more slowly to differences between the actual demand and the forecasted demand. The most commonly used values of α are in the range from .01 to .50. However, the determination of α is usually judgmental and subjective and will often be based on trial-and-error experimentation. An inaccurate estimate of α can limit the usefulness of this forecasting technique.

To demonstrate the computation of an exponentially smoothed forecast, we will use an

example. PM Computer Services assembles customized personal computers from generic parts. The company was formed and is operated by two part-time State University students, Paulette Tyler and Maureen Becker, and has had steady growth since it started. The company assembles computers mostly at night, using other part-time students as labor. Paulette and Maureen purchase generic computer parts in volume at a discount from a variety of sources whenever they see a good deal.

It is therefore important that they develop a good forecast of demand for their computers so that they will know how many computer component parts to purchase and stock.

The company has accumulated the demand data in [Table 15.3](#) for its computers for the past 12 months, from which it wants to compute exponential smoothing forecasts, using smoothing constants (α) equal to .30 and .50.

Table 15.3. Demand for

personal computers

Period

Month

Demand

1

January

37

2

February

40

3

March

41

4

April

37

5

May

45

6

June

50

7

July

43

8	August	47
9	September	56
10	October	52
11	November	55
12	December	54

[Page 676]

To develop the series of forecasts for the data in [Table](#)

15.3, we will start with period 1 (January) and compute the forecast for period 2 (February) by using $\alpha = .30$. The formula for exponential smoothing also requires a forecast for period 1, which we do not have, so we will use the demand for period 1 as both demand and *the forecast for period 1*. Other ways to determine a starting forecast include averaging the first three or four periods and making a subjective estimate. Thus, the forecast for February is

$$F_2 = \alpha D_1 + (1 - \alpha)F_1 = (.30)(37) + (.70)(37) = 37 \text{ units}$$

The forecast for period 3 is computed similarly:

$$F_3 = \alpha D_2 + (1 - \alpha)F_2 = (.30)(40) + (.70)(37) = 37.9 \text{ units}$$

The remaining monthly forecasts are shown in [Table 15.4](#). The final forecast is for period 13, January, and is the forecast of interest to PM Computer Services:

$$F_{13} = \alpha D_{12} + (1 - \alpha)F_{12} = (.30)(54) + (.70)(50.84) = 51.79$$

units

[Table 15.4](#) also includes the forecast values by using $\alpha = .50$. Both exponential smoothing forecasts are shown in [Figure 15.3](#), together with the actual data.

Figure 15.3. Exponential smoothing forecasts

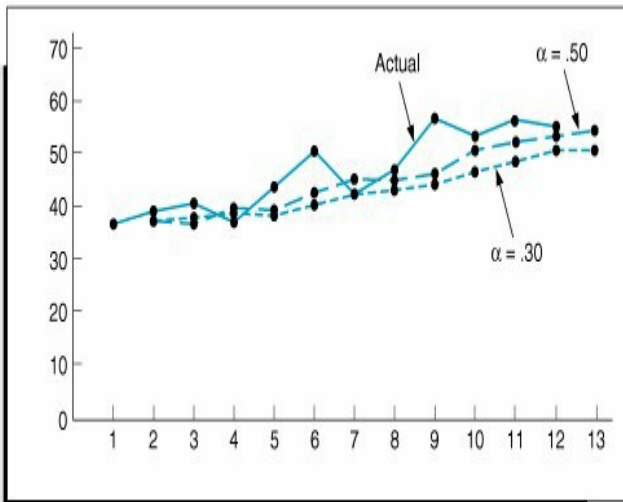


Table 15.4. Exponential smoothing forecasts, $\alpha = .30$ and $\alpha = .50$

Forecast, F_t
+ 1

Period	Month	Demand	$\alpha =$.30	$\alpha =$.50
---------------	--------------	---------------	--------------------------	--------------------------

1	January	37		
2	February	40	37.00	37.00
3	March	41	37.90	38.50
4	April	37	38.83	39.75
5	May	45	38.28	38.37
6	June	50	40.29	41.68

7	July	43	43.20	45.84
8	August	47	43.14	44.42
9	September	56	44.30	45.71
10	October	52	47.81	50.85
11	November	55	49.06	51.42
12	December	54	50.84	53.21
13	January		51.79	53.61

In [Figure 15.3](#), the forecast using the higher smoothing constant, $\alpha = .50$, reacts more strongly to changes in demand than does the forecast with $\alpha = .30$, although both smooth out the random fluctuations in the forecast. Notice that both forecasts lag behind the actual demand. For example, a pronounced downward change in demand in July is not reflected in the forecast until August. If these changes mark a change in trend (i.e., a long-term upward or downward movement) rather than just a

random fluctuation, then the forecast will always lag behind this trend. We can see a general upward trend in delivered orders throughout the year. Both forecasts tend to be consistently lower than the actual demand; that is, the forecasts lag behind the trend.

[Page 677]

Based on simple observation of the two forecasts in [Figure 15.3](#), $\alpha = .50$ seems to be the more accurate of the two, in the sense that it seems to

follow the actual data more closely. (Later in this chapter we will discuss several quantitative methods for determining forecast accuracy.) In general, when demand is relatively stable, without any trend, using a small value for α is more appropriate to simply smooth out the forecast. Alternatively, when actual demand displays an increasing (or decreasing) trend, as is the case in [Figure 15.3](#), a larger value of α is generally better. It will react more quickly to the more recent upward or

downward movements in the actual data. In some approaches to exponential smoothing, the accuracy of the forecast is monitored in terms of the difference between the actual values and the forecasted values. If these differences become larger, then α is changed (higher or lower) in an attempt to adapt the forecast to the actual data. However, the exponential smoothing forecast can also be adjusted for the effects of a trend.

As we noted with the moving

average forecast, the forecaster sometimes needs a forecast for more than one period into the future. In our PM Computer Services example, the final forecast computed was for 1 month, January. A forecast for 2 or 3 months could have been computed by grouping the demand data into the required number of periods and then using these values in the exponential smoothing computations. For example, if a 3-month forecast were needed, demand for January, February, and March could be summed

and used to compute the forecast for the next 3-month period, and so on, until a final 3-month forecast resulted. Alternatively, if a trend were present, the final period forecast could be used for an extended forecast by adjusting it by a trend factor.

Adjusted Exponential Smoothing

The **adjusted exponential smoothing** forecast consists of the exponential smoothing

forecast with a trend adjustment factor added to it. The formula for the adjusted forecast is

Adjusted exponential smoothing is the exponential smoothing forecast with an adjustment for a trend added to it.

$$AF_{t+1} = F_{t+1} + T_{t+1}$$

where

T = an exponentially smoothed trend factor

The trend factor is computed much the same as the exponentially smoothed forecast. It is, in effect, a forecast model for trend:

$$T_{t+1} = \beta(F_{t+1} - F_t) + (1 - \beta)T_t$$

where

T_t = the last period trend factor

β = a smoothing constant for trend

Like α , β is a value between zero and one. It reflects the weight given to the most recent trend data. Also like α , β is often determined subjectively, based on the judgment of the forecaster. A high β reflects trend changes more than a low β . It is not uncommon for β to equal α in this method.

The closer β is to one, the stronger a trend is reflected.

Notice that this formula for the trend factor reflects a weighted measure of the increase (or decrease) between the current forecast, F_{t+1} , and the previous forecast, F_t .

[Page 678]

As an example, PM Computer Services now wants to develop an adjusted exponentially smoothed forecast, using the same 12 months of demand shown in [Table 15.3](#). It will use

the exponentially smoothed forecast with $\alpha = .50$ computed in [Table 15.4](#) with a smoothing constant for trend, β , of .30.

The formula for the adjusted exponential smoothing forecast requires an initial value for T_t to start the computational process. This initial trend factor is most often an estimate determined subjectively or based on past data by the forecaster. In this case, because we have a relatively long sequence of demand data (i.e., 12 months), we will start with

the trend, T_t , equal to zero. By the time the forecast value of interest, F_{13} , is computed, we should have a relatively good value for the trend factor.

The adjusted forecast for February, AF_2 , is the same as the exponentially smoothed forecast because the trend computing factor will be zero (i.e., F_1 and F_2 are the same and $T_2 = 0$). Thus, we will compute the adjusted forecast for March, AF_3 , as follows, starting with the determination

of the trend factor, T_3 :

$$T_3 = \beta(F_3 - F_2) + (1 - \beta)T_2 = (.30)(38.5 - 37.0) + (.70)(0) = 0.45$$

and

$$AF_3 = F_3 + T_3 = 38.5 + 0.45 = 38.95$$

This adjusted forecast value for period 3 is shown in [Table 15.5](#), with all other adjusted forecast values for the 12-month period plus the forecast for period 13, computed as follows:

$$T_{13} = \beta(F_{13} - F_{12}) + (1 - \beta)T_{12} =$$

$$(.30)(53.61 - 53.21) + (.70)(1.77) = 1.36$$

and

$$AF_{13} = F_{13} + T_{13} = 53.61 + 1.36 = 54.96 \text{ units}$$

Table 15.5. Adjusted exponential smoothed forecast values

Period	Month	Demand	Forecast (F_{t+1})	Trend (T_{t+1})
1	January	37	37.00	
2	February	40	37.00	0.00

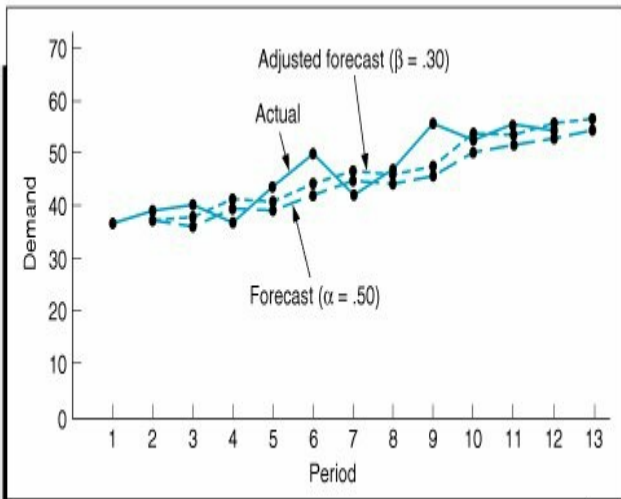
3	March	41	38.50	0.45
4	April	37	39.75	0.69
5	May	45	38.37	0.07
6	June	50	41.68	1.04
7	July	43	45.84	1.97
8	August	47	44.42	0.95
9	September	56	45.71	1.05
10	October	52	50.85	2.28

11	November	55	51.42	1.76
12	December	54	53.21	1.77
13	January		53.61	1.36

The adjusted exponentially smoothed forecast values shown in [Table 15.5](#) are compared with the exponentially smoothed forecast values and the actual data in [Figure 15.4](#).

Figure 15.4. Adjusted, exponentially smoothed forecast

(This item is displayed on page
679 in the print version)



Notice that the adjusted forecast is consistently higher than the exponentially

smoothed forecast and is thus more reflective of the generally increasing trend of the actual data. However, in general, the pattern, or degree of smoothing, is very similar for both forecasts.

Linear Trend Line

Linear regression is most often thought of as a causal method of forecasting in which a mathematical relationship is developed between demand and some other factor that

causes demand behavior. However, when demand displays an obvious trend over time, a least squares regression line, or **linear trend line**, can be used to forecast demand.

*A **linear trend line** is a linear regression model that relates demand to time.*

Management Science

Application: Seasonal

Forecasting at Dell

Dell is the largest computer company in the world. Its founder, Michael Dell, formed the company based on the idea of selling personal computers directly to customers and bypassing retailers; Dell Inc. sells computer systems directly to customers via phone and Internet orders. Once an order is received and processed, it is sent to an assembly plant in Austin, Texas, where Dell builds, tests, and packages a product within 8 hours.

Dell's direct sales model is dependent upon its arrangements with its suppliers, many of which are located in Southeast Asia, with shipping times

to Austin of 7 to 30 days. To compensate for these long lead times, Dell requires its suppliers to keep specified (i.e., target) inventory levels typically 10 days of inventory on hand at small warehouses, called "revolvers" (short for *revolving inventory*). In this type of vendor-managed inventory (VMI) arrangement, Dell's suppliers determine how much inventory to order and when to order to meet Dell's target inventory levels at the revolvers.

To help its suppliers make good ordering decisions, Dell shares its demand forecast with them on a monthly basis. Dell uses a 6-month rolling forecast developed by its marketing department that is updated weekly. Buyers receive weekly forecasts from commodity teams that break down the forecasts

for specific parts and components. These forecasts reflect product-specific trends and seasonality. For home computer systems, Christmas is the major sales period of the year. Other high-demand periods include the back-to-school season and the end of the year, when the government makes big purchases.



Source: R. Kapuscinski, R. Zhang, P. Carboneau, R. Moore, and B. Reeves, "Inventory Decisions in Dell's Supply Chain," *Interfaces* 34, no. 3 (May/June 2004): 191205.

[Page 680]

A linear trend line relates a dependent variable, which for our purposes is demand, to one independent variable, time, in the form of a linear equation, as follows:

$$y = a + bx$$

where

a = intercept (at period 0)

b = slope of the line

x = the time period

y = forecast for demand for period x

These parameters of the linear trend line can be calculated by using the least squares formulas for linear regression:

$$b = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2}$$

$$a = \bar{y} - b\bar{x}$$

where

n = number of periods

$$\bar{x} = \frac{\sum x}{n}$$

$$\bar{y} = \frac{\sum y}{n}$$

As an example, consider the demand data for PM Computer Services shown in [Table 15.3](#). They appear to follow an increasing linear trend. As such, the company wants to compute a linear trend line as an alternative to the

exponential smoothing and adjusted exponential smoothing forecasts shown in [Tables 15.4](#) and [15.5](#). The values that are required for the least squares calculations are shown in [Table 15.6](#).

Table 15.6. Least squares calculations

x (period)	y (demand)	xy	x^2
1	37	37	1
2	40	80	4

3	41	123	9
4	37	148	16
5	45	225	25
6	50	300	36
7	43	301	49
8	47	376	64
9	56	504	81
10	52	520	100
11	55	605	121

12	54	648	144
78	557	3,867	650

Using these values for $\sum x$ and $\sum y$ and the values from [Table 15.6](#), the parameters for the linear trend line are computed as follows:

$$\bar{x} = \frac{78}{12} = 6.5$$

$$\bar{y} = \frac{557}{12} = 46.42$$

$$b = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2} = \frac{3,867 - (12)(6.5)(46.42)}{650 - 12(6.5)^2} = 1.72$$

$$a = \bar{y} - b\bar{x} = 46.42 - (1.72)(6.5) = 35.2$$

[Page 681]

Therefore, the linear trend line is

$$y = 35.2 + 1.72x$$

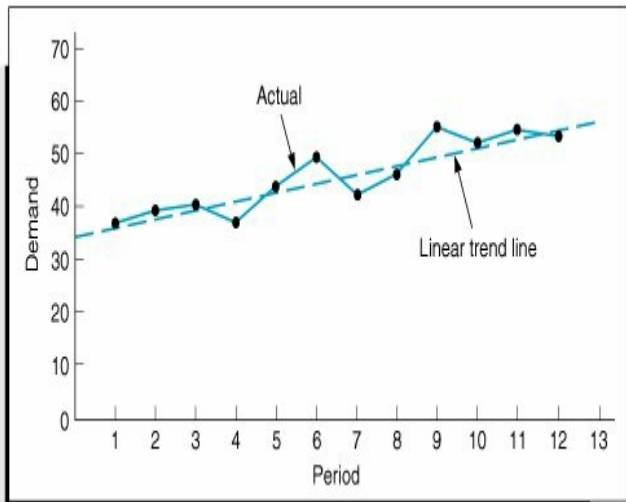
To calculate a forecast for period 13, $x = 13$ would be substituted in the linear trend line:

$$y = 35.2 + 1.72(13) = 57.56$$

[Figure 15.5](#) shows the linear trend line in comparison to the actual data. The trend line visibly appears to closely reflect the actual data (i.e., to be a "good fit") and would thus be a good forecast model for this problem. However, a disadvantage of the linear trend line is that it will not adjust to a change in the trend as the exponential smoothing forecast methods will (i.e., it is assumed that all future forecasts will follow a straight line). This limits the use of this

method to a shorter time frame in which the forecaster is relatively certain that the trend will not change.

Figure 15.5. Linear trend line



A linear trend line will not adjust to a change in trend as

*will exponential
smoothing.*

Seasonal Adjustments

As we mentioned at the beginning of this chapter, a seasonal pattern is a repetitive up-and-down movement in demand. Many demand items exhibit seasonal behavior. Clothing sales follow annual seasonal patterns, with demand

for warm clothes increasing in the fall and winter and declining in the spring and summer, as the demand for cooler clothing increases. Demand for many retail items including toys, sports equipment, clothing, electronic appliances, hams, turkeys, wine, and fruit increases during the Christmas season. Greeting card demand increases in conjunction with various special days, such as Valentine's Day and Mother's Day. Seasonal patterns can also occur on a monthly, weekly, or even daily

basis. Some restaurants have higher demand in the evening than at lunch, or on weekends as opposed to weekdays. Traffic and sales picks up at shopping malls on Friday and Saturday.

There are several methods available for reflecting seasonal patterns in a time series forecast. We will describe one of the simpler methods using a **seasonal factor**, which is a numerical value that is multiplied by the normal forecast to get a seasonally adjusted forecast.

*It is possible to adjust for seasonality by multiplying the normal forecast by a **seasonal factor**.*

One method for developing a demand for seasonal factors is dividing the actual demand for each seasonal period by the total annual demand, according to the following formula:

$$S_i = \frac{D_i}{\sum D}$$

The resulting seasonal factors between zero and one are, in effect, the portion of total annual demand assigned to each season. These seasonal factors are thus multiplied by the annual forecasted demand to yield seasonally adjusted forecasts for each period.

As an example, Wishbone Farms is a company that raises turkeys, which it sells to a meat-processing company throughout the year. However,

the peak season obviously occurs during the fourth quarter of the year, October to December. Wishbone Farms has experienced a demand for turkeys for the past 3 years as shown in [Table 15.7](#).

Table 15.7. Demand for turkeys at Wishbone Farms

Year	Demand (1,000s)				Total
	QUARTER 1	QUARTER 2	QUARTER 3	QUARTER 4	
2003	12.6	8.6	6.3	17.5	45

2004	14.1	10.3	7.5	18.2	50
2005	15.3	10.6	8.1	19.6	53
Total	42.0	29.5	21.9	55.3	14

Because we have 3 years of demand data, we can compute the seasonal factors by dividing total quarterly demand for the 3 years by total demand across all 3 years:

$$S_1 = \frac{D_1}{\sum D} = \frac{42.0}{148.7} = 0.28$$

$$S_2 = \frac{D_2}{\sum D} = \frac{29.5}{148.7} = 0.20$$

$$S_3 = \frac{D_3}{\sum D} = \frac{21.9}{148.7} = 0.15$$

$$S_4 = \frac{D_4}{\sum D} = \frac{55.3}{148.7} = 0.37$$

Next, we want to multiply the forecasted demand for the next year, 2006, by each of the seasonal factors to get the forecasted demand for each quarter. However, to accomplish this, we need a demand forecast for 2006. In this case, because the demand data in [Table 15.7](#) seem to exhibit a

generally increasing trend, we compute a linear trend line for the 3 years of data in [Table 15.7](#) to use as a rough forecast estimate:

$$y = 40.97 + 4.30x = 40.97 + 4.30(4) = 58.17$$

Thus, the forecast for 2006 is 58.17, or 58,170 turkeys.

Using this annual forecast of demand, the seasonally adjusted forecasts, SF_i , for 2006 are as follows:

$$SF_1 = (S_1)(F_5) = (.28)(58.17)$$

$$= 16.28$$

$$SF_2 = (S_2)(F_5) = (.20)(58.17) \\ = 11.63$$

$$SF_3 = (S_3)(F_5) = (.15)(58.17) \\ = 8.73$$

$$SF_4 = (S_4)(F_5) = (.37)(58.17) \\ = 21.53$$

Comparing these quarterly forecasts with the actual demand values in [Table 15.7](#) shows them to be relatively good forecast estimates, reflecting both the seasonal variations in the data and the

general upward trend.

[Page 683]

Management Science

Application: Forecasting

Service Calls at FedEx

FedEx is the world's largest express delivery company, with annual revenues of \$14 billion. With a workforce of 145,000 employees, 600 planes, and 42,000 ground vehicles, it delivers more than 3 million packages per day to 210 countries. To support its global delivery network, FedEx has 51 customer service call centers, including 16 in the United States and 35 overseas. The customer service centers in the United States handle about 500,000 calls daily for requests for pickup, drop-off locations, package tracking, package rating, and so on. FedEx's service

level goal is to answer 90% of all calls within 20 seconds, and it has some of the highest customer satisfaction levels in the industry.

FedEx has three major service call networks: domestic for calls related to packages sent and delivered within the United States; international for packages sent overseas from the United States; and freight for calls related to packages over 150 pounds. For each of these call networks, FedEx uses four different forecasts, encompassing different forecasting horizons: strategic plan, business plan, tactical forecast, and operational forecast. The strategic plan is for a 5-year horizon and includes forecasts for the number of calls to service representatives, average call handling time, staffing requirements, and technology calls. This plan is revised and updated once

a year. The business plan includes the same forecasts as the strategic plan, except it is for a 1-year horizon, with updates and revisions as management requires. The tactical forecast is a daily forecast of the number of calls offered to service representatives, provided once per month and rolled over for 6 months. The goal for the tactical forecast is to be within a 2% error per month and within a 4% error on a daily basis. The operational forecast is a daily forecast of the number of calls offered to service representatives and the average call handle time per half hour. It is prepared on a weekly basis and forecasted 1 month in advance.

The forecasting models used include exponential smoothing, linear regression, and time series, with adjustments for trend and

seasonality. For example, the tactical forecast employs a time series model based on 8 years of historical data, with adjustments for seasonalities that is, month, week, day, and day of month. Trend adjustments are also used.



Source: W. Xu, "Long Range Planning

for Call Centers at FedEx," *Journal of Business Forecasting* 18, no. 4 (Winter 1999-2000): 7-11.

Forecast Accuracy

It is not probable that a forecast will be completely accurate; forecasts will always deviate from the actual demand. This difference between the forecast and the actual is referred to as the **forecast error**. Although some amount of forecast error is inevitable, the objective of forecasting is for the error to be as slight as possible. Of course, if the degree of error is

not small, this may indicate either that the forecasting technique being used is the wrong one or that the technique needs to be adjusted by changing its parameters (for example, α in the exponential smoothing forecast).

*The **forecast error** is the difference between the forecast and actual demand.*

There are a variety of different measures of forecast error, and in this section we discuss several of the most popular ones: mean absolute deviation (*MAD*), mean absolute percent deviation (*MAPD*), cumulative error (*E*), average error or bias ($\bar{E}$), and mean squared error (*MSE*).

Mean Absolute Deviation

The **mean absolute deviation**

(MAD) is one of the most popular and simplest-to-use measures of forecast error.

MAD is an average of the difference between the forecast and actual demand, as computed by the following formula:

$$MAD = \frac{\sum |D_t - F_t|}{n}$$

where

t = the period number

D_t = demand in period t

F_t = the forecast for period t

n = the total number of periods

$||$ = the absolute value

MAD is the average,
absolute difference
between the forecast
and the demand.

In our examples for PM Computer Services, forecasts were developed using exponential smoothing (with $\alpha = .30$ and with $\alpha = .50$), adjusted exponential smoothing ($\alpha = .50, \beta = .30$), and a linear trend line for the demand data. The company wants to compare the accuracy of these different forecasts by using *MAD*.

We will compute *MAD* for all four forecasts; however, we will only present the computational

detail for the exponential smoothing forecast with $\alpha = .30$. [Table 15.8](#) shows the values necessary to compute *MAD* for the exponential smoothing forecast.

Table 15.8. Computational values for MAD and error

Period	Demand, D_t	Forecast, F_t ($\alpha = .30$)	Error ($D_t - F_t$)	$ D_t - F_t $	Er (F
1	37	37.00			
2	40	37.00	3.00	3.00	9

3	41	37.90	3.10	3.10	9
4	37	38.83	1.83	1.83	3
5	45	38.28	6.72	6.72	45
6	50	40.29	9.71	9.71	94
7	43	43.20	0.20	0.20	0
8	47	43.14	3.86	3.86	14
9	56	44.30	11.70	11.70	130
10	52	47.81	4.19	4.19	17

11	55	49.06	5.94	5.94	35
12	54	50.84	3.16	3.16	9
	520 ^[*]		49.31	53.41	37

[*] Because the computation of MAD will be based on the 11 periods 2 through 12, $\sum D_t = 520$, excluding the demand value of 37.

Using the data in [Table 15.8](#), MAD is computed as

$$MAD = \frac{\sum |D_t - F_t|}{n} = \frac{53.41}{11} = 4.85$$

[Page 685]

In general, the smaller the value of *MAD*, the more accurate the forecast, although, viewed alone, *MAD* is difficult to access. In this example, the data values were relatively small, and the *MAD* value of 4.85 should be judged accordingly. Overall, it would seem to be a "low" value (i.e., the forecast appears to be relatively accurate). However, if

the magnitude of the data values were in the thousands or millions, then a *MAD* value of a similar magnitude might not be bad either. The point is that you cannot compare a *MAD* value of 4.85 with a *MAD* value of 485 and say the former is good and the latter is bad; they depend to a certain extent on the relative magnitude of the data.

The lower the value of MAD relative to the magnitude of the data, the more

*accurate the
forecast.*

One benefit of *MAD* is being able to compare the accuracy of several different forecasting techniques, as we are doing in this example. The *MAD* values for the remaining forecasts are

- Exponential smoothing ($\alpha = .50$): *MAD* = 4.04
- Adjusted exponential

smoothing ($\alpha = .50$, $\beta = .30$): $MAD = 3.81$

- Linear trend line: $MAD = 2.29$

When we compare all four forecasts, the linear trend line has the lowest MAD value, of 2.29. It would seem to be the most accurate, although it does not appear to be significantly better than the adjusted exponential smoothing forecast. Furthermore, we can deduce from these MAD values that increasing α from .30 to

.50 enhanced the accuracy of the exponentially smoothed forecast. The adjusted forecast is even more accurate.

A variation of MAD is the **mean absolute percent deviation (MAPD)**. It measures the absolute error as a percentage of demand rather than per period. As a result, it eliminates the problem of interpreting the measure of accuracy relative to the magnitude of the demand and forecast values, as MAD does. $MAPD$ is computed according to the following formula:

$$MAPD = \frac{\sum |D_t - F_t|}{\sum D_t}$$

MAPD is absolute error as a percentage of demand.

Using the data from [Table 15.8](#) for the exponential smoothing forecast ($\alpha = .30$) for PM Computer Services, *MAPD* is computed as

$$MAPD = \frac{53.41}{520} = .103, \text{ or } 10.3\%$$

A lower *MAPD* implies a more accurate forecast. The *MAPD* values for our other three forecasts are

- Exponential smoothing ($\alpha = .50$): *MAPD* = 8.5%
- Adjusted exponential smoothing ($\alpha = .50, \beta = .30$): *MAPD* = 8.1%
- Linear trend line: *MAPD* = 4.9%

Cumulative Error

Cumulative error is computed simply by summing the forecast errors, as shown in the following formula:

$$E = \sum e_t$$

Cumulative error is the sum of the forecast errors.

A relatively large positive value indicates that the forecast is probably consistently lower

than the actual demand or is biased low. A large negative value implies that the forecast is consistently higher than actual demand or is biased high. Also, when the errors for each period are scrutinized and there appears to be a preponderance of positive values, this shows that the forecast is consistently less than actual, and vice versa.

[Page 686]

The cumulative error for the exponential smoothing forecast

($\alpha = .30$) for PM Computer Services can be read directly from [Table 15.8](#); it is simply the sum of the values in the "Error" column:

$$E = \sum e_t = 49.31$$

This relatively large-value positive error for cumulative error and the fact that the individual errors for each period in [Table 15.8](#) are positive indicate that this forecast is frequently below the actual demand. A quick glance back at the plot of the exponential smoothing ($\alpha =$

.30) forecast in [Figure 15.3](#) visually verifies this result.

The cumulative errors for the other forecasts are

- Exponential smoothing ($\alpha = .50$): $E = 33.21$
- Adjusted exponential smoothing ($\alpha = .50, \beta = .30$): $E = 21.14$

We did not show the cumulative error for the linear trend line. E will always be near zero for the linear trend

line; thus, it is not a good measure on which to base comparisons with other forecast methods.

A measure closely related to cumulative error is the **average error**. It is computed by averaging the cumulative error over the number of time periods:

$$\bar{E} = \frac{\sum e_t}{n}$$

Average error is the per-period average of cumulative error.

For example, the average error for the exponential smoothing forecast ($\alpha = .30$) is computed as follows (notice that the value of 11 was used for n because we used actual demand for the first-period forecast, resulting in no error, i.e., $D_1 = F_1 = 37$):

$$\bar{E} = \frac{49.31}{11} = 4.48$$

The average error is interpreted similarly to the cumulative error. A positive

value indicates low bias, and a negative value indicates high bias. A value close to zero implies a lack of bias.

Large $+\bar{E}$ indicates that a forecast is biased low; large $-\bar{E}$ indicates that a forecast is biased high.

Another measure of forecast accuracy related to error is

mean squared error (MSE).

With *MSE*, each individual error value is squared, and then these values are summed and averaged. The last column in [Table 15.8](#) shows the sum of the squared forecast errors (i.e., 376.04) for the PM Computer example forecast ($\alpha = 0.30$). The *MSE* is computed as

$$MSE = \frac{376.04}{11} = 34.18$$

As with other measures of forecast accuracy, the smaller the *MSE*, the better.

Table 15.9 summarizes the measures of forecast accuracy we have discussed in this section for the four example forecasts we developed in the previous section for PM Computer Services. The results are consistent for all four forecasts, indicating that for the PM Computer Services example data, a larger value of a is preferable for the exponential smoothing forecast. The adjusted forecast is more accurate than the exponential smoothing forecasts, and the linear trend

is more accurate than all the others. Although these results are example specific, they do indicate how the different forecast measures for accuracy can be used to adjust a forecasting method or select the best method.

[Page 687]

Table 15.9. Comparison of forecasts for PM Computer Services

Forecast	<i>MAD</i>	<i>MAPD</i>	(%)	<i>E</i>	$\bar{E}$	<i>MS</i>
-----------------	-------------------	--------------------	------------	-----------------	-----------------------------	------------------

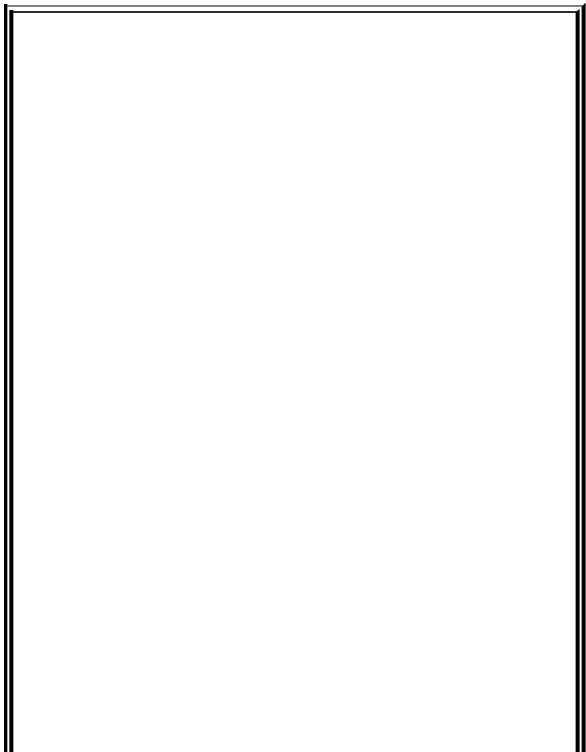
Exponential smoothing ($\alpha = .30$)	4.85	10.3	49.31	4.48	34.1
---	------	------	-------	------	------

Exponential smoothing ($\alpha = .50$)	4.04	8.5	33.21	3.02	24.1
---	------	-----	-------	------	------

Adjusted exponential smoothing	3.81	8.1	21.14	1.92	21.1
--------------------------------	------	-----	-------	------	------

($\alpha = .50, \beta = .30$)

Linear trend line	2.29	4.9			8.6
-------------------	------	-----	--	--	-----



Management Science

Application: Demand

Forecasting at National Car Rental

In 1993, due to large financial losses, National Car Rental was threatened with liquidation by its parent company, General Motors. The car rental industry had been in turmoil since the 1980s, when automobile manufacturers purchased almost all the major rental car companies and then flooded them with their excess auto production. The excess supply of cheap cars led to low prices, especially during periods of low demand, which eroded profits. When in the 1990s demand for cars increased, thus raising the cost of

cars to the rental car companies, profits fell further.

Car rental companies rely largely on corporate clients that have large numbers of employees who travel. Rental car demand peaks at midweek, forcing car rental companies to turn away customers. They also have a large excess fleet that is idle on weekends, which, in turn, allows occasional renters, such as leisure customers, to book multiple reservations with no prepayment or cancellation or no-show penalty. This can result in no-shows that exceed 50% of reservations.

Prior to 1993, no demand forecasts had existed for National. To save itself from liquidation, National Car Rental committed over \$10 million to design and build a revenue

management system that would manage capacity, pricing, and reservations. The system encompasses a number of management science and analytic models. A critical component of the system was the development of a comprehensive set of demand and revenue forecasts. Demand forecasts are made for the length of the rental period and the number of cars in use on a specific date. These demand forecasts are based on a combination of long- and short-term forecasts. The long-term forecast is a time series model with seasonal factors. The system generates forecasts for all days within a 60-day booking period. The system also generates forecasts for daily non-reservation customers and for no-shows.

The complete revenue management

system produced immediate results, improving revenues by \$56 million in the first year alone. Rather than liquidate, General Motors sold National Car Rental in 1995 for \$1.2 billion.



Source: M. K. Geraghty and E. Johnson, "Revenue Management

Saves National Car Rental,"
Interfaces 27, no. 1
(JanuaryFebruary 1997): 10727.

Time Series Forecasting Using Excel

All the time series forecasts we have presented can also be developed with Excel spreadsheets. [Exhibit 15.1](#) shows an Excel spreadsheet set up to compute the exponentially smoothed forecast and adjusted exponentially smoothed forecast for the PM Computer Services example summarized in [Table 15.5](#). Notice that the

formula for computing the trend factor in cell D10 is shown on the formula bar on the top of the spreadsheet. The adjusted forecast in column E is computed by typing the formula **=C9 +D9** in cell E9 and copying it to cells **E10:E20** (using the "Copy" and "Paste" options that appear after clicking the right mouse button).

Exhibit 15.1.

=G21/11

The exponential smoothing forecast can also be developed directly from Excel without "customizing" a spreadsheet and entering our own formulas, as we did in [Exhibit 15.1](#). From the "Tools" menu at the top of the spreadsheet, select the "Data Analysis" option. (If your "Tools" menu does not include this menu item, you should add it by accessing the "Add-ins" option from the "Tools" menu or by loading from the original Excel or office software.)

[Exhibit 15.2](#) shows the Data Analysis window and the "Exponential Smoothing" menu item you should select and then click on "OK." The resulting Exponential Smoothing window is shown in [Exhibit 15.3](#). The input range includes the demand values in column B in [Exhibit 15.1](#), the damping factor is alpha (α), which in this case is 0.5, and the output should be placed in column C in [Exhibit 15.1](#). Clicking on "OK" will result in the same forecast values in column C of [Exhibit 15.1](#) that

we computed using our own exponential smoothing formula. Note that the "Data Analysis" group of analysis tools does not have an adjusted exponential smoothing selection; that is the reason we developed our own customized spreadsheet in [Exhibit 15.1](#). The "Data Analysis" tools also have a "Moving Average" menu item from which a moving average forecast can be computed.

Exhibit 15.2.

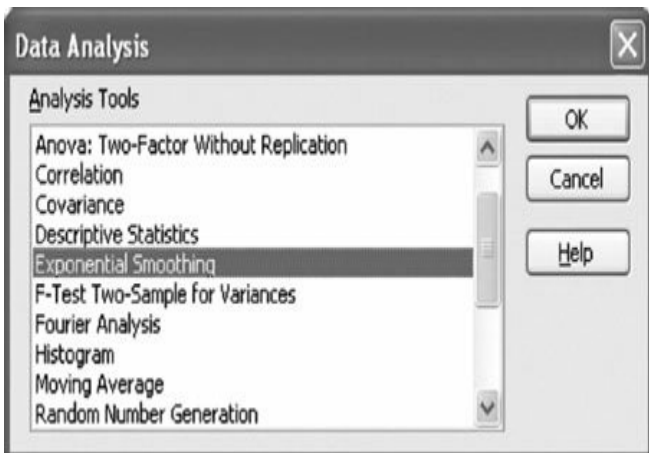


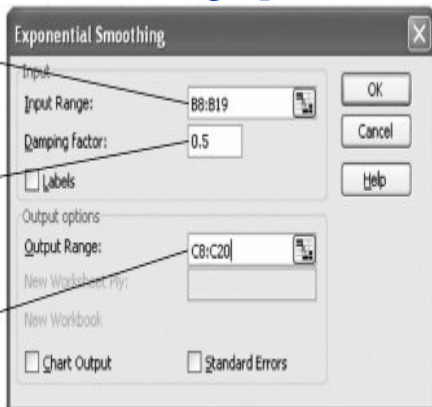
Exhibit 15.3.

[\[View full size image\]](#)

Demand values

$\alpha = 0.5$

Cells in which the
forecasted values will
be placed



The image shows the 'Exponential Smoothing' dialog box in Microsoft Excel. The dialog box has a title bar with a close button (X). It contains several input fields and checkboxes. The 'Input' section has an 'Input Range' field set to 'B8:B19' and a 'Damping factor' field set to '0.5'. There is an unchecked checkbox for 'Labels'. The 'Output options' section has an 'Output Range' field set to 'C8:C20'. Below this are fields for 'New Worksheet Ply:' and 'New Workbook:'. At the bottom, there are two unchecked checkboxes: 'Chart Output' and 'Standard Errors'. On the right side of the dialog box, there are three buttons: 'OK', 'Cancel', and 'Help'. Three callout boxes with lines pointing to specific fields provide additional context: 'Demand values' points to the 'Input Range' field, ' $\alpha = 0.5$ ' points to the 'Damping factor' field, and 'Cells in which the forecasted values will be placed' points to the 'Output Range' field.

Exponential Smoothing

Input

Input Range: B8:B19

Damping factor: 0.5

☐ Labels

Output options

Output Range: C8:C20

New Worksheet Ply:

New Workbook:

☐ Chart Output ☐ Standard Errors

OK

Cancel

Help

Excel can also be used to develop seasonally adjusted forecasts. [Exhibit 15.4](#) shows an Excel spreadsheet set up to

develop the seasonally adjusted forecast for the demand for turkeys at Wishbone Farms. You will notice that the seasonally adjusted forecasts for each quarter are slightly different from the forecasts computed manually (e.g., SF_1 with Excel equals 16.43, whereas SF_1 computed manually equals 16.28). This is due to the rounding done in the manual computations.

Exhibit 15.4.

[\[View full size image\]](#)

[illegible]

[Page 690]

Computing the Exponential Smoothing

Forecast with Excel QM

In [Chapter 1](#) we introduced Excel QM, a set of spreadsheet macros that we have also used in several other chapters. Excel QM includes a spreadsheet macro for exponential smoothing. After it is activated, the Excel QM menu is accessed by clicking on "QM" on the menu bar at the top of the spreadsheet. Clicking on "Forecasting" from this menu results in a Spreadsheet Initialization window, in which you enter the problem title and

the number of periods of past demand. Clicking on "OK" will result in the spreadsheet shown in [Exhibit 15.5](#). Initially, this spreadsheet will have example values in the (shaded) data cells, **B7** and **B10:B21**. Thus, the first step in using this macro is to type in the data for our PM Computer Services problem: alpha (α) = 0.50 in cell B7 and our demand values in cells **B10:B21**. The forecast results are computed automatically from formulas already embedded in the spreadsheet. The resulting

next-period forecast for January is shown in cell B26, and the monthly forecasts are shown in cells **D10:D21**. (Do not confuse this exponentially smoothed forecast and the values for *MAD* and the like with the *adjusted* exponentially smoothed forecast and *MAD* value in [Exhibit 15.1](#).)

Exhibit 15.5.

[\[View full size image\]](#)

Click "QM" to access forecasting macros.

Microsoft Excel - Exhibit15.5.xls

File Edit Insert Format Tools Data QM Window Help

Formulas & Functions

fx

Q1 = C21 + BE9 * (Q21 - C21)

PM Computer Services

Forecasting Exponential smoothing

Enter alpha (between 0 and 1) then enter the past demands in the shaded area.

Alpha 0.5

Period	Demand	Forecast	Error	Absolute	Squared
Month 1	37	37			
Month 2	40	37	3	3	9
Month 3	41	39.5	1.5	1.5	2.25
Month 4	37	39.75	-2.75	2.75	7.5625
Month 5	45	39.375	5.625	5.625	31.690625
Month 6	50	41.5625	8.4375	8.4375	71.1875
Month 7	43	45.84375	-2.84375	2.84375	8.088906
Month 8	47	44.42188	2.57813	2.57813	6.64673
Month 9	50	45.71094	4.28906	4.28906	18.39681
Month 10	50	50.85547	-0.85547	0.85547	0.73183
Month 11	55	51.42773	3.57227	3.57227	12.76108
Month 12	54	53.21387	0.78613	0.78613	0.61800
Total		32.21387	44.40137	271.08827	
Average		3.01944	4.03549	24.54439	
		Bias	MAD	MSE	
				SE	

Next period 53.61

Input problem data in cells B7 and B10:B21.

Time Series Forecasting Using QM for Windows

QM for Windows has the capability to perform forecasting for all the time series methods we have described so far. QM for Windows has modules for moving averages, exponential smoothing and adjusted exponential smoothing, and linear regression.

To demonstrate the forecasting

capability of QM for Windows, we will generate the exponential smoothing ($\alpha = .30$) forecast computed manually for PM Computer Services ([Table 15.4](#)). The solution output is shown in [Exhibit 15.6](#).

Exhibit 15.6.

(This item is displayed on page 691 in the print version)

[\[View full size image\]](#)

Details and Error Analysis

PM Computer Services Example Solution

	Demand(y)	Forecast	Error	Error	Error^2	Pct Error
1	37					
2	40	37	3	3	9	.075
3	41	37.9	3.1	3.1	9.61	.0756
4	37	38.83	-1.83	1.83	3.3489	.0495
5	45	38.281	6.719	6.719	45.1449	.1493
6	50	40.2967	9.7033	9.7033	94.154	.1941
7	43	43.2077	-.2077	.2077	.0431	.0048
8	47	43.1454	3.8546	3.8546	14.8581	.082
9	56	44.3018	11.6982	11.6982	136.8486	.2089
10	52	47.8112	4.1888	4.1888	17.5457	.0806
11	55	49.0679	5.9321	5.9321	35.1902	.1079
12	54	50.8475	3.1525	3.1525	9.9382	.0584
TOTALS	557		49.3108	53.3862	375.6818	1.086
AVERAGE	46.4167		4.4828	4.8533	34.1529	.0987
Next period forecast		51.7933	(Bias)	(MAD)	(MSE)	(MAPE)
				Std err	6.4608	

Notice that the solution

summary includes the forecast per period and the forecast for the next period (13), as well as three measures of forecast accuracy: average error (bias), mean absolute deviation (*MAD*), and mean squared error (*MSE*).

[Page 691]

The least squares module or the simple linear regression module in QM for Windows can be used to develop a linear trend line forecast. Using the least squares module, the

solution summary for the linear trend line forecast we developed for PM Computer Services is shown in [Exhibit 15.7](#).

Exhibit 15.7.

[\[View full size image\]](#)

Forecasting Results



PM Computer Services Example Summary

Measure	Value	Future Period	Forecast
Error Measures		13	57.6212
Bias (Mean Error)	0	14	59.345
MAD (Mean Absolute Deviation)	2.2892	15	61.0688
MSE (Mean Squared Error)	8.6672	16	62.7925
Standard Error (denom= $n-2=10$)	3.225	17	64.5163
MAPE (Mean Absolute Percent Error)	.0499	18	66.2401
Regression line		19	67.9639
Demand(y) = 35.21213		20	69.6876
+ 1.7238 * Time(x)		21	71.4114
Statistics		22	73.1352
Correlation coefficient	.8963	23	74.859
Coefficient of determination (r^2)	.8034	24	76.5827
		25	78.3065
		26	80.0303

Regression Methods

The time series techniques of exponential smoothing and moving average relate a single variable being forecast (such as demand) to *time*. In contrast, *regression* is a forecasting technique that measures the relationship of one variable to one or more other variables. For example, if we know that something has caused product demand to behave in a certain way in the past, we might like

to identify that relationship. If the same thing happens again in the future, we can then predict what demand will be. For example, there is a well-known relationship between increased demand in new housing and lower interest rates. Correspondingly, a whole myriad of building products and services display increased demand if new housing starts increase. Similarly, an increase in sales of DVD players results in an increase in demand for DVDs.

The simplest form of regression is linear regression, which you will recall we used previously to develop a linear trend line for forecasting. In the following section we will show how to develop a regression model for variables related to items other than time.

Linear Regression

Simple **linear regression** relates one dependent variable

to one independent variable in the form of a linear equation:

The diagram shows the linear equation $y = a + bx$ with four arrows pointing to its components: 'dependent variable' points to y , 'intercept' points to a , 'slope' points to b , and 'independent variable' points to x .

Linear regression
relates demand
(dependent variable)
to an independent
variable.

To develop the linear equation,

the slope, b , and the intercept, a , must first be computed by using the following least squares formulas:

$$a = \bar{y} - b\bar{x}$$

$$b = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2}$$

where

$$\bar{x} = \frac{\sum x}{n} = \text{mean of the } x \text{ data}$$

$$\bar{y} = \frac{\sum y}{n} = \text{mean of the } y \text{ data}$$

We will consider regression within the context of an example. The State University

athletic department wants to develop its budget for the coming year, using a forecast for football attendance. Football attendance accounts for the largest portion of its revenues, and the athletic director believes attendance is directly related to the number of wins by the team. The business manager has accumulated total annual attendance figures for the past 8 years:

Wins	Attendance
4	36,300

6	40,100
---	--------

6	41,200
---	--------

8	53,000
---	--------

6	44,000
---	--------

7	45,600
---	--------

5	39,000
---	--------

7	47,500
---	--------

Given the number of returning starters and the strength of the schedule, the athletic director believes the team will win at least seven games next year. He wants to develop a simple regression equation for these data to forecast attendance for this level of success.

[Page 693]

The computations necessary to compute a and b , using the least squares formulas, are summarized in [Table 15.10](#). (Note that the magnitude of y

has been reduced to make manual computation easier.)

Table 15.10. Least squares computations

x (wins)	y (attendance, 1,000s)	xy	x^2
4	36.3	145.2	16
6	40.1	240.6	36
6	41.2	247.2	36
8	53.0	424.0	64

6	44.0	264.0	36
7	45.6	319.2	49
5	39.0	195.0	25
7	47.5	332.5	49
49	346.7	2,167.7	311

$$\bar{x} = \frac{49}{8} = 6.125$$

$$\bar{y} = \frac{346.9}{8} = 43.34$$

$$b = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2}$$
$$= \frac{(2,167.7) - (8)(6.125)(43.34)}{(311) - (8)(6.125)^2} = 4.06$$

$$a = \bar{y} - b\bar{x} = 43.34 - (4.06)(6.125) = 18.46$$

Substituting these values for a and b into the linear equation line, we have

$$y = 18.46 + 4.06x$$

Thus, for $x = 7$ (wins), the forecast for attendance is

$$y = 18.46 + 4.06(7) = 46.88$$

or 46,880

The data points with the regression line are shown in [Figure 15.6](#). Observing the regression line relative to the data points, it would appear that the data follow a distinct upward linear trend, which would indicate that the forecast should be relatively accurate. In fact, the *MAD* value for this forecasting model is 1.41, which suggests an accurate forecast.

Correlation

Correlation in a linear regression equation is a measure of the strength of the relationship between the independent and dependent variables. The formula for the correlation coefficient is

Correlation is a measure of the strength of the relationship between independent and dependent variables.

$$r = \frac{n \sum xy - \sum x \sum y}{\sqrt{[n \sum x^2 - (\sum x)^2][n \sum y^2 - (\sum y)^2]}}$$

The value of r varies between -1.00 and +1.00, with a value of ± 1.00 indicating a strong linear relationship between the variables. If $r = 1.00$, then an increase in the independent variable will result in a corresponding linear increase in the dependent variable. If $r = -1.00$, an increase in the independent variable will result in a linear decrease in the

dependent variable. A value of r near zero implies that there is little or no linear relationship between variables.

[Page 694]

Figure 15.6. Linear regression line

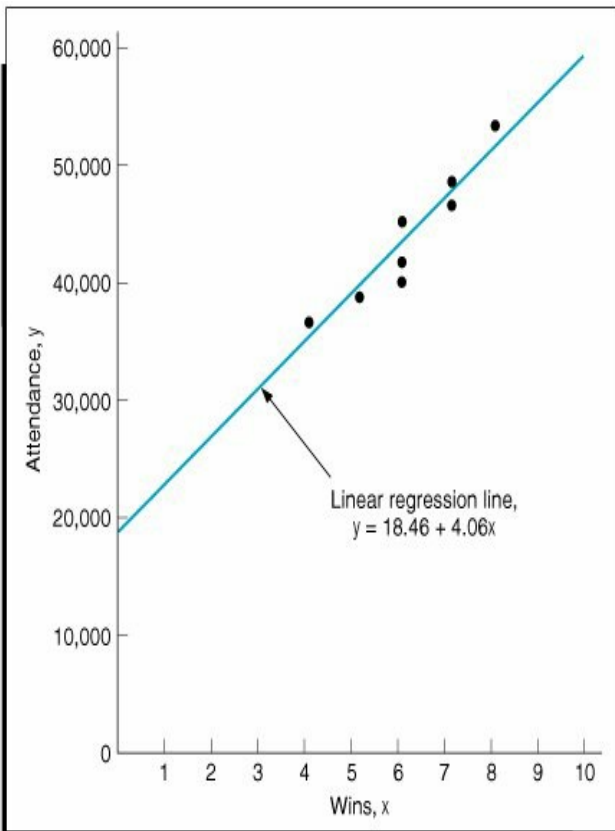
Attendance, y

60,000
50,000
40,000
30,000
20,000
10,000
0

1 2 3 4 5 6 7 8 9 10

Wins, x

Linear regression line,
 $y = 18.46 + 4.06x$



We can determine the correlation coefficient for the linear regression equation determined in our State University example by substituting most of the terms calculated for the least squares formula (except for Σy^2) into the formula for r :

$$r = \frac{(8)(2,167.7) - (49)(346.7)}{\sqrt{[(8)(311) - (49)^2][(8)(15,224.7) - (346.7)^2]}}$$
$$= .948$$

This value for the correlation

coefficient is very close to one, indicating a strong linear relationship between the number of wins and home attendance.

Another measure of the strength of the relationship between the variables in a linear regression equation is the **coefficient of determination**. It is computed by simply squaring the value of r . It indicates the percentage of the variation in the dependent variable that is a result of the behavior of the independent variable. For our example, $r =$

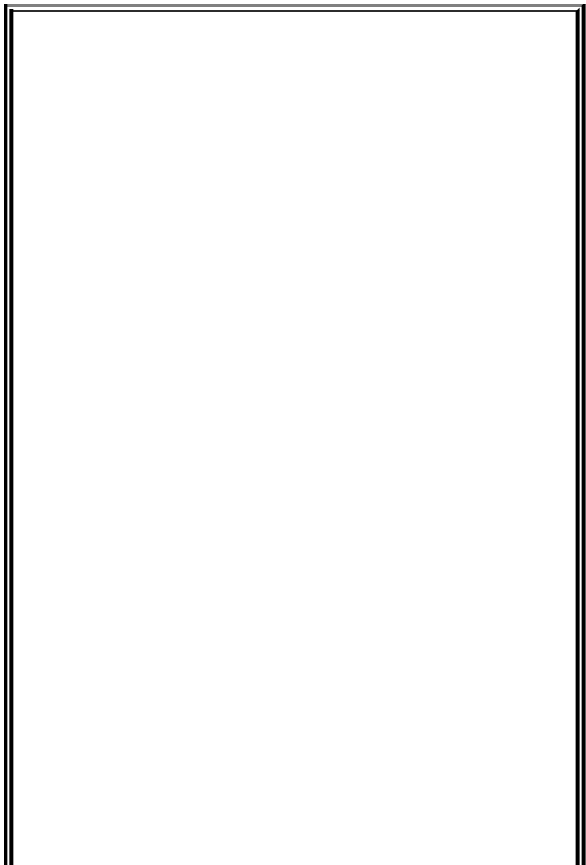
.948; thus, the coefficient of determination is

*The **coefficient of determination** is the percentage of the variation in the dependent variable that results from the independent variable.*

$$\begin{aligned} r^2 &= (.948)^2 \\ &= .899 \end{aligned}$$

This value for the coefficient of

determination means that 89.9% of the amount of variation in attendance can be attributed to the number of wins by the team (with the remaining 10.1% due to other unexplained factors, such as weather, a good or poor start, publicity, etc.). A value of one (or 100%) would indicate that attendance totally depends on wins. However, because 10.1% of the variation is a result of other factors, some amount of forecast error can be expected.



Management Science

Application: Forecasting Daily Demand in the Gas Industry

Vermont Gas Systems is a natural gas utility that serves approximately 26,000 business, industrial, and residential customers in 13 towns and cities in northwestern Vermont. Demand forecasts are a critical part of Vermont Gas Systems's supply chain that stretches across Canada from suppliers in western Canada to storage facilities along the TransCanada pipeline to Vermont Gas Systems's pipeline. Gas orders must be specified to suppliers at least 24 hours in advance. Vermont Gas Systems has storage capacity

available for a buffer inventory of only 1 hour of gas use, so an accurate daily forecast of gas demand is essential.

Vermont Gas Systems uses regression to forecast daily gas demand. In its forecast models, gas demand is the dependent variable, and factors such as weather information and industrial customer demand are independent variables. During the winter, customers use more gas for heat, making an accurate weather forecast a very important factor. Detailed 3-day weather forecasts are provided to Vermont Gas Systems five times per day from a weather forecasting service. Individual regression forecasts are developed for 24 large-use industrial and municipal customers, such as factories, hospitals, and schools. End-use

demand is the total potential capacity of all natural gas appliances in the system. It changes daily, as new customers move into a new house, apartment, or business, adding new appliances or equipment to the system. The utility uses only the most recent 30 days of demand data in developing its forecast models, and it updates the models on a weekly basis. Vermont Gas Systems interprets the results of the forecast model and supplements them with its individual knowledge of the supply chain distribution system and customer usage to develop an overall, accurate daily forecast of gas demand.

Columbia Gas Company in Ohio, a subsidiary of Virginia-based Columbia Energy Group, is the largest natural gas utility in Ohio, with nearly 1.3 million customers in more than 1,000

communities. Columbia employs two types of daily forecast: the design day forecast and the daily operational forecast. The design day forecast is used to determine the amount of gas supply, transportation capacity, and storage capacity that Columbia requires to meet its customer needs. It is very important that the design day forecast be accurate; if it is not, Columbia may not contract for enough gas from its suppliers, which could create shortages and put its customers at risk. The daily operational forecast is used to ensure that scheduled supplies are balanced with forecasted demands over the next 5-day period. It is used to balance supply and demand on a daily basis. The forecasting process is similar for the two types of forecasts. Columbia uses multiple regression analysis, based on daily

demand for 2 years, and several weather-related independent variables to develop the parameters of a time series forecast model for the design day forecast and the daily operational forecast.



Source: M. Flock, "Forecasting Winter Daily Gas Demand at Vermont Gas Systems," *Journal of Business Forecasting* 13, no. 1 (Spring 1994): 2; and H. Catron, "Daily Demand Forecasting at Columbia Gas," *Journal of Business Forecasting* 19, no. 2 (Summer 2000): 105.

Regression Analysis with Excel

[Exhibit 15.8](#) shows a spreadsheet set up to develop the linear regression forecast for our State University athletic department example. Notice that Excel computes the slope directly with the formula **=SLOPE(B5:B12, A5:A12)** entered in cell E7 and shown on the formula bar at the top of the spreadsheet. The formula

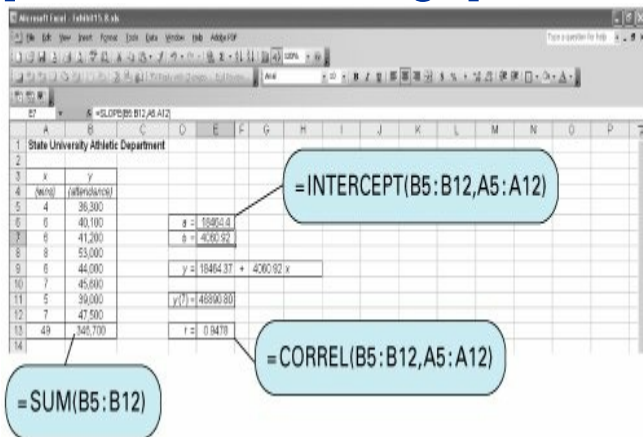
for the intercept in cell E6 is
=INTERCEPT(B5:B12,A5:A12)

The values for the slope and intercept are subsequently entered in cells E9 and G9 to form the linear regression equation. The correlation coefficient in cell E13 is computed by using the formula **=CORREL(B5:B12,A5:A12)**. Although it is not shown on the spreadsheet, the coefficient of determination (r^2) could be computed by using the formula **=RSQ(B5:B12,A5:A12)**.

Exhibit 15.8.

(This item is displayed on page 696 in the print version)

[\[View full size image\]](#)



The same linear regression equation could be computed in Excel if we had developed and entered the mathematical formulas for computing the slope and intercept we developed in the previous section, although that would have been more time-consuming and tedious.

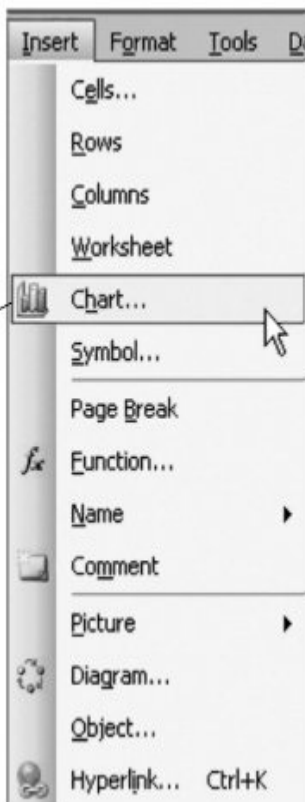
[Page 696]

It is also possible to develop a scatter diagram of our example data similar to the chart shown in [Figure 15.6](#) by using the

Chart Wizard in Excel. First, cover the example data in cells **A5:B12** on the spreadsheet in [Exhibit 15.8](#). Next click on "Insert" on the toolbar at the top of the spreadsheet. This will result in the menu shown in [Exhibit 15.9](#).

Exhibit 15.9.

Click on "Chart"
to access the
Chart Wizard.



Select "Chart" from this menu, which will access the Chart Wizard window. In the Chart Wizard window select the "XY (Scatter)" chart from the "Chart Type" menu, as shown in [Exhibit 15.10](#).

Clicking on "Next" on the window in [Exhibit 15.10](#) will provide a preliminary plot of the example data. (If you forgot to cover your example data cells earlier, you will be asked to do so at this point;

this is the range **A5:B12**.) Clicking on "Next" will enable you to add or delete chart legends, title the chart and the axes, and generally customize your chart. Clicking on "Finish" will display the chart on your spreadsheet so that you can position it, reduce it, expand it, or work on it some more.

[Exhibit 15.11](#) shows our spreadsheet with the scatter diagram chart for our example data.

Exhibit 15.10.

[\[View full size image\]](#)

Chart Wizard - Step 1 of 4 - Chart Type



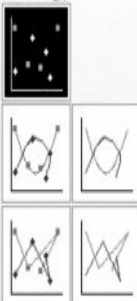
Standard Types

Custom Types

Chart type:



Chart sub-type:



Scatter. Compares pairs of values.

Press and Hold to View Sample

Cancel

< Back

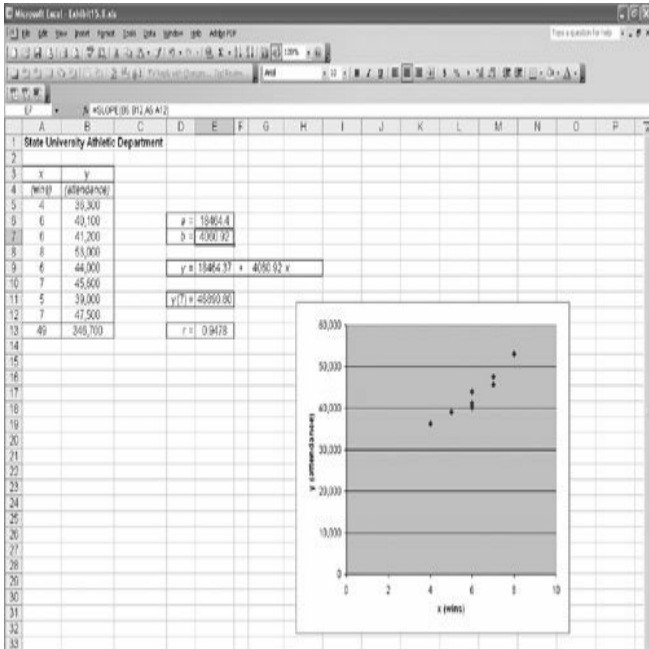
Next >

Finish

Click on "XY (Scatter)"
to develop a scatter
diagram.

Exhibit 15.11.

[\[View full size image\]](#)



A linear regression forecast can also be developed directly with Excel by using the "Data Analysis" option from the "Tools" menu we accessed previously to develop an exponentially smoothed forecast. [Exhibit 15.12](#) shows the "Regression" selection from the Data Analysis window and [Exhibit 15.13](#) shows the Regression window. We first enter the cells from [Exhibit 15.8](#) that include the y values (for attendance), **B5:B12**. Next, we enter the x value cells, **A5:A12**. The output

range is the location on the spreadsheet where you want to put the output results. This range needs to be large (18 cells by 9 cells) and must not overlap with anything else on the spreadsheet. Clicking on "OK" will result in the spreadsheet shown in [Exhibit 15.14](#). (Note that the "Summary Output" section has been slightly edited i.e., moved around so that all the results could be included on the screen in [Exhibit 15.14](#).)

Exhibit 15.12.

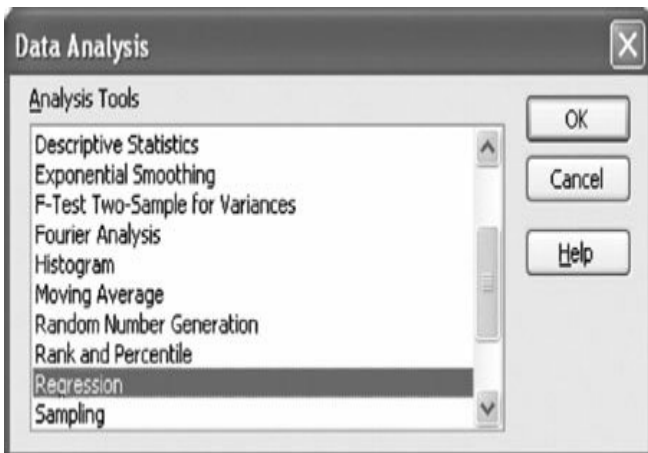


Exhibit 15.13.

Regression



Input

Input Y Range:

\$B\$5:\$B\$12



Input X Range:

\$A\$5:\$A\$12



☐ Labels

☐ Constant is Zero

☒ Confidence Level:

95 %

OK

Cancel

Help

Output options

☒ Output Range:

\$A\$15:\$J\$20



☐ New Worksheet Ply:

☐ New Workbook

Residuals

☐ Residuals

☐ Residual Plots

☐ Standardized Residuals

☐ Line Fit Plots

Normal Probability

☐ Normal Probability Plots

Exhibit 15.14.

(This item is displayed on page
699 in the print version)

[\[View full size image\]](#)

Microsoft Excel - Exhibit 15.14.xls													
File Edit View Insert Format Tools Data Window Help													
Type a question for help													

information, the explanation and use of which are beyond the scope of this text. The essential items that interest us are the intercept and slope (labeled "X Variable 1") in the "Coefficients" column at the bottom of the spreadsheet and the "Multiple R" (or correlation coefficient) value shown under "Regression Statistics."

[Page 699]

Note the Excel QM also has a spreadsheet macro for regression analysis that can be

accessed similarly to the exponentially smoothed forecast in [Exhibit 15.15](#).

Exhibit 15.15.



Forecasting Results



State University Athletic Department Summary

Measure	Value
Error Measures	
Bias (Mean Error)	.001
MAD (Mean Absolute Deviation)	1,412.644
MSE (Mean Squared Error)	2,537,298.0
Standard Error (denom= $n-2-0=6$)	1,839.311
MAPE (Mean Absolute Percent Error)	.033
Regression line	
Dpendnt var, $Y = 18,464.37$	
$+ 4,060.919 * X_1$	
Statistics	
Correlation coefficient	.9478
Coefficient of determination (r^2)	.8983

Regression Analysis with QM for Windows

QM for Windows has the capability to perform linear regression, as demonstrated earlier. To demonstrate this program module, we will use our State University athletic department example. The program output, including the linear equation and correlation coefficient, is shown in [Exhibit 15.15](#).

Multiple Regression with Excel

Another causal method of forecasting is multiple regression, a more powerful extension of linear regression. Linear regression relates a dependent variable such as demand to one other independent variable, whereas multiple regression reflects the relationship between a dependent variable and *two or more* independent variables. A multiple regression model has

the following general form:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_k x_k$$

where

β_0 = the intercept

$\beta_1 \dots \beta_k$ = parameters representing the contribution of the independent variables

$x_1 \dots x_k$ = independent variables

Multiple regression
*relates demand to
two or more
independent
variables.*

For example, the demand for new housing (y) in a region or an urban area might be a function of several independent variables, including interest rates, population, housing

prices, and personal income. Development and computation of the multiple regression equation, including the compilation of data, are quite a bit more complex than linear regression. Therefore, the only viable means for forecasting using multiple regression problems is by using a computer.

To demonstrate the capability to solve multiple regression problems with Excel spreadsheets, we will expand our State University athletic department example for

forecasting attendance at football games that we used to demonstrate linear regression. Instead of attempting to predict attendance based on only one variable (wins), we will include a second variable for advertising and promotional expenditures, as follows:

Wins	Promotion	Attendance
4	\$29,500	36,300
6	55,700	40,100

6	71,300	41,200
8	87,000	53,000
6	75,000	44,000
7	72,000	45,600
5	55,300	39,000
7	81,600	47,500

We will use the "Data Analysis" option (add-in) from the "Tools"

menu at the top of the spreadsheet that we used in the previous section to develop our linear regression equation, and then we will use the "Regression" option from the "Data Analysis" menu. The resulting spreadsheet, with the multiple regression statistics, is shown in [Exhibit 15.16](#).

Exhibit 15.16.

(This item is displayed on page 701 in the print version)

[\[View full size image\]](#)

Microsoft Excel - Tab11115.16.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question or help

100%

and

A14

	A	B	C	D	E	F	G	H	I	J	K	L
1	State University Athletic Department											
2												
3	X_1	X_2	Y									
4	(year)	(# promotion)	(attendance)									
5	4	28,500	36,200									
6	6	55,700	40,100									
7	6	71,300	41,200									
8	6	97,000	63,000									
9	6	75,000	44,000									
10	7	72,000	45,000									
11	5	55,300	39,000									
12	7	81,800	47,500									
13	48	527,400	346,700									
14												
15	SUMMARY OUTPUT											
16												
17	Regression Statistics			ANOVA								
18	Multiple R	0.94918										
19	R Square	0.90065										
20	Adjusted R Square	0.88133										
21	Standard Error	1939.69736										
22	Observations	8										
23												
24												
25		Coefficients	Standard Error	t Stat	P-value	Lower 95%	Upper 95%	Lower 95.0%	Upper 95.0%			
26	Intercept	19394.4236	4159.2818	4.6130	0.0068	8454.0780	29734.7692	8454.0780	29734.7692			
27	(year)	35.803864	1489.9812	2.3940	0.0638	-294.8217	1416.8145	-294.8217	1416.8145			
28	(# promotion)	0.0393	0.1013	0.3940	0.7007	-0.2238	0.2634	-0.2238	0.2634			

r^2 , the coefficient of determination

Note that the data need to be set up on the spreadsheet so that the x variables are in adjacent columns (in this case, columns A and B). Then we enter the "Input x Range" as **A4:B12**, as shown in [Exhibit 15.17](#). Notice that we have also included cells A4, B4, and C4, which include our variable headings (i.e., "wins," "\$ promotion," and "attendance"), in the input ranges. By clicking on "Labels," headings can be placed on our spreadsheet in

cells A27 and A28.

Exhibit 15.17.

(This item is displayed on page
701 in the print version)

[\[View full size image\]](#)

Regression



Input

Input Y Range:

\$C\$4:\$C\$12



Input X Range:

\$A\$4:\$B\$12



☒ Labels

☐ Constant is Zero

☐ Confidence Level:

95

%

OK

Cancel

Help

Output options

☒ Output Range:

A15:I28



☐ New Worksheet Ply:

☐ New Workbook

Residuals

☐ Residuals

☐ Residual Plots

☐ Standardized Residuals

☐ Line Fit Plots

Normal Probability

☐ Normal Probability Plots

Includes x_1 and x_2 columns

The regression coefficients for

our x variables, wins and promotion, are shown in cells B27 and B28 in [Exhibit 15.16](#). Thus, the multiple regression equation is formulated as

$$y = 19,094.42 + 3,560.99x_1 + .0368x_2$$

[Page 701]

This equation can now be used to forecast attendance based on both projected football wins and promotional expenditure. For example, if the athletic department expects the team

to win seven games and plans to spend \$60,000 on promotion and advertising, the forecasted attendance is

$$\begin{aligned}y &= 19,094.42 + 3,560.99(7) \\&\quad + .0368(60,000) \\&= 46,229.35\end{aligned}$$

[Page 702]

If the promotional expenditure is held constant, every win will

increase attendance by 3,560.99, whereas if the wins are held constant, every \$1,000 of advertising money spent will increase attendance by 36.8 fans. This would seem to suggest that number of wins has a more significant impact on attendance than promotional expenditures.

The coefficient of determination, r^2 , shown in cell B19 in [Exhibit 15.16](#), is .90, which suggests that 90% of the amount of variation in attendance can be attributed to

the number of wins and the promotional expenditures. However, as we have already noted, the number of wins probably accounts for a larger part of the variation in attendance.

A problem often encountered in multiple regression is *multicollinearity*, or the amount of "overlapping" information about the dependent variable that is provided by several independent variables. This problem usually occurs when the independent variables are highly correlated, as in this

example, in which wins and promotional expenditures are both positively correlated; that is, more wins coincide with higher promotional expenditures and vice versa. (Possibly the athletic department increased promotional expenditures when it thought it would have a better team that would achieve more wins.) Multicollinearity and how to cope with it are beyond the scope of this text and this brief section on multiple regression; however, most statistics texts discuss this

topic in detail.

Summary

We have presented several methods of forecasting that are useful for different time frames. Time series and regression methods can be used to develop forecasts encompassing horizons of any length time, although they tend to be used most frequently for short- and medium-range forecasts. These quantitative forecasting techniques are generally easy to understand,

simple to use, and not especially costly, unless the data requirements are substantial. They also have exhibited a good track record of performance for many companies that have used them. For these reasons, regression methods, and especially times series, are widely popular.

When managers and students are first introduced to forecasting methods, they are sometimes surprised and disappointed at the lack of exactness of the forecasts.

However, they soon learn that forecasting is not easy and exactness is not possible. Forecasting is a lot like playing horseshoes: It's great to get a "ringer," but you can win by just getting close, although those who have the skill and experience to get ringers will beat those who just get close. Often forecasts are used as inputs to other decision models for example, in inventory models, the subject of the next chapter. The primary factor in determining the amount of inventory a firm

should order (i.e., prepare to have on hand) is the demand that will occur in the futureforecasted demand.

References

Box, G. E. P., and Jenkins, G. M. *Time Series Analysis: Forecasting and Control*, 2nd ed. Oakland, CA: Holden-Day, 1976.

Brown, R. G. *Statistical Forecasting for Inventory Control*. New York: McGraw-Hill, 1959.

Gardner, E. S., and Dannenbring, D. G. "Forecasting with Exponential Smoothing: Some Guidelines for Model Selection." *Decision Sciences* 11, no. 2 (1980): 37083.

Makridakis, S., Wheelwright, S. C., and McGee, V. E.

Forecasting: Methods and Applications, 2nd ed. New York: John Wiley & Sons, 1983.

Example Problem Solutions

The following problem provides an example of the computation of exponentially smoothed and adjusted exponentially smoothed forecasts.

Problem Statement

A computer software firm has experienced the following demand for its Personal Finance

software package:

Period

Units

1

56

2

61

3

55

4

70

5

66

6

65

7

72

8

75

Develop an exponential smoothing forecast, using $\alpha = .40$, and an adjusted exponential smoothing forecast, using $\alpha = .40$ and $\beta = .20$. Compare the accuracy of the two forecasts, using *MAD* and cumulative error.

Solution

Compute the Exponential Smoothing Forecast

$$F_{t+1} = \alpha D_t + (1 - \alpha) F_t$$

For period 2 the forecast (assuming that $F_1 = 56$)

$$F_2 = \alpha D_1 + (1 - \alpha) F_1 = (.40)(61) + (.60)(56) = 56.4$$

Step 1. $(.60)(56) = 56$

For period 3 the forecast

$$F_3 = (.40)(61) + (.60)(56.4) = 56.84$$

The remaining forecasts

computed similarly and
in the table on page 70.

Compute the Adjusted Exponential Smoothing Forecast

$$\begin{aligned}AF_{t+1} &= F_{t+1} + T_{t+1} \\T_{t+1} &= \beta(F_{t+1} - F_t) + (1 - \beta)T_t\end{aligned}$$

Starting with the forecast
period 3 (because $F_1 =$
will assume that $T_2 = 0$

$$\begin{aligned}T_3 &= \beta(F_3 - F_2) + (1 - \beta)T_2 = (.20)(58 - 56) + (.80)(0) = .40 \\AF_3 &= F_3 + T_3 = 58 + .40 = 58.40\end{aligned}$$

The remaining adjusted
are computed similarly
shown in the following t

Period	D_t	F_t	AF_t	L
--------	-------	-------	--------	-----

1	56			
---	----	--	--	--

Step 2.

2	61	56.00	56.00	5
---	----	-------	-------	---

3 55 58.00 58.40 3

4 70 56.80 56.88 1

5 66 62.08 63.20 3

6 65 63.65 64.86 1

7 72 64.18 65.26 7

8 75 67.31 68.80 7

9 70.39 72.19

Compute the *MAD* Value

Step 3.
$$MAD(F_t) = \frac{\sum |D_t - F_t|}{n} = \frac{41.97}{7} = 5.99$$
$$MAD(AF_t) = \frac{\sum |D_t - AF_t|}{n} = \frac{37.39}{7} = 5.34$$

Compute the Cumulative

$$E(F_t) = 35.97$$
$$E(AF_t) = 30.60$$

Because both *MAD* and cumulative error are less

Step 4. adjusted forecast, it would be the most accurate

The following problem provides an example of the computation of linear regression forecasts

Problem Statement

A local building products store has accumulated sales data for two-by-four lumber (in board

feet) and the number of building permits in its area for the past 10 quarters:

Quarter	Building Permits, x	Lumber Sales (1,000s of bd. ft.), y
1	8	12.6
2	12	16.3
3	7	9.3
4	9	11.5
5	15	18.1

6	6	7.6
7	5	6.2
8	8	14.2
9	10	15.0
10	12	17.8

[Page 705]

Develop a linear regression

model for these data and determine the strength of the linear relationship by using correlation. If the model appears to be relatively strong, determine the forecast for lumber, given 10 building permits in the next quarter.

Solution

**Compute the
Components of the
Linear Regression
Equation**

Step 1.

$$\bar{x} = \frac{92}{10} = 9.2$$

$$\bar{y} = \frac{128.6}{10} = 12.86$$

$$b = \frac{\sum xy - n\bar{x}\bar{y}}{\sum x^2 - n\bar{x}^2} = \frac{(1,290.3) - (10)(9.2)(12.86)}{(932) - (10)(9.2)^2}$$

$$b = 1.25$$

$$a = \bar{y} - b\bar{x} = 12.86 - (1.25)(9.2)$$

$$a = 1.36$$

Develop the Linear Regression Equation

Step 2.

$$y = a + bx$$

$$y = 1.36 + 1.25x$$

Compute the

Correlation Coefficient

$$\begin{aligned} r &= \frac{n \sum xy - \sum x \sum y}{\sqrt{[n \sum x^2 - (\sum x)^2][n \sum y^2 - (\sum y)^2]}} \\ &= \frac{(10)(1,290.3) - (92)(128.6)}{\sqrt{[(10)(932) - (92)^2][(10)(1,810.48) - (128.6)^2]}} \\ &= .925 \end{aligned}$$

Step 3.

Thus, there appears to be a strong linear relationship.

**Calculate the
Forecast for $x = 10$
Permits**

Step 4.

$$\begin{aligned}y &= a + bx \\&= 1.36 + 1.25(10) \\&= 13.86 \text{ or } 1,386 \text{ bd. ft.}\end{aligned}$$

Problems

The Saki motorcycle dealer in the Minn wants to make an accurate forecast of Saki Super TXII motorcycle during the Because the manufacturer is in Japan, i motorcycles back or reorder if the prop ordered a month ahead. From sales re accumulated the following data for the

[Page 706]

Month	Motorcycle Sales
-------	------------------

January

9

February	7
----------	---

March	10
-------	----

April	8
-------	---

May	7
-----	---

June	12
------	----

1.

July	10
------	----

August	11
--------	----

September	12
-----------	----

October	10
---------	----

November	14
----------	----

December	16
----------	----

- 1.** Compute a 3-month moving average demand for April through January
- 2.** Compute a 5-month moving average June through January.
- 3.** Compare the two forecasts computed using *MAD*. Which one should the January of the next year?

The manager of the Carpet City outlet wants an accurate forecast of the demand for September

biggest seller). If the manager does not order carpet from the carpet mill, customers must come from one of Carpet City's many competitors. The manager has collected the following demand data for the next four months:

Month	Demand for Soft Shag Carpet (1,000 yd.)
1	8
2	12
3	7
4	9

2.

5

15

6

11

7

10

8

12

-
- 1.** Compute a 3-month moving average for months 4 through 9.
 - 2.** Compute a weighted 3-month moving average forecast for months 4 through 9. Assign weights of .55, .33, and .12 to the months in the moving average, with the most recent month.
 - 3.** Compare the two forecasts by using the mean absolute deviation. Which forecast appears to be more accurate?

The Fastgro Fertilizer Company distributes fertilizer to various lawn and garden shops. The company has its quarterly production schedule on a first-come, first-served basis. The number of tons of fertilizer that many tons of fertilizer will be demanded by the company has gathered the following data over the past 3 years from its sales records:

[Page 707]

Year	Quarter	Demand for Fertilizer (tons)
1	1	105
	2	150
	3	93

3.

	4	121
2	5	140
	6	170
	7	105
	8	150
3	9	150
	10	170
	11	110

- 1.** Compute a three-quarter moving forecast for quarters 4 through 13 and compute the cumulative error for each quarter.
- 2.** Compute a five-quarter moving forecast for quarters 6 through 13 and compute the cumulative error for each quarter.
- 3.** Compute a weighted three-quarter moving forecast, using weights of .50, .30, and .20 for the most recent, next recent, and moving average respectively, and compute the cumulative error for each quarter.
- 4.** Compare the forecasts developed using cumulative error. Which forecast is most accurate? Do any of them exhibit a consistent bias?

- Graph the demand data in [Problem 3](#). (
4. trends, cycles, or seasonal patterns?

The chairperson of the department of r
State University wants to forecast the
who will enroll in production and operat
(POM) next semester, in order to deter
sections to schedule. The chair has acc
following enrollment data for the past e

Semester	Students Enrolled in POM
1	400
2	450
3	350

5.

4	420
5	500
6	575
7	490
8	650

- 1.** Compute a three-semester moving average for semesters 4 through 9.
- 2.** Compute the exponentially smoothed forecast ($\alpha = .20$) for the enrollment data.

3. Compare the two forecasts by using the more accurate of the two.

The manager of the Petroco Service Station forecast the demand for unleaded gasoline that the proper number of gallons can be supplied by the distributor. The owner has accumulated historical data on demand for unleaded gasoline from the past 10 months:

[Page 708]

Month	Gasoline Demanded (gallons)
October	800
November	725

December	630
----------	-----

January	500
---------	-----

6. February	645
--------------------	-----

March	690
-------	-----

April	730
-------	-----

May	810
-----	-----

June	1,200
------	-------

July	980
------	-----

1. Compute an exponentially smoothed forecast using a value of $\alpha = .30$.
2. Compute an adjusted exponential forecast (with $\alpha = .30$ and $\beta = .20$).
3. Compare the two forecasts by using the mean absolute deviation to indicate which seems to be more accurate.

The Victory Plus Mutual Fund of growth and income has the following average monthly price for the past two years:

Month	Fund Price
1	62.7
2	63.9

3 68.0

4 66.4

5 67.2

7. 6 65.8

7 68.2

8 69.3

9 67.2

10 70.1

Compute the exponentially smoothed forecast with $\alpha = .40$, the adjusted exponential smoothing forecast with $\alpha = .40$ and $\beta = .30$, and the linear trend line forecast. Compare the accuracy of the three forecasts using the cumulative error and *MAD*, and indicate which appears to be most accurate.

The Bayside Fountain Hotel is adjacent to the Coliseum, a 24,000-seat arena that is used for professional basketball and ice hockey games and hosts a variety of concerts, trade shows, and conventions throughout the year. The hotel has experienced the following occupancy rates for the past 10 years since the coliseum opened:

Year	Occupancy Rate (%)
-------------	-------------------------------

8.

1	83
2	78
3	75
4	81
5	86
6	85
7	89
8	90

[Page 709]

Compute an exponential smoothing for an adjusted exponential smoothing for $\beta = .20$, and a linear trend line for three forecasts, using *MAD* and average indicate which seems to be most accurate.

Emily Andrews has invested in a science mutual fund. Now she is considering liquidating her investment and investing in another fund. She would like to know the price of the science and technology fund one month before making a decision. She has the following data on the average price of the fund over the past 20 months:

Month	Fund Price
-------	------------

1	\$63 1/4
---	----------

2	60 1/8
---	--------

3	61 3/4
---	--------

4	64 1/4
---	--------

5	59 3/8
---	--------

6	57 7/8
---	--------

7	62 1/4
---	--------

8	65 1/8
---	--------

9 68 $\frac{1}{4}$

10 65 $\frac{1}{2}$

9. 11 68 $\frac{1}{8}$

12 63 $\frac{1}{4}$

13 64 $\frac{3}{8}$

14 68 $\frac{5}{8}$

15 70 $\frac{1}{8}$

16 72 $\frac{3}{4}$

17 74 $\frac{1}{8}$

18 71 $\frac{3}{4}$

19 75 $\frac{1}{2}$

20 76 $\frac{3}{4}$

- 1.** Using a 3-month average, forecast the fund price for month 21.
- 2.** Using a 3-month weighted average, with the most recent month weighted 0.60, the month before weighted 0.30, and the third month weighted 0.10, forecast the fund price for month 21.
- 3.** Compute an exponentially smoothed forecast for month 21.

= 0.40, and forecast the fund price

4. Compare the forecasts in (a), (b) and (c) using the MAD , and indicate the most accurate forecast.

Eurotronics manufactures components for electronic products such as computers, radios at plants in Belgium, Germany, and France. The components are transported by truck to Hamburg, Germany, and then shipped overseas to customers in Mexico, the United States, and the Pacific Rim. Eurotronics must reserve space on ships months and sometimes years in advance. This requires an accurate forecast of the space needed. Following are the number of cubic feet of space the company has used in each of the past five years.

[Page 710]

Month	Space (1,000s ft. ³)
-------	-------------------------------------

1 10.6

2 12.7

3 9.8

4 11.3

5 13.6

6 14.4

7 12.2

10. 8 16.7

9	18.1
---	------

10	19.2
----	------

11	16.3
----	------

12	14.7
----	------

13	18.2
----	------

14	19.6
----	------

15	21.4
----	------

16	22.8
----	------

17	20.6
----	------

Develop a forecasting model that you think will provide the company with relatively accurate forecasts for the next year and indicate the forecasting methods required for the next 3 months.

The Whistle Stop Cafe in Weems, Georgia, is famous for its popular homemade ice cream, made fresh daily from the back of the cafe. People drive all the way from Macon to buy the ice cream. The two owners of the cafe want to develop a forecasting model to help plan their ice cream production operation and determine the number of employees they need to sell the ice cream at the cafe. They have accumulated the following data for their ice cream for the past 12 quarters.

Year	Quarter	Ice Cream Sales (gallons)
2003	1	350
	2	510
	3	750
	4	420
2004	5	370
	6	480
	7	860

	8	50
2005	9	45
	10	55
	11	82
	12	57

Develop an adjusted exponential smoothing model with $\alpha = .50$ and $\beta = .50$ to forecast demand and forecast accuracy using cumulative error (E) and bias (B). Does there appear to be any bias in the

12. For the demand data in [Problem 11](#), develop an adjusted forecast for 2004. (Use a line model to develop a forecast estimate for 2004. Which model do you perceive to be more accurate: a linear trend model, an exponential smoothing model from [Problem 11](#), or a seasonally adjusted forecast?)

13. Develop a seasonally adjusted forecast for fertilizer demand data for fertilizer found in [Problem 3](#). Use a trend line model to compute a forecast for demand in year 4.

Monaghan's Pizza delivery service has been operating on weekdays during the past month and has delivered 100 pizzas at four different time periods per

Time Period	1	2	3	4	5
-------------	---	---	---	---	---

10:00 A.M.3:00 P.M.	62	49	53	35	43
---------------------	----	----	----	----	----

14. 3:00 P.M.7:00 P.M. 73 55 81 77 60

7:00 P.M.11:00 P.M.	42	38	45	50	29
---------------------	----	----	----	----	----

11:00 P.M.12:00 A.M.	35	40	36	39	26
-------------------------	----	----	----	----	----

Develop a seasonally adjusted forecast pizza demand and forecast demand for periods for a single upcoming day.

The Cat Creek Mining Company mines has experienced the following demand past 8 years:

Year	Coal Sales (tons)
1	4,260
2	4,510
3	4,050
4	3,720
5	3,900
6	3,470

15.

7	2,890
---	-------

8	3,100
---	-------

Develop an adjusted exponential smoothing model ($\alpha = .30$, $\beta = .20$) and a linear trend line model. Compare the forecast accuracy of the two by using the Mean Absolute Deviation. The linear trend line forecast seems to be more accurate.

The Northwoods Outdoor Company is a retail operation that specializes in outdoor recreation equipment. Demand for its items is very seasonal, with a peak during the Christmas season and during the spring. The following data for orders per season over the past 5 years:

Quarter

2001

2002

2

JanuaryMarch

18.6

18.1

2

AprilJune

23.5

24.7

2

JulySeptember

20.4

19.5

2

OctoberDecember

41.9

46.3

4

- 1.** Develop a seasonally adjusted forecast for these order data. Forecast demand for 2006 (using a linear trend line for orders in 2006).
- 2.** Develop a separate linear trend line for each of the four seasons and forecast demand for 2006.
- 3.** Which of the two approaches used appears to be the more accurate? Justify your selection.

Metro Food Vending operates vending machines in schools, government buildings, the airport, bus stations, college campuses, retail businesses and agencies around town, and uses mobile vending trucks for building and construction sites. The company believes its sandwich sales follow a seasonal pattern. It has accumulated the following sales per season during the past 4 years.

Sandwich Sales

17.

Season	2002	2003
Fall	42.7	44.3
Winter	36.9	42.7
Spring	51.3	55.6
Summer	62.9	64.8

Develop a seasonally adjusted forecast sandwich sales data. Forecast demand 2006 by using a linear trend line estimate. Do the data appear to have a seasonal

The emergency room at the new Community Hospital selected every other week during the past year to observe the number of patients during the week (Friday through Sunday) and the weekend (Monday through Thursday). They typically observe greater patient traffic on weekends than on weekdays.

Number of Patients

Week	Weekend	Weekdays
------	---------	----------

1

116

83

18.

2	126	92
3	125	97
4	132	91
5	128	103
6	139	88
7	145	96
8	137	106
9	151	95

Develop a seasonally adjusted forecast number of patients during each part of 11.

[Page 713]

Aztec Industries has developed a forecast was used to forecast during a 10-month forecasts and actual demand were as f

Month	Actual Demand	Forecast I
-------	---------------	------------

1

160

170

19.

2	150	165
3	175	157
4	200	166
5	190	183
6	220	186
7	205	203
8	210	204
9	200	207
10	220	203

Measure the accuracy of the forecast by calculating the mean absolute error and cumulative error. Does the forecast appear to be accurate?

RAP Computers assembles personal computers. For generic parts it purchases at a discount from a supplier. The company orders these parts via phone orders it receives from the supplier. In response to the company's ads in trade magazines, the supplier's business has developed an exponential smoothing model to forecast future computer demand. The actual demand for the company's computers for the past 12 months, as well as a forecast, are shown in the following table.

Month	Demand	Forecast
March	120	

20.

April	110	120.0
May	150	116.0
June	130	129.6
July	160	129.7
August	165	141.8
September	140	151.1
October	155	146.7
November		150.0

1. Using a measure of forecast accuracy, ascertain whether the forecast appears to be more accurate.
2. Determine whether a 3-month moving average will provide a better forecast.

Develop an exponential smoothing forecast for the demand data in [Problem 1](#). Compare

21. with the 3-month moving average from [Problem 1](#), using *MAD*, and indicate which is to be more accurate.

The Jersey Dairy Products Company produces cheese, which it sells to supermarkets and food service companies. Because of concerns about the economy in cheese, the company has seen demand decline during the past decade. It is now

introducing some alternative low-fat da
wants to determine how much availabl
will have next year. The company has c
exponential smoothing forecast with α
cheese. The actual demand and the for
model are as follows:

[Page 714]

Year	Demand (1,000 lb.)	Forecast
1	16.8	
2	14.1	16.8
3	15.3	15.7

22.

4	12.7	15.5
5	11.9	14.4
6	12.3	13.4
7	11.5	12.9
8	10.8	12.4

Assess the accuracy of the forecast model and cumulative error. If the exponential model does not appear to be accurate, a linear trend model would provide a more accurate forecast.

The manager of the Ramona Inn Hotel Stadium believes that how well the local professional baseball team is playing has an effect on the occupancy rate at the hotel during the season. Following are the number of victories for the Blue Sox (out of a 162-game schedule) for the past 8 years and the corresponding occupancy rates:

Year	Blue Sox Wins	Occupancy (%)
1	75	83
2	70	78
3	85	86
4	91	85

5	87	89
6	90	93
7	87	92
8	67	91

Develop a linear regression model for the data to forecast the occupancy rate for next year if the team win 88 games.

Carpet City wants to develop a means to increase carpet sales. The store manager believes that carpet sales are directly related to the number of new homes starts in town. The manager has gathered the following data:

county records on monthly house cons
from store records on monthly sales. T
follows:

Monthly Carpet Sales (1,000 yd.)	Monthly Construction Permits
5	21
10	35
4	10
3	12
8	16

24.

2

9

12

41

11

15

9

18

14

26

[Page 715]

- 1.** Develop a linear regression model to forecast carpet sales if 30 construction

new homes are filed.

2. Determine the strength of the causal relationship between monthly sales and new home construction by using correlation.

The manager of Gilley's Ice Cream Parlor wants an accurate forecast of the demand for ice cream. If the store orders ice cream from a distributor a week in advance and the store orders too little, it loses business; if it orders too much, the extra must be thrown away. The manager believes that a major determinant of ice cream demand is temperature (i.e., the hotter the weather, the more ice cream people buy). Using an almanac, the manager determined the average daytime temperature for 10 weeks, selected at random, and from sales records determined the ice cream consumption for the same 10 weeks. These data are summarized as follows:

Week	Average Temperature (degrees)	Ice Cream (gallons)
------	-------------------------------------	------------------------

25.

1	73	110
2	65	95
3	81	135
4	90	160
5	75	97
6	77	105
7	82	120
8	93	175

9

86

140

10

79

121

-
1. Develop a linear regression model to forecast the ice cream consumption weekly daytime temperature is expressed in degrees.
 2. Determine the strength of the linear relationship between temperature and ice cream consumption using correlation.

26. Compute the coefficient of determination for [Problem 25](#) and explain its meaning.

Administrators at State University believe the number of freshman applications they experienced are directly related to tuition. They have collected the following enrollment data over the past decade:

Year	Freshman Applications	Annual Tuition
1	6,050	\$3,600
2	4,060	3,600
3	5,200	4,000
4	4,410	4,400

5	4,380	4,500
27. 6	4,160	5,700
7	3,560	6,000
8	2,970	6,000
9	3,280	7,500
10	3,430	8,000

[Page 716]

1. Develop a linear regression model to forecast the number of applications

University if tuition increases to \$9,000 per year. If tuition is lowered to \$7,000 per year, how many freshman applications are expected?

2. Determine the strength of the line between freshman applications and tuition.
3. Describe the various planning decisions at State University that would be affected by a change in freshman applications.

Develop a linear trend line model for the freshman applications data at State University in the past 10 years.

28. 1. Does this forecast appear to be more accurate than the linear regression forecast in [Problem 27](#)? Justify your answer.
 2. Compute the correlation coefficient for the trend line forecast and explain its meaning.
29. Explain the numerical value of the slope of the trend line.

regression equation in [Problem 25](#).

Some members of management of the Firm believe that demand for its products is determined by the level of promotional activities of local department stores where cosmetics are sold. However, others in management believe that other factors, such as local income, are stronger determinants of demand behavior. To test these hypotheses, data for local annual promotional expenditures for department store products and local annual unit sales for department store cosmetics have been collected from 20 stores selected from different localities:

Store	Annual Unit Sales (1,000s)	Annual Promotional Expenditure (\$1,000s)
1	3.5	\$12.6
2	7.2	15.5

3	3.1	10.8
---	-----	------

4	1.6	8.7
---	-----	-----

5	8.9	20.3
---	-----	------

6	5.7	21.9
---	-----	------

7	6.3	25.6
---	-----	------

8	9.1	14.3
---	-----	------

30.

9	10.2	15.1
---	------	------

10	7.3	18.7
----	-----	------

11	2.5	9.6
12	4.6	12.7
13	8.1	16.3
14	2.5	8.1
15	3.0	7.5
16	4.8	12.4
17	10.2	17.3
18	5.1	11.2

19	11.3	18.5
20	10.4	16.7

Based on these data, does it appear that the relationship between sales and production expenditures is sufficient to warrant using a regression forecasting model? Explain your answer.

Employees at Precision Engine Parts Company are paid according to exact design specifications. Employees are paid according to a piecework system wherein the faster they work and the more they produce, the greater their chances for promotion. Management suspects that this method of payment contributes to an increased number of defective parts. A specific part requires a normal, standard 10 minutes to produce. The quality control

checked the actual average times to pr
10 different employees during 20 days
during the past month and determined
percentage of defective parts, as follow

[Page 717]

Average Time (min.)	Defective (%)	Average (min.)
21.6	3.1	20
22.5	4.6	18
23.1	2.7	21
24.6	1.8	23

31.

22.8	3.5	23
23.7	3.2	24
20.9	3.7	19
19.7	4.5	19
24.5	0.8	21
26.7	1.2	20

Develop a linear regression model relating production time to percentage defects. Determine whether a relationship exists and the p

defective items that would be expected production time of 23 minutes.

Apperson and Fitz is a chain of clothing to high school and college students. It | catalog and operates a Web site that fe provocatively attired males and female very expensive to maintain, and compa not sure whether the number of hits at sales (i.e., people may be looking at th only). The Web master has accumulate for hits per month and orders placed at past 20 months:

Month	Hits (1,000s)	Orders (1,000s)
1	34.2	7.6
2	28.5	6.3

3	36.7	8.9
---	------	-----

4	42.3	5.7
---	------	-----

5	25.8	5.9
---	------	-----

6	52.3	6.3
---	------	-----

7	35.2	7.2
---	------	-----

8	27.9	4.1
---	------	-----

9	31.4	3.7
---	------	-----

32.

10	29.4	5.9
----	------	-----

11	46.7	10.8
12	43.5	8.7
13	52.6	9.3
14	61.8	6.5
15	37.3	4.8
16	28.9	3.1
17	26.4	6.2
18	39.4	5.9

19

44.7

7.2

20

46.3

5.5

[Page 718]

Develop a linear regression model for t
indicate whether there appears to be a
between Web site hits and orders. Wha
forecast for orders with 50,000 hits pe

The Gametime Hat Company manufact
that have various team logos in an ass
and colors. The company has had mon
past 24 months as follows:

Month Demand (1,000s)

1	8.2
---	-----

2	7.5
---	-----

3	8.1
---	-----

4	9.3
---	-----

5	9.1
---	-----

6	9.5
---	-----

7	10.4
---	------

8	9.7
---	-----

9	10.2
---	------

10	10.6
----	------

11	8.2
----	-----

<u>33.</u> 12	9.9
----------------------	-----

13	10.3
----	------

14	10.5
----	------

15	11.7
----	------

16	9.8
----	-----

17	10.8
----	------

18	11.3
----	------

19	12.6
----	------

20	11.5
----	------

21	10.8
----	------

22	11.7
----	------

23	12.5
----	------

24	12.8
----	------

Develop a forecast model using the method you think is best and justify your selection by using appropriate measures) of forecast accuracy.

Infoworks is a large computer discount store in a small town where State University is located. The store has collected historical data on computer sales for the past 10 years, as follows:

Year	Personal Computers Sold	Price
1	1,045	
2	1,610	

3 860

4 1,211

5 975

34. 6 1,117

7 1,066

8 1,310

9 1,517

10 1,246

1. Develop a linear trend line forecast demand in year 11.
2. Develop a linear regression model sales to computer sales in order to demand in year 11 if 1,300 comp
3. Compare the forecasts developed indicate which one appears to be

35. Develop an exponential smoothing model the data in [Problem 34](#) to forecast price 11 and compare its accuracy to the linear forecast developed in (a).

Arrow Air is a regional East Coast airline data for the percentage available seats

flights for four quarters(1) JanuaryMarch
 JulySeptember, and (4) OctoberDecem
 years. Arrow Air also has collected data
 percentage fare discount for each of th
 follows:

Year	Quarter	Average Fare Discount (%)	Se Occup (%
1	1	63	21
	2	75	34
	3	76	18
	4	58	26

36.

2	1	59	18
	2	62	40
	3	81	25
	4	76	30
3	1	65	23
	2	70	28
	3	78	30
	4	69	35

4	1	59	20
	2	61	35
	3	83	26
	4	71	30
5	1	60	25
	2	66	37
	3	86	25
	4	74	30

1. Develop a seasonally adjusted forecast for seat occupancy. Forecast seat occupancy by using a linear trend line forecast for seat occupancy in year 6.
2. Develop linear regression models for seat occupancy to discount fares in order to increase occupancy for each quarter in year 6. Discount of 20% for quarter 1, 36% for quarter 2, 25% for quarter 3, and 30% for quarter 4.
3. Compare the forecasts developed in 1 and 2 and indicate which one appears to be more accurate.

Develop an adjusted exponential smoothing model ($\alpha = .40$ and $\beta = .40$) for the data in [Problem 37](#).

37. Use the adjusted exponential smoothing model developed in Problem 37 to forecast seat occupancy and compare it to the seasonally adjusted model developed in Problem 37.

The consumer loan department at Cent
Trust wants to develop a forecasting m
determine its potential loan application
coming year. Because adjustable-rate b
based on government long-term treas
department collected the following data
treasury note interest rates for the pas

[Page 720]

Year	Rate	Year	Rate
1	5.77	9	9.71
2	5.85	10	11.55
3	6.92	11	14.44

38.

4	7.82	12	12.92
5	7.49	13	10.45
6	6.67	14	11.89
7	6.69	15	9.64
8	8.29	16	7.06

Develop an appropriate forecast model to forecast treasury note rates in the future and how accurate it appears to be compared to the actual rates.

The busiest time of the day at the Taco restaurant is between 11:00 A.M. and 2:00 P.M.

service is very labor dependent, and a providing quick service is the number of hand during this 3-hour period. To determine employees it needs during each hour of period, Taco Town requires an accurate Following are the number of customers Town during each hour of the lunch per weekdays:

Hour			
Day	1112	121	12
1	90	125	87
2	76	131	93

3	87	112	99
---	----	-----	----

4	83	149	78
---	----	-----	----

5	71	156	83
---	----	-----	----

6	94	178	89
---	----	-----	----

7	56	101	124
---	----	-----	-----

39.

8	63	91	66
---	----	----	----

9	73	146	119
---	----	-----	-----

10	101	104	96
----	-----	-----	----

11	57	114	106
12	68	125	95
13	75	206	102
14	94	117	118
15	103	145	122
16	67	121	93
17	94	113	76
18	83	166	94

19	79	124	87
----	----	-----	----

20	81	118	115
----	----	-----	-----

Develop a forecast model that you believe is most accurate to forecast Taco Town's customer demand for 2019 and explain why you selected this model.

[Page 721]

The Wellton Fund is a balanced mutual fund that has a mix of stocks and bonds. Following are the share prices of the fund and Dow Jones Industrial Average for a 20-year period:

Year	Share Price	DJIA
------	-------------	------

1	\$14.75	1,046
---	---------	-------

2	15.06	1,258
---	-------	-------

3	14.98	1,211
---	-------	-------

4	15.73	1,546
---	-------	-------

5	16.11	1,895
---	-------	-------

6	16.07	1,938
---	-------	-------

7	16.78	2,168
---	-------	-------

8	17.69	2,753
---	-------	-------

40.

9	16.90	2,633
10	17.81	3,168
11	19.08	3,301
12	20.40	3,754
13	19.39	3,834
14	24.43	5,117
15	26.46	6,448
16	29.45	7,908
17	29.35	9,181

18	27.96	11,497
----	-------	--------

19	28.21	10,786
----	-------	--------

20	27.26	10,150
----	-------	--------

Develop a linear regression model for the data to forecast the fund share price for a DJIA of 10,000. Does there appear to be a strong relationship between the fund share price and the DJIA?

The Valley United Soccer Club has boys' soccer teams at all age levels up to 18 years old. The club has been successful and grown in popularity over the years. However, an obstacle to its continued growth is the lack of practice and game soccer fields in the area.

tried to make a case to the town council recreation committee that it needs more space to accommodate the increasing number of kids who play on club teams. The number of kids on club soccer teams and the town's population over the past 15 years are as follows:

Year	Club Soccer Players	Town Population
1	146	18,060
2	135	18,021
3	159	18,110
4	161	18,125

5	176	18,240
6	190	18,231
7	227	18,306
8	218	18,477
<u>41.</u> 9	235	18,506
10	231	18,583
11	239	18,609
12	251	18,745

13	266	19,003
14	301	19,062
15	327	19,114

[Page 722]

The soccer club wants to develop a for demonstrate to the town council its ex future.

- 1.** Develop a linear trend line forecast number of soccer players the club year.
- 2.** The town planning department ha club that the town expects to gro 19,300 by next year and to 20,00

Develop a linear regression model using the population as a predictor of the number of players, and compare this forecast to the one developed in part (a). Which model should the club use to support its decision on how many fields?

The Port of Savannah is considering adding a new container terminal. The port has experienced an increase in container throughput during the past 10 years, as TEUs (i.e., 20-foot equivalent units, the standard measure for containers):

Year	TEUs (1,000s)
-------------	----------------------

1	526.1
---	-------

2	549.4
---	-------

3 606.0

4 627.0

5 695.7

42. 6 734.9

7 761.1

8 845.4

9 1,021.1

10 1,137.1

11 1,173.6

12

1,233.4

1. Develop a linear trend line forecast
forecast the number of TEUs for y
2. How strong is the linear relationst

The admission data for freshmen at Tec
years are as follows:

Year	Applicants	Offers	% Offers	Accept
------	------------	--------	-------------	--------

1	13,876	11,200	80.7	4,1
---	--------	--------	------	-----

2	14,993	11,622	77.8	4,3
3	14,842	11,579	78.0	4,7
4	16,285	13,207	81.1	5,0
5	16,922	11,382	73.2	4,5
6	16,109	11,937	74.1	4,6
7	15,883	11,616	73.1	4,6
8	18,407	11,539	62.7	4,6
9	18,838	13,138	69.7	5,0
10	17,756	11,952	67.3	4,8

43.

Tech's admission objective is a class of freshmen, and Tech wants to forecast the offers it will likely have to make in order to meet its admission objective.

- 1.** Develop a linear trend line to forecast the number of applicants and percentage of acceptance based on these results to estimate the percentage of offers that Tech should expect to make.

[Page 723]

- 2.** Develop a linear trend line to forecast the number of offers that Tech should expect to receive. Compare this result with the result from the linear forecast. Do you think is more accurate?
- 3.** Assume that Tech receives 18,300 offers. How many offers do you think Tech should expect to receive?

get 5,000 acceptances?

The State of Virginia has instituted a set of standardized learning (SOL) tests in math, history, English, and science that all high school students must pass before they are allowed to graduate and receive their diplomas. The school superintendent of Loudoun County believes the tests are unfair because the scores are closely related to teacher salaries. He thinks the more years a teacher has been at a school, the higher the salary. The superintendent has sampled 12 other counties in the state and accumulated data on average teacher salary and average

School	Average SOL Score	Average Teacher Salary	Average Teacher Tenure
1	81	\$34,300	

2	78	28,700	:
---	----	--------	---

3	76	26,500
---	----	--------

4	77	36,200
---	----	--------

5	84	35,900
---	----	--------

6	86	32,500	:
---	----	--------	---

44.	7	79	31,800
------------	---	----	--------

8	91	38,200	:
---	----	--------	---

9	68	27,100
---	----	--------

10	73	31,500
----	----	--------

11	90	37,600	:
12	85	40,400	:

1. Using Excel or QM for Windows, determine the regression equation for these data.
2. What is the coefficient of determination for the regression equation? Do you think the superintendent is correct in his beliefs?
3. Montgomery County has an average teacher salary of \$74,000, with an average teacher salary of \$74,000 and an average teacher tenure of 7.8 years. The superintendent has proposed to raise the average teacher salary by \$30,000 as well as a benefits program.

of increasing the average tenure t suggested that if the board passe the average SOL score will increas correct, according to the forecast

Tech administrators believe their freshmen influenced by two variables: tuition and applicant pool of eligible high school seniors. The following data for an 8-year period rates (per semester) and the sizes of the applicant pool for each year:

Tuition	Applicant Pool	Applicants
\$ 900	76,200	11,060
1,250	78,050	10,900

1,375	67,420	8,670
-------	--------	-------

1,400	70,390	9,050
-------	--------	-------

1,550	62,550	7,400
-------	--------	-------

45.

1,625	59,230	7,100
-------	--------	-------

1,750	57,900	6,300
-------	--------	-------

1,930	60,080	6,100
-------	--------	-------

[Page 724]

1. Using Excel, develop the multiple i

for these data.

2. What is the coefficient of determination for the regression equation?
3. Determine the forecast for freshman tuition rate of \$1,500 per semester for applicants of 60,000.

In [Problem 34](#), Infoworks believes its price is related to the average price of its printers. The following historical data on average printer prices over the last 5 years, as follows:

Year	Average Printer Price
1	\$475

2 490

3 520

4 420

5 410

46.

6 370

7 350

8 300

9 280

- 1.** Using Excel, develop the multiple regression equation for these data.
- 2.** What is the coefficient of determination for the regression equation?
- 3.** Determine the forecast for printer sales if the average personal computer sales of 1,500 units and the average printer price of \$300.

The manager of the Bayville police department wants to develop a forecast model for the number of police cars, based on mileage in the fleet of the cars. The following data have been collected for different cars:

Miles Driven	Car Age (yr.)	Maintenance Cost
-----------------	---------------------	---------------------

16,320	7	\$1,200
--------	---	---------

15,100	8	1,400
--------	---	-------

18,500	8	1,820
--------	---	-------

10,200	3	900
--------	---	-----

9,175	3	650
-------	---	-----

12,770	7	1,150
--------	---	-------

8,600	3	875
-------	---	-----

47.

8,000	2	875
-------	---	-----

7,900	3	900
-------	---	-----

- 1.** Using Excel, develop a multiple regression equation for these data.
- 2.** What is the coefficient of determination for the regression equation?
- 3.** Forecast the annual maintenance cost for a car that is 5 years old and will be driven 15,000 miles per year.

The dean of the college of business at a large university is planning a fund-raising campaign. One of the selling points for the college's use with potential donors is that increased private endowment will improve its ranking among business schools, as published each year.

The Global News and Business Report. demonstrate that there is a relationship and the rankings. He has collected the showing the private endowments (\$1,000,000s) annual budgets (\$1,000,000s) from sources for eight of Tech's peer institutions the ranking of each school:

[Page 725]

Private Endowment (\$1,000,000s)	Annual Budget (\$1,000,000s)
\$ 2.5	\$ 8.1
52.0	26.0
12.7	7.5

48.

63.0

33.0

46.0

12.0

27.1

16.1

23.3

17.0

46.4

14.9

48.9

21.8

- 1.** Using Excel, develop a linear regression model to estimate the amount of the private endowment and forecast a ranking for a private

\$70 million. Does there appear to be a relationship between the endowment and the annual budget?

- 2.** Using Excel, develop a multiple regression model using all these data, including private endowment, and forecast a range of annual budget for an endowment of \$70 million and an endowment of \$40 million. How does this forecast compare to the forecast in part (a)?

Case Problem

FORECASTING AT STATE UNIVERSITY

During the past few years, the legislature has severely reduced funding for State University. In reaction, the administration at State has significantly raised tuition each year for the past 5 years. Perceived as a bargain 5 years ago, State is now considered one of the more expensive state-supported universities.

This has led some parents and students to question the value of a State education, and applications for admission have declined. Because a portion of state educational funding is based on a formula tied to enrollments, State has maintained its enrollment levels by going deeper into its applicant pool and accepting less-qualified students.

On top of these problems, a substantial increase in the college-age population is expected in the next decade. Key members of the state

legislature have told the university administration that State will be expected to absorb additional students during the next decade. However, because of the economic outlook and the budget situation, the university should not expect any funding increases for additional facilities, classrooms, dormitory rooms, or faculty. The university already has a classroom deficit in excess of 25%, and class sizes are above the averages of the peer institutions.

The president of the university, Alva McMahon, established several task forces consisting of faculty and administrators to address these problems. These groups made several wide-ranging general recommendations, including the implementation of appropriate management practices and more in-depth, focused planning.

Discuss in general terms how forecasting might be used for university planning to address these specific problem areas. Include in your discussion the

types of forecasting methods
that might be used.

Case Problem

THE UNIVERSITY BOOKSTORE STUDENT COMPUTER PURCHASE PROGRAM

The University Bookstore is owned and operated by State University through an independent corporation with its own board of directors. The bookstore has three locations on or near the State University campus. It stocks a range of items, including textbooks, trade books, logo apparel,

drawing and educational supplies, and computers and related products, including printers, modems, and software. The bookstore has a program to sell personal computers to incoming freshmen and other students at a substantial educational discount, partly passed on from computer manufacturers. This means that the bookstore just covers computer costs, with a very small profit margin remaining.

Each summer all incoming freshmen and their parents

come to the State campus for a 3-day orientation program. The students come in groups of 100 throughout the summer. During their visit the students and their parents are given details about the bookstore's computer purchase program. Some students place their computer orders for the fall semester at this time, whereas others wait until later in the summer. The bookstore also receives orders from returning students throughout the summer. This program presents a challenging management problem for the

bookstore.

Orders come in throughout the summer, many only a few weeks before school starts in the fall, and the computer suppliers require at least 6 weeks for delivery. Thus, the bookstore must forecast computer demand to build up inventory to meet student demand in the fall. The student computer program and the forecast of computer demand have repercussions all along the bookstore supply chain. The bookstore has a warehouse near campus where it must

store all computers because it has no storage space at its retail locations. Ordering too many computers not only ties up the bookstore's cash reserves, it also takes up limited storage space and limits inventories for other bookstore products during the bookstore's busiest sales period. Because the bookstore has such a low profit margin on computers, its bottom line depends on these other products. Because competition for good students has increased, the university has become very quality

conscious and insists that all university facilities provide exemplary student service, which for the bookstore means meeting all student demands for computers when fall semester starts. The number of computers ordered also affects the number of temporary warehouse and bookstore workers who must be hired for handling and assisting with PC installations. The number of truck trips from the warehouse to the bookstore each day of fall registration is also affected by computer sales.

The bookstore student computer purchase program has been in place for 14 years. Although the student population has remained stable during this period, computer sales have been somewhat volatile. Following are the historical sales data for computers during the first month of fall registration:

<i>Year</i>	<i>Computers Sold</i>
--------------------	------------------------------

1	518
---	-----

2	651
---	-----

3 708

4 921

5 775

6 810

7 856

8 792

9 877

10 693

11	841
12	1,009
13	902
14	1,103

Develop an appropriate forecast model for bookstore management to use to forecast computer demand for next fall semester and indicate how accurate it appears to be. What

other forecasts might be useful to the bookstore?

Case Problem

VALLEY SWIM CLUB

The Valley Swim Club has 300 stockholders, each holding one share of stock in the club. A share of club stock allows the shareholder's family to use the club's heated outdoor pool during the summer, upon payment of annual membership dues of \$175. The club has not issued any stock in years, and only a few of the existing

shares come up for sale each year. The board of directors administers the sale of all stock. When a shareholder wants to sell, he or she turns the stock in to the board, which sells it to the person at the top of the waiting list. For the past few years, the length of the waiting list has remained relatively steady, at approximately 20 names.

However, during the past winter two events occurred that have suddenly increased the demand for shares in the club. The winter was especially severe,

and subzero weather and heavy ice storms caused both the town and the county pools to buckle and crack. The problems were not discovered until maintenance crews began to prepare the pools for the summer, and repairs cannot be completed until the fall. Also during the winter, the manager of the local country club had an argument with her board of directors and one night burned down the clubhouse. Although the pool itself was not damaged, the dressing room facilities, showers, and snack

bar were destroyed. As a result of these two events, the Valley Swim Club was inundated with applications to purchase shares. The waiting list suddenly grew to 250 people as the summer approached.

[Page 727]

The board of directors of the swim club had refrained from issuing new shares in the past because there never was a very great demand, and the demand that did exist was usually absorbed within a year by stock

turnover. In addition, the board has a real concern about overcrowding. It seemed like the present membership was about right, and there were very few complaints about overcrowding, except on holidays such as Memorial Day and the Fourth of July. However, at a recent board meeting, a number of new applicants had attended and asked the board to issue new shares. In addition, a number of current shareholders suggested that this might be an opportunity for the club to raise

some capital for needed repairs and to improve some of the existing facilities. This was tempting to the board.

Although it had set the share price at \$500 in the past, the board could set it at a much higher level now. In addition, an increase in attendance could create a need for more lifeguards.

Before the board of directors could make a decision on whether to sell more shares and, if so, how many, the board members felt they needed more information. Specifically,

they would like a forecast of the average number of people (family members, guests, etc.) who might attend the pool each day during the summer, with the current number of shares.

The board of directors has the following daily attendance records for June through August from the previous summer; it thinks the figures would provide accurate estimates for the upcoming summer:

M-	W-	F-193	Su-	T-177	Th-
----	----	-------	-----	-------	-----

139	380		399		238
T-273	Th-367	Sa-378	M-197	W-161	F-224
W-172	F-359	Su-461	T-273	Th-308	Sa-368
Th-275	Sa-463	M-242	W-213	F-256	Su-541
F-337	Su-578	T-177	Th-303	Sa-391	M-235
Sa-402	M-287	W-245	F-262	Su-400	T-218
Su-		Th-	Sa-		W-

487 T-247 390 447 M-224 271

M-198 W-356 F-284 Su-399 T-239 Th-259

T-310 Th-322 Sa-417 M-275 W-274 F-232

W-347 F-419 Su-474 T-241 Th-205 Sa-317

Th-393 Sa-516 M-194 W-190 F-361 Su-369

F-421 Su-478 T-207 Th-243 Sa-411 M-361

Sa- 595	M- 303	W- 215	F-277	Su- 419
------------	-----------	-----------	-------	------------

Su- 497	T-223	Th- 304	Sa- 241	M-258
------------	-------	------------	------------	-------

M- 341	W- 315	F-331	Su- 384	T-130
-----------	-----------	-------	------------	-------

T-291	Th- 258	Sa- 407	M-246	W- 195
-------	------------	------------	-------	-----------

Develop a forecasting model to forecast daily demand during the summer.

Chapter 16.

Inventory

Management

[Page 729]

Inventory analysis is one of the most popular topics in management science. One reason is that almost all types of business organizations have inventory. Although we tend to think of inventory only in terms of stock on a store shelf, it can take on a variety of forms, such

as partially finished products at different stages of a manufacturing process, raw materials, resources, labor, or cash. In addition, the purpose of inventory is not always simply to meet customer demand. For example, companies frequently stock large inventories of raw materials as a hedge against strikes. Whatever form inventory takes or whatever its purpose, it often represents a significant cost to a business firm. It is estimated that the average annual cost of

manufactured goods inventory in the United States is approximately 30% of the total value of the inventory. Thus, if a company has \$10.0 million worth of products in inventory, the cost of holding the inventory (including insurance, obsolescence, depreciation, interest, opportunity costs, storage costs, etc.) would be approximately \$3.0 million. If the amount of inventory could be reduced by half to \$5.0 million, then \$1.5 million would be saved in inventory costs, a significant cost reduction.

In this chapter we describe the classic economic order quantity models, which represent the most basic and fundamental form of inventory analysis.

These models provide a means for determining how much to order (the order quantity) and when to place an order so that inventory-related costs are minimized. The underlying assumption of these models is that demand is known with certainty and is constant. In addition, we will describe models for determining the order size and reorder points

(when to place an order) when demand is uncertain.

Elements of Inventory Management

Inventory is defined as a stock of items kept on hand by an organization to use to meet customer demand. Virtually every type of organization maintains some form of inventory. A department store carries inventories of all the retail items it sells; a nursery has inventories of different plants, trees, and flowers; a rental car agency has

inventories of cars; and a major league baseball team maintains an inventory of players on its minor league teams. Even a family household will maintain inventories of food, clothing, medical supplies, personal hygiene products, and so on.

Inventory is a stock
of items kept on
hand to meet
demand.

The Role of Inventory

A company or an organization keeps stocks of inventory for a variety of important reasons. The most prominent is holding finished goods inventories to meet customer demand for a product, especially in a retail operation. However, customer demand can also be in the form of a secretary going to a storage closet to get a printer cartridge or paper, or a carpenter getting a board or nail from a storage shed. A level of inventory is normally

maintained that will meet anticipated or expected customer demand. However, because demand is usually not known with certainty, additional amounts of inventory, called **safety**, or *buffer*, **stocks**, are often kept on hand to meet unexpected variations in excess of expected demand.

Additional stocks of inventories are sometimes built up to meet seasonal or cyclical demand. Companies will produce items when demand is low to meet high seasonal demand for

which their production capacity is insufficient. For example, toy manufacturers produce large inventories during the summer and fall to meet anticipated demand during the Christmas season. Doing so enables them to maintain a relatively smooth production flow throughout the year. They would not normally have the production capacity or logistical support to produce enough to meet all of the Christmas demand during that season. Correspondingly, retailers might find it necessary to keep large stocks of

inventory on their shelves to meet peak seasonal demand, such as at Christmas, or for display purposes to attract buyers.

[Page 730]

A company will often purchase large amounts of inventory to take advantage of price discounts, as a hedge against anticipated future price increases, or because it can get a lower price by purchasing in volume. For example, Wal-Mart has long been known to

purchase an entire manufacturer's stock of soap powder or other retail items because it can get a very low price, which it subsequently passes on to its customers. Companies will often purchase large stocks of items when a supplier liquidates to get a low price. In some cases, large orders will be made simply because the cost of an order may be very high, and it is more cost-effective to have higher inventories than to make a lot of orders.

Many companies find it

necessary to maintain in-process inventories at different stages in a manufacturing process to provide independence between operations and to avoid work stoppages or delays.

Inventories of raw materials and purchased parts are kept on hand so that the production process will not be delayed as a result of missed or late deliveries or shortages from a supplier. Work-in-process inventories are kept between stages in the manufacturing process so that production can

continue smoothly if there are temporary machine breakdowns or other work stoppages. Similarly, a stock of finished parts or products allows customer demand to be met in the event of a work stoppage or problem with the production process.

Demand

A crucial component and the basic starting point for the management of inventory is customer demand. Inventory

exists for the purpose of meeting the demand of customers. Customers can be inside the organization, such as a machine operator waiting for a part or a partially completed product to work on, or outside the organization, such as an individual purchasing groceries or a new stereo. As such, an essential determinant of effective inventory management is an accurate forecast of demand. For this reason the topics of forecasting ([Chapter 15](#)) and inventory management are directly

interrelated.

In general, the demand for items in inventory is classified as dependent or independent.

Dependent demand items are typically component parts, or materials, used in the process of producing a final product. For example, if an automobile company plans to produce 1,000 new cars, it will need 5,000 wheels and tires (including spares). In this case the demand for wheels is dependent on the production of cars; that is, the demand for one item is a function of

demand for another item.

Dependent demand
items are used
internally to produce
a final product.

Alternatively, cars are an example of an **independent demand** item. In general, independent demand items are final or finished products that are not a function of, or dependent upon, internal

production activity.

Independent demand is usually external, and, thus, beyond the direct control of the organization. In this chapter we will focus on the management of inventory for independent demand items.

Independent demand items are final products demanded by an external customer.

Inventory Costs

There are three basic costs associated with inventory: carrying (or holding) costs, ordering costs, and shortage costs. **Carrying costs** are the costs of holding items in storage. These vary with the level of inventory and occasionally with the length of time an item is held; that is, the greater the level of inventory over time, the higher the carrying cost(s). Carrying costs can include the cost of losing the use of funds tied up

in inventory; direct storage costs, such as rent, heating, cooling, lighting, security, refrigeration, record keeping, and logistics; interest on loans used to purchase inventory; depreciation; obsolescence as markets for products in inventory diminish; product deterioration and spoilage; breakage; taxes; and pilferage.

*Inventory costs include **carrying, ordering, and shortage costs.***

Carrying costs are normally specified in one of two ways. The most general form is to assign total carrying costs, determined by summing all the individual costs mentioned previously, on a per-unit basis per time period, such as a month or a year. In this form, carrying costs would commonly be expressed as a per-unit dollar amount on an annual basis (for example, \$10 per

year). Alternatively, carrying costs are sometimes expressed as a percentage of the value of an item or as a percentage of average inventory value. It is generally estimated that carrying costs range from 10% to 40% of the value of a manufactured item.

Ordering costs are the costs associated with replenishing the stock of inventory being held. These are normally expressed as a dollar amount per order and are independent of the order size. Thus, ordering costs vary with the

number of orders made (i.e., as the number of orders increases, the ordering cost increases). Costs incurred each time an order is made can include requisition costs, purchase orders, transportation and shipping, receiving, inspection, handling and placing in storage, and accounting and auditing.

Ordering costs generally react inversely to carrying costs. As the size of orders increases, fewer orders are required, thus reducing annual ordering costs. However, ordering larger

amounts results in higher inventory levels and higher carrying costs. In general, as the order size increases, annual ordering costs decrease and annual carrying costs increase.

Shortage costs, also referred to as *stockout costs*, occur when customer demand cannot be met because of insufficient inventory on hand. If these shortages result in a permanent loss of sales for items demanded but not provided, shortage costs include the loss of profits.

Shortages can also cause customer dissatisfaction and a loss of goodwill that can result in a permanent loss of customers and future sales. In some instances the inability to meet customer demand or lateness in meeting demand results in specified penalties in the form of price discounts or rebates. When demand is internal, a shortage can cause work stoppages in the production process and create delays, resulting in downtime costs and the cost of lost production (including indirect

and direct production costs).

Costs resulting from immediate or future lost sales because demand could not be met are more difficult to determine than carrying or ordering costs. As a result, shortage costs are frequently subjective estimates and many times no more than educated guesses.

Shortages occur because it is costly to carry inventory in stock. As a result, shortage costs have an inverse relationship to carrying costs; as the amount of inventory on

hand increases, the carrying cost increases, while shortage costs decrease.

The objective of **inventory management** is to employ an inventory control system that will indicate how much should be ordered and when orders should take place to minimize the sum of the three inventory costs described here.

*The purpose of
**inventory
management** is to
determine how much*

and when to order.

Inventory Control Systems

An inventory system is a structure for controlling the level of inventory by determining how much to order (the level of replenishment) and when to order. There are two basic types of inventory systems: a *continuous* (or *fixed order quantity*) system and a *periodic* (or *fixed time period*) *system*. The primary difference between the two systems is

that in a continuous system, an order is placed for the same constant amount whenever the inventory on hand decreases to a certain level, whereas in a periodic system, an order is placed for a variable amount after an established passage of time.

Continuous Inventory Systems

In a *continuous inventory system*, alternatively referred to as a *perpetual system* or a

fixed order quantity system, a continual record of the inventory level for every item is maintained. Whenever the inventory on hand decreases to a predetermined level, referred to as the *reorder point*, a new order is placed to replenish the stock of inventory. The order that is placed is for a "fixed" amount that minimizes the total inventory carrying, ordering, and shortage costs. This fixed order quantity is called the *economic order quantity*; its determination will be discussed in greater detail in

a later section.

[Page 732]

In a continuous inventory system, a constant amount is ordered when inventory declines to a predetermined level.

A positive feature of a continuous system is that the

inventory level is closely and continuously monitored so that management always knows the inventory status. This is especially advantageous for critical inventory items such as replacement parts or raw materials and supplies. However, the cost of maintaining a continual record of the amount of inventory on hand can also be a disadvantage of this type of system.

A simple example of a continuous inventory system is a ledger-style checkbook that

many of us use on a daily basis. Our checkbook comes with 300 checks; after the 200th check has been used (and there are 100 left), there is an order form for a new batch of checks that has been inserted by the printer. This form, when turned in at the bank, initiates an order for a new batch of 300 checks from the printer. Many office inventory systems use "reorder" cards that are placed within stacks of stationery or at the bottom of a case of pens or paper clips to signal when a

new order should be placed. If you look behind the items on a hanging rack in a Kmart store, you will see a card indicating that it is time to place an order for this item, for an amount indicated on the card.

A more sophisticated example of a continuous inventory system is a computerized checkout system with a laser scanner, used by many supermarkets and retail stores. In this system a laser scanner reads the Universal Product Code (UPC), or bar code, off the product package, and the

transaction is instantly recorded and the inventory level updated. Such a system is not only quick and accurate, but it also provides management with continuously updated information on the status of inventory levels. Although not as publicly visible as supermarket systems, many manufacturing companies, suppliers, and distributors also use bar code systems and handheld laser scanners to inventory materials, supplies, equipment, in-process parts, and finished goods.

Because continuous inventory systems are much more common than periodic systems, models that determine fixed order quantities and the time to order will receive most of our attention in this chapter.

Periodic Inventory Systems

In a **periodic inventory system**, also referred to as a *fixedtime period system* or *periodic review system*, the inventory on hand is counted at

specific time intervals for example, every week or at the end of each month. After the amount of inventory in stock is determined, an order is placed for an amount that will bring inventory back up to a desired level. In this system the inventory level is not monitored at all during the time interval between orders, so it has the advantage of requiring little or no record keeping. However, it has the disadvantage of less direct control. This typically results in larger inventory levels for a

periodic inventory system than in a continuous system, to guard against unexpected stockouts early in the fixed period. Such a system also requires that a new order quantity be determined each time a periodic order is made.

*In a **periodic inventory system**, an order is placed for a variable amount after a fixed passage of time.*

Periodic inventory systems are often found at a college or university bookstore. Textbooks are normally ordered according to a periodic system, wherein a count of textbooks in stock (for every course) is made after the first few weeks or month during the semester or quarter. An order for new textbooks for the next semester is then made according to estimated course enrollments for the next term (i.e., demand) and the number remaining in stock. Smaller

retail stores, drugstores, grocery stores, and offices often use periodic systems; the stock level is checked every week or month, often by a vendor, to see how much (if anything) should be ordered.

[Page 733]

Time Out: For Ford Harris

The earliest published derivation of the classic economic lot size model is credited to Ford Harris of Westinghouse Corporation in 1915. His equation determined a minimum sum of inventory costs and setup costs, given demand that was known and constant and a rate of production that was assumed to be higher than demand.

Economic Order Quantity Models

You will recall that in a continuous or fixed order quantity system, when inventory reaches a specific level, referred to as the *reorder point*, a fixed amount is ordered. The most widely used and traditional means for determining how much to order in a continuous system is the **economic order quantity (EOQ)** model, also referred to

as the economic lot size model.

EOQ is a continuous inventory system.

The function of the EOQ model is to determine the optimal order size that minimizes total inventory costs. There are several variations of the EOQ model, depending on the assumptions made about the inventory system. In this and following sections we will

describe three model versions:
the basic EOQ model, the EOQ
model with noninstantaneous
receipt, and the EOQ model
with shortages.

The Basic EOQ Model

The simplest form of the economic order quantity model on which all other model versions are based is called the basic EOQ model. It is essentially a single formula for determining the optimal order size that minimizes the sum of carrying costs and ordering costs. The model formula is derived under a set of simplifying and restrictive assumptions, as follows:

- Demand is known with certainty and is relatively constant over time.
- No shortages are allowed.
- Lead time for the receipt of orders is constant.
- The order quantity is received all at once.

EOQ is the optimal order quantity that will minimize total inventory costs.

Assumptions of the EOQ model include constant demand, no shortages, constant lead time, and instantaneous order receipt.

The graph in [Figure 16.1](#) reflects these basic model assumptions.

Figure 16.1. The inventory order cycle

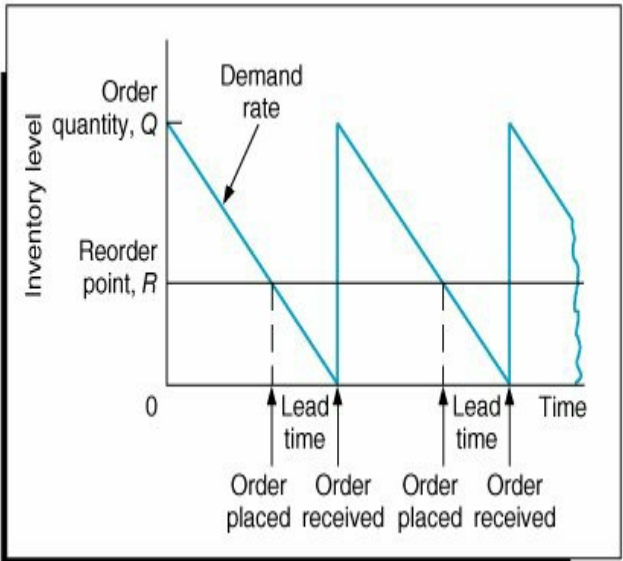


Figure 16.1 describes the continuous inventory order cycle system inherent in the EOQ model. An order quantity, Q , is received and is used up over time at a constant rate. When the inventory level decreases to the reorder point, R , a new order is placed, and a period of time, referred to as the *lead time*, is required for delivery. The order is received all at once, just at the moment

when demand depletes the entire stock of inventory (and the inventory level reaches zero), thus allowing no shortages. This cycle is continuously repeated for the same order quantity, reorder point, and lead time.

As we mentioned earlier, Q is the order size that minimizes the sum of carrying costs and holding costs. These two costs react inversely to each other in response to an increase in the order size. As the order size increases, fewer orders are required, causing the ordering

cost to decline, whereas the average amount of inventory on hand increases, resulting in an increase in carrying costs. Thus, in effect, the optimal order quantity represents a compromise between these two conflicting costs.

Carrying Cost

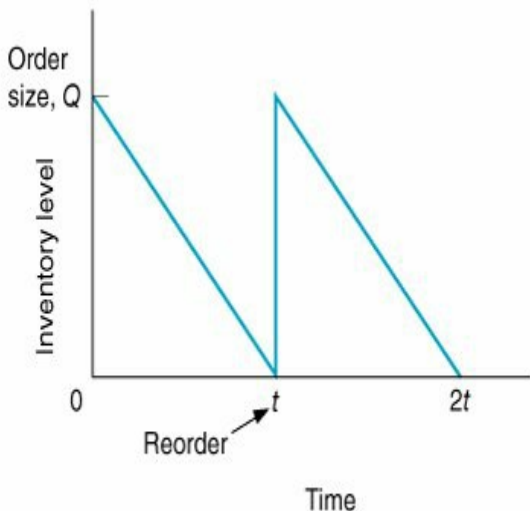
Carrying cost is usually expressed on a per-unit basis for some period of time (although it is sometimes given as a percentage of average

inventory). Traditionally, the carrying cost is referred to on an annual basis (i.e., per year).

The total carrying cost is determined by the amount of inventory on hand during the year. The amount of inventory available during the year is illustrated in [Figure 16.2](#).

Figure 16.2.

Inventory usage



In [Figure 16.2](#), Q represents

the size of the order needed to replenish inventory, which is what a manager wants to determine. The line connecting Q to time, t , in our graph represents the rate at which inventory is depleted, or *demand*, during the time period, t . Demand is assumed to be *known with certainty* and is thus constant, which explains why the line representing demand is straight. Also, notice that inventory never goes below zero; shortages do not exist. In addition, when the inventory level does reach

zero, it is assumed that an order arrives immediately after an infinitely small passage of time, a condition referred to as **instantaneous receipt**. This is a simplifying assumption that we will maintain for the moment.

Referring to [Figure 16.2](#), we can see that the amount of inventory is Q , the size of the order, for an infinitely small period of time because Q is always being depleted by demand. Similarly, the amount of inventory is zero for an infinitely small period of time

because the only time there is no inventory is at the specific time t . Thus, the amount of inventory available is somewhere between these two extremes. A logical deduction is that the amount of inventory available is the *average inventory* level, defined as

[Page 735]

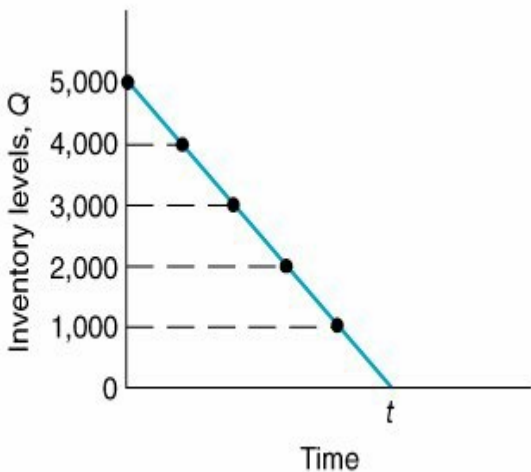
$$\text{average inventory} = \frac{Q}{2}$$

To verify this relationship, we can specify any number of points/values of Q over the

entire time period, t , and divide by the number of points. For example, if $Q = 5,000$, the six points designated from 5,000 to 0, as shown in [Figure 16.3](#), are summed and divided by 6:

$$\begin{aligned}\text{average inventory} &= \frac{5,000 + 4,000 + 3,000 + 2,000 + 1,000 + 0}{6} \\ &= 2,500\end{aligned}$$

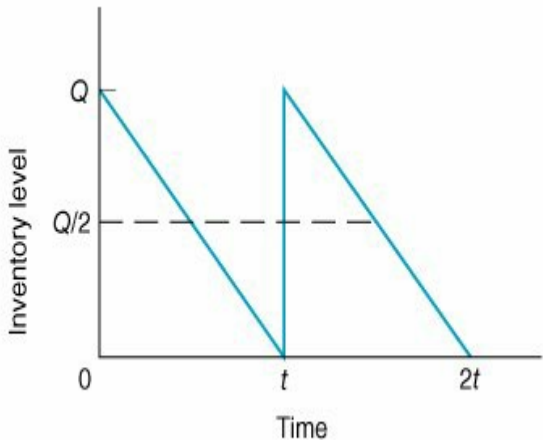
Figure 16.3. Levels of Q



Alternatively, we can sum just the two extreme points (which

also encompass the range of time, t) and divide by 2. This also equals 2,500. This computation is the same, in principle, as adding Q and 0 and dividing by 2, which equals $Q/2$. This relationship for average inventory is maintained, regardless of the size of the order, Q , or the frequency of orders (i.e., the time period, t). Thus, the average inventory on an *annual basis* is also $Q/2$, as shown in [Figure 16.4](#).

Figure 16.4. Annual average inventory



Now that we know that the amount of inventory available *on an annual basis* is the average inventory, $Q/2$, we can determine the total annual carrying cost by multiplying the average number of units in inventory by the carrying cost per unit per year, C_c :

$$\text{annual carrying cost} = C_c \frac{Q}{2}$$

Ordering Cost

The total annual ordering cost is computed by multiplying the cost per order, designated as C_o , by the number of orders per year. Because annual demand is assumed to be known and constant, the number of orders will be D/Q , where Q is the order size:

$$\text{annual ordering cost} = C_o \frac{D}{Q}$$

The only variable in this equation is Q ; both C_o and D

are constant parameters. In other words, demand is known with certainty. Thus, the relative magnitude of the ordering cost is dependent upon the order size.

Total Inventory Cost

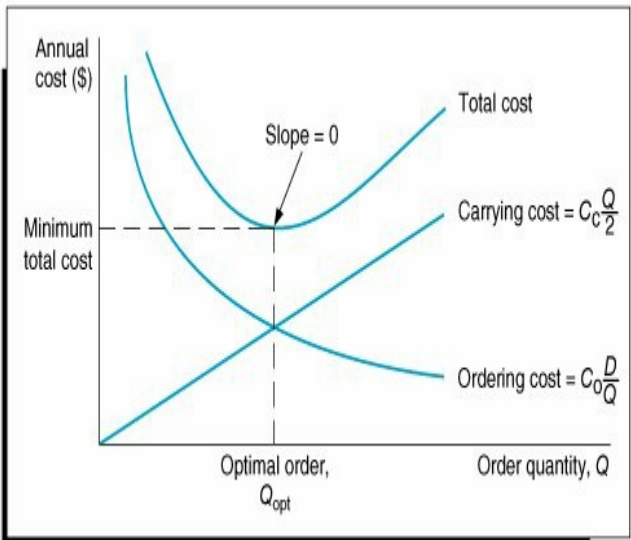
The total annual inventory cost is simply the sum of the ordering and carrying costs:

$$TC = C_o \frac{D}{Q} + C_c \frac{Q}{2}$$

These cost functions are shown

in [Figure 16.5](#). Notice the inverse relationship between ordering cost and carrying cost, resulting in a convex total cost curve.

Figure 16.5. The EOQ cost model



Observe the general upward trend of the total carrying cost curve. As the order size Q

(shown on the horizontal axis) increases, the total carrying cost (shown on the vertical axis) increases. This is logical because larger orders will result in more units carried in inventory. Next, observe the ordering cost curve in [Figure 16.5](#). As the order size, Q , increases, the ordering cost *decreases* (just the opposite of what occurred with the carrying cost). This is logical because an increase in the size of the orders will result in fewer orders being placed each year. Because one cost increases as

the other decreases, the result of summing the two costs is a convex total cost curve.

[Page 737]

The optimal order quantity occurs at the point in [Figure 16.5](#) where the total cost curve is at a minimum, which also coincides exactly with the point where the ordering cost curve intersects with the carrying cost curve. This enables us to determine the optimal value of Q by equating the two cost functions and solving for Q , as

follows:

$$C_o \frac{D}{Q} = C_c \frac{Q}{2}$$
$$Q^2 = \frac{2C_o D}{C_c}$$
$$Q_{\text{opt}} = \sqrt{\frac{2C_o D}{C_c}}$$

The optimal value of Q corresponds to the lowest point on the total cost curve.

Alternatively, the optimal value

of Q can be determined by differentiating the total cost curve with respect to Q , setting the resulting function equal to zero (the slope at the minimum point on the total cost curve), and solving for Q , as follows:

$$TC = C_o \frac{D}{Q} + C_c \frac{Q}{2}$$

$$\frac{\delta TC}{\delta Q} = -\frac{C_o D}{Q^2} + \frac{C_c}{2}$$

$$0 = -\frac{C_o D}{Q^2} + \frac{C_c}{2}$$

$$Q_{\text{opt}} = \sqrt{\frac{2C_o D}{C_c}}$$

The total minimum cost is

determined by substituting the value for the optimal order size, Q_{opt} , into the total cost equation:

$$TC_{\min} = \frac{C_o D}{Q_{\text{opt}}} + C_o \frac{Q_{\text{opt}}}{2}$$

We will use the following example to demonstrate how the optimal value of Q is computed. The I-75 Carpet Discount Store in north Georgia stocks carpet in its warehouse and sells it through an adjoining showroom. The store keeps several brands and styles of carpet in stock; however, its

biggest seller is Super Shag carpet. The store wants to determine the optimal order size and total inventory cost for this brand of carpet, given an estimated annual demand of 10,000 yards of carpet, an annual carrying cost of \$0.75 per yard, and an ordering cost of \$150. The store would also like to know the number of orders that will be made annually and the time between orders (i.e., the order cycle), given that the store is open every day except Sunday, Thanksgiving Day, and

Christmas Day (which is not on a Sunday).

We can summarize the model parameters as follows:

$$C_c = \$0.75$$

$$C_o = \$150$$

$$D = 10,000 \text{ yd.}$$

[Page 738]

The optimal order size is computed as follows:

$$Q_{\text{opt}} = \sqrt{\frac{2C_o D}{C_c}}$$

$$= \sqrt{\frac{2(150)(10,000)}{(0.75)}}$$

$$Q_{\text{opt}} = 2,000 \text{ yd.}$$

The total annual inventory cost is determined by substituting Q_{opt} into the total cost formula, as follows:

$$TC_{\text{min}} = C_o \frac{D}{Q_{\text{opt}}} + C_c \frac{Q_{\text{opt}}}{2}$$

$$= (150) \frac{10,000}{2,000} + (0.75) \frac{(2,000)}{2}$$

$$= \$750 + 750$$

$$TC_{\text{min}} = \$1,500$$

The number of orders per year is computed as follows:

$$\text{number of orders per year} = \frac{D}{Q_{\text{opt}}} = \frac{10,000}{2,000} = 5$$

Given that the store is open 311 days annually (365 days minus 52 Sundays, plus Thanksgiving and Christmas), the order cycle is determined as follows:

$$\text{order cycle time} = \frac{311 \text{ days}}{D/Q_{\text{opt}}} = \frac{311}{5} = 62.2 \text{ store days}$$

It should be noted that the optimal order quantity

determined in this example, and in general, is an approximate value because it is based on estimates of carrying and ordering costs as well as uncertain demand (although all these parameters are treated as known, certain values in the EOQ model). Thus, in practice it is acceptable to round off the Q values to the nearest whole number. The precision of a decimal place generally is neither necessary nor appropriate. In addition, because the optimal order quantity is computed from a

square root, errors or variations in the cost parameters and demand tend to be dampened. For instance, if the order cost had actually been a third higher, or \$200, the resulting optimal order size would have varied by about 15% (i.e., 2,390 yards instead of 2,000 yards). In addition, variations in both inventory costs will tend to offset each other because they have an inverse relationship. As a result, the EOQ model is relatively robust, or resilient to errors in the cost estimates and

demand, which has tended to enhance its popularity.

The EOQ model is robust; because Q is a square root, errors in the estimation of D , C_c , and C_o are dampened.

EOQ Analysis Over Time

One aspect of inventory

analysis that can be confusing is the time frame encompassed by the analysis. Therefore, we will digress for just a moment to discuss this aspect of EOQ analysis.

Recall that previously we developed the EOQ model "regardless of order size, Q , and time, t ." Now we will verify this condition. We will do so by developing our EOQ model on a *monthly basis*. First, demand is equal to 833.3 yards per month (which we determined by dividing the annual demand of 10,000 yards by 12 months).

Next, by dividing the annual carrying cost, C_c , of \$0.75 by 12, we get the monthly (per-unit) carrying cost: $C_c =$ \$0.0625. (The ordering cost of \$150 is not related to time.) We thus have the values

[Page 739]

$D = 833.3$ yd. per month

$C_c = \$0.0625$ per yd. per month

$C_o = \$150$ per order

which we can substitute into our EOQ formula:

$$\begin{aligned} Q_{\text{opt}} &= \sqrt{\frac{2C_o D}{C_c}} \\ &= \sqrt{\frac{2(150)(833.3)}{(0.0625)}} = 2,000 \text{ yd.} \end{aligned}$$

This is the same optimal order size that we determined on an annual basis. Now we will compute total monthly inventory cost:

$$\begin{aligned}
 \text{total monthly inventory cost} &= C_c \frac{Q_{\text{opt}}}{2} + C_o \frac{D}{Q_{\text{opt}}} \\
 &= (\$0.0625) \frac{(2,000)}{2} + (\$150) \frac{(833.3)}{(2,000)} \\
 &= \$125 \text{ per month}
 \end{aligned}$$

To convert this monthly total cost to an annual cost, we multiply it by 12 (months):

$$\begin{aligned}
 \text{total annual inventory cost} &= \\
 &(\$125)(12) = \$1,500
 \end{aligned}$$

This brief example demonstrates that regardless of the time period encompassed by EOQ analysis, the economic order quantity (Q_{opt}) is the

same.

The EOQ Model with Noninstantaneous Receipt

A variation of the basic EOQ model is achieved when the assumption that orders are received all at once is relaxed. This version of the EOQ model is known as the **noninstantaneous receipt model**, also referred to as the *gradual usage*, or **production lot size, model**. In this EOQ

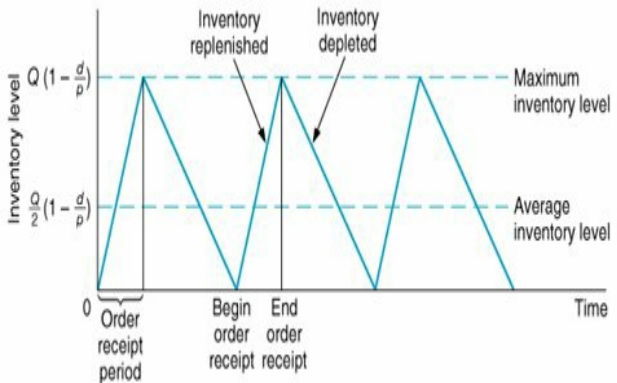
variation, the order quantity is received gradually over time and the inventory level is depleted at the same time it is being replenished. This is a situation most commonly found when the inventory user is also the producer, as, for example, in a manufacturing operation where a part is produced to use in a larger assembly. This situation can also occur when orders are delivered gradually over time or the retailer and producer of a product are one and the same. The noninstantaneous receipt

model is illustrated graphically in [Figure 16.6](#), which highlights the difference between this variation and the basic EOQ model.

Figure 16.6. The EOQ model with noninstantaneous order receipt

(This item is displayed on page 740 in the print version)

[\[View full size image\]](#)



*The
**noninstantaneous
receipt model**
relaxes the*

assumption that Q is received all at once.

The ordering cost component of the basic EOQ model does not change as a result of the gradual replenishment of the inventory level because it is dependent only on the number of annual orders. However, the carrying cost component is not the same for this model variation because average inventory is different. In the

basic EOQ model, average inventory was half the maximum inventory level, or $Q/2$, but in this variation, the maximum inventory level is not simply Q ; it is an amount somewhat lower than Q , adjusted for the fact that the order quantity is depleted during the order receipt period.

[Page 740]

To determine the average inventory level, we define the following parameters that are unique to this model:

p = daily rate at which the order is received over time, also known as the *production rate*

d = the daily rate at which inventory is demanded

The demand rate cannot exceed the production rate because we are still assuming that no shortages are possible, and if $d = p$, then there is no order size because items are used as fast as they are produced. Thus, for this model, the production rate must exceed the demand rate, or $p >$

d.

Observing [Figure 16.6](#), the time required to receive an order is the order quantity divided by the rate at which the order is received, or Q/p . For example, if the order size is 100 units and the production rate, p , is 20 units per day, the order will be received in 5 days. The amount of inventory that will be depleted or used up during this time period is determined by multiplying by the demand rate, or $(Q/p)d$. For example, if it takes 5 days to receive the order and during

this time inventory is depleted at the rate of 2 units per day, then a total of 10 units is used. As a result, the maximum amount of inventory that is on hand is the order size minus the amount depleted during the receipt period, computed as follows and shown earlier in [Figure 16.6](#):

$$\begin{aligned}\text{maximum inventory level} &= Q - \frac{Q}{p}d \\ &= Q\left(1 - \frac{d}{p}\right)\end{aligned}$$

Because this is the maximum inventory level, the average

inventory level is determined by dividing this amount by 2, as follows:

$$\begin{aligned}\text{average inventory level} &= \frac{1}{2} \left[Q \left(1 - \frac{d}{p} \right) \right] \\ &= \frac{Q}{2} \left(1 - \frac{d}{p} \right)\end{aligned}$$

[Page 741]

The total carrying cost, using this function for average inventory, is

$$\text{total carrying cost} = C_c \frac{Q}{2} \left(1 - \frac{d}{p} \right)$$

Thus, the total annual inventory cost is determined according to the following formula:

$$TC = C_o \frac{D}{Q} + C_c \frac{Q}{2} \left(1 - \frac{d}{p} \right)$$

The total inventory cost is a function of two other costs, just as in our previous EOQ model. Thus, the minimum inventory cost occurs when the total cost curve is lowest and where the carrying cost curve and ordering cost curve intersect (see [Figure 16.5](#)). Therefore, to find optimal Q_{opt} , we equate

total carrying cost with total ordering cost:

$$C_c \frac{Q}{2} \left(1 - \frac{d}{p}\right) = C_o \frac{D}{Q}$$

$$C_c \frac{Q^2}{2} \left(1 - \frac{d}{p}\right) = C_o D$$

$$Q_{\text{opt}} = \sqrt{\frac{2C_o D}{C_c(1 - d/p)}}$$

For our previous example we will now assume that the I-75 Carpet Discount Store has its own manufacturing facility, in which it produces Super Shag carpet. We will further assume that the ordering cost, C_o , is

the cost of setting up the production process to make Super Shag carpet. Recall that $C_c = \$0.75$ per yard and $D = 10,000$ yards per year. The manufacturing facility operates the same days the store is open (i.e., 311 days) and produces 150 yards of the carpet per day. The optimal order size, the total inventory cost, the length of time to receive an order, the number of orders per year, and the maximum inventory level are computed as follows:

$$C_o = \$150$$

$$C_c = \$0.75 \text{ per unit}$$

$$D = 10,000 \text{ yd.}$$

$$d = \frac{10,000}{311} = 32.2 \text{ yd. per day}$$

$$p = 150 \text{ yd. per day}$$

The optimal order size is determined as follows:

$$\begin{aligned} Q_{\text{opt}} &= \sqrt{\frac{2C_o D}{C_c \left(1 - \frac{d}{p}\right)}} \\ &= \sqrt{\frac{2(150)(10,000)}{0.75 \left(1 - \frac{32.2}{150}\right)}} \end{aligned}$$

$$Q_{\text{opt}} = 2,256.8 \text{ yd.}$$

This value is substituted into the following formula to determine total minimum annual inventory cost:

$$\begin{aligned} TC_{\min} &= C_o \frac{D}{Q} + C_c \frac{Q}{2} \left(1 - \frac{d}{p} \right) \\ &= (150) \frac{(10,000)}{(2,256.8)} + (0.75) \frac{(2,256.8)}{2} \left(1 - \frac{32.2}{150} \right) \\ &= \$1,329 \end{aligned}$$

The length of time to receive an order for this type of manufacturing operation is commonly called the length of the *production run*. It is

computed as follows:

$$\text{production run length} = \frac{Q}{p} = \frac{2,256.8}{150} = 15.05 \text{ days}$$

The number of orders per year is actually the number of production runs that will be made, computed as follows:

$$\begin{aligned}\text{number of production runs} &= \frac{D}{Q} \\ &= \frac{10,000}{2,256.8} \\ &= 4.43 \text{ runs}\end{aligned}$$

Finally, the maximum inventory level is computed as follows:

$$\begin{aligned}\text{maximum inventory level} &= Q\left(1 - \frac{d}{p}\right) \\ &= 2,256.8\left(1 - \frac{32.2}{150}\right) \\ &= 1,722 \text{ yd.}\end{aligned}$$

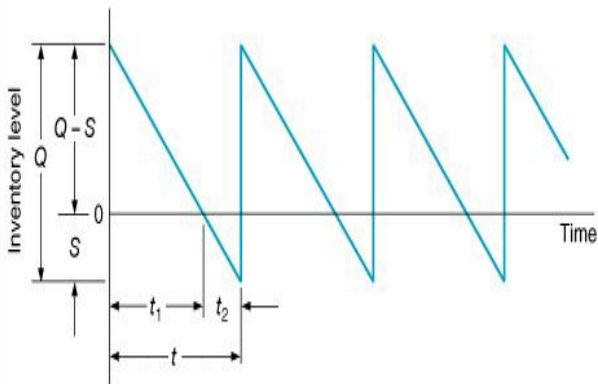
The EOQ Model with Shortages

One of the assumptions of our basic EOQ model is that shortages and back ordering are not allowed. The third model variation that we will describe, the EOQ model with shortages, relaxes this assumption. However, it will be assumed that all demand not met because of inventory shortage can be back ordered and delivered to the customer

later. Thus, all demand is eventually met. The EOQ model with shortages is illustrated in [Figure 16.7](#).

Figure 16.7. The EOQ model with shortages

(This item is displayed on page 743 in the print version)



The EOQ model with shortages relaxes the assumption that shortages cannot

exist.

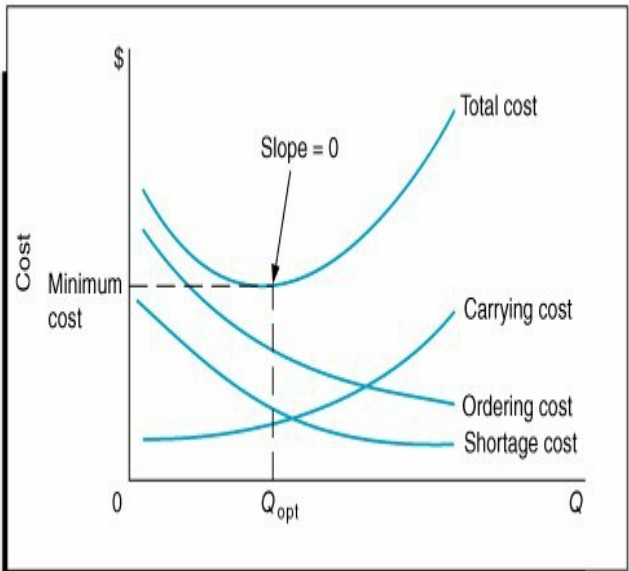
Because back-ordered demand, or shortages, S , are filled when inventory is replenished, the maximum inventory level does not reach Q , but instead a level equal to $Q - S$. It can be seen from [Figure 16.7](#) that the amount of inventory on hand ($Q - S$) decreases as the amount of the shortage increases, and vice versa. Therefore, the cost associated with shortages,

which we described earlier in this chapter as primarily the cost of lost sales and lost customer goodwill, has an inverse relationship to carrying costs. As the order size, Q , increases, the carrying cost increases and the shortage cost declines. This relationship between carrying and shortage cost as well as ordering cost is shown in [Figure 16.8](#).

[Page 743]

Figure 16.8. Cost

model with shortages



We will forgo the lengthy derivation of the individual cost components of the EOQ model with shortages, which requires the application of plane geometry to the graph in [Figure 16.8](#). The individual cost functions are provided as follows, where S equals the shortage level and C_s equals the annual per-unit cost of shortages:

$$\text{total shortage costs} = C_s \frac{S^2}{2Q}$$

$$\text{total carrying cost} = C_c \frac{(Q-S)^2}{2Q}$$

$$\text{total ordering cost} = C_o \frac{D}{Q}$$

Combining these individual cost components results in the total inventory cost formula:

$$TC = C_s \frac{S^2}{2Q} + C_c \frac{(Q-S)^2}{2Q} + C_o \frac{D}{Q}$$

[Page 744]

You will notice in [Figure 16.8](#) that the three cost component

curves do not intersect at a common point, as was the case in the basic EOQ model. As a result, the only way to determine the optimal order size *and the optimal shortage level, S* , is to differentiate the total cost function with respect to Q and S , set the two resulting equations equal to zero, and solve them simultaneously. Doing so results in the following formulas for the optimal order quantity and shortage level:

$$Q_{\text{opt}} = \sqrt{\frac{2C_o D}{C_c} \left(\frac{C_s + C_c}{C_s} \right)}$$

$$S_{\text{opt}} = Q_{\text{opt}} \left(\frac{C_c}{C_c + C_s} \right)$$

For example, we will now assume that the I-75 Carpet Discount Store allows shortages and the shortage cost, C_s , is \$2 per yard per year. All other costs and demand remain the same ($C_c = \$0.75$, $C_o = \$150$, and $D = 10,000$ yd.). The optimal order size and shortage level and total minimum annual inventory cost are computed as follows:

$$C_o = \$150$$

$$C_c = \$0.75 \text{ per yd.}$$

$$C_s = \$2 \text{ per yd.}$$

$$D = 10,000 \text{ yd.}$$

$$\begin{aligned} Q_{\text{opt}} &= \sqrt{\frac{2C_o D \left(\frac{C_s + C_c}{C_s} \right)}{C_c}} \\ &= \sqrt{\frac{2(150)(10,000)}{0.75} \left(\frac{2 + 0.75}{2} \right)} \\ &= 2,345.2 \text{ yd} \end{aligned}$$

$$\begin{aligned} S_{\text{opt}} &= Q_{\text{opt}} \left(\frac{C_c}{C_c + C_s} \right) \\ &= 2,345.2 \left(\frac{0.75}{2 + 0.75} \right) \\ &= 639.6 \text{ yd} \end{aligned}$$

$$\begin{aligned} TC_{\min} &= \frac{C_s S^2}{2Q} + \frac{C_c (Q - S)^2}{2Q} + C_o \frac{D}{Q} \\ &= \frac{(2)(639.6)^2}{2(2,345.2)} + \frac{(0.75)(1,705.6)^2}{2(2,345.2)} + \frac{(150)(10,000)}{2,345.2} \\ &= \$174.44 + 465.16 + 639.60 \\ &= \$1,279.20 \end{aligned}$$

Several additional parameters of the EOQ model with shortages can be computed for this example, as follows:

$$\text{number of orders} = \frac{D}{Q} = \frac{10,000}{2,345.2} = 4.26 \text{ orders per year}$$

$$\text{maximum inventory level} = Q - S = 2,345.2 - 639.6 = 1,705.6 \text{ yd.}$$

[Page 745]

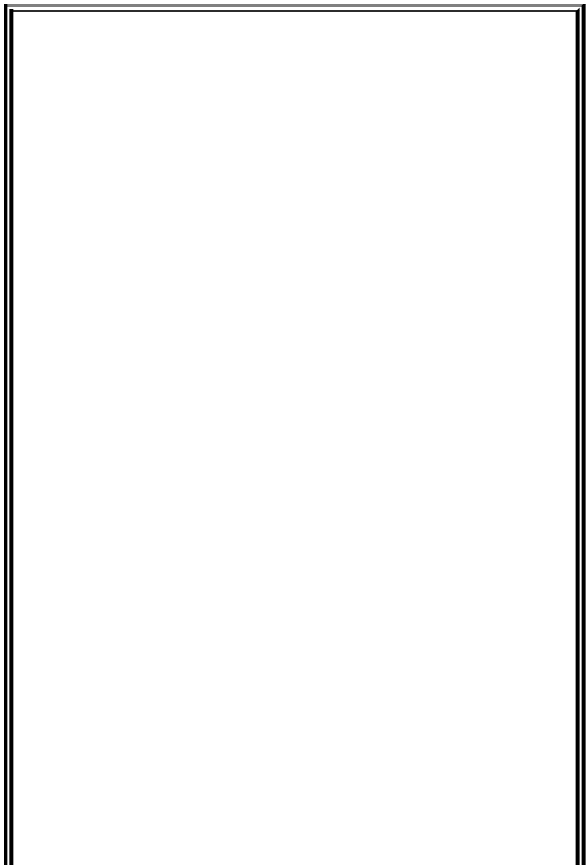
The time between orders, identified as t in [Figure 16.7](#), is computed as follows:

$$t = \frac{\text{days per year}}{\text{number of orders}} = \frac{311}{4.26} = 73.0 \text{ days between orders}$$

The time during which inventory is on hand, t_1 in [Figure 16.7](#), and the time during which there is a shortage, t_2 in [Figure 16.7](#), during each order cycle can be computed using the following formulas:

$$\begin{aligned}t_1 &= \frac{Q - S}{D} \\&= \frac{2,345.2 - 639.6}{10,000} \\&= 0.171 \text{ year, or } 53.2 \text{ days}\end{aligned}$$

$$\begin{aligned}t_2 &= \frac{S}{D} \\&= \frac{639.6}{10,000} \\&= 0.064 \text{ year, or } 19.9 \text{ days}\end{aligned}$$



Management Science

Application: Determining Inventory Ordering Policy at Dell

Dell Inc., the world's largest computer-systems company, bypasses retailers and sells directly to customers via phone or the Internet. After an order is processed, it is sent to one of its assembly plants in Austin, Texas, where the product is built, tested, and packaged within 8 hours.

Dell carries very little components inventory itself. Technology changes occur so fast that holding inventory can be a huge liability; some components lose 0.5% to 2.0% of

their value per week. In addition, many of Dell's suppliers are located in Southeast Asia, and their shipping times to Austin range from 7 days for air transport to 30 days for water and ground transport. To compensate for these factors, Dell's suppliers keep inventory in small warehouses called "revolvers" (for revolving inventory), which are a few miles from Dell's assembly plants. Dell keeps very little inventory at its own plants, so it withdraws inventory from the revolvers every few hours, while most Dell's suppliers deliver to their revolvers three times per week.

The cost of carrying inventory by Dell's suppliers is ultimately reflected in the final price of a computer. Thus, in order to maintain a competitive price advantage in the market, Dell strives to help its suppliers reduce inventory costs. Dell has a vendor-

managed inventory (VMI) arrangement with its suppliers, which decide how much to order and when to send their orders to the revolvers. Dell's suppliers order in batches (to offset ordering costs), using a continuous ordering system with a batch order size, Q , and a reorder point, R , where R is the sum of the inventory on order and a safety stock. The order size estimate, based on long-term data and forecasts, is held constant. Dell sets target inventory levels for its suppliers typically 10 days of inventory and keeps track of how much suppliers deviate from these targets and reports this information back to suppliers so that they can make adjustments accordingly.



Source: R. Kapuscinski, R. Zhang, P. Carbonneau, R. Moore, and B. Reeves, "Inventory Decisions in Dell's Supply Chain," *Interfaces* 34, no. 3 (May/June 2004): 191205.

EOQ Analysis with QM for Windows

QM for Windows has modules for all the EOQ models we have presented, including the basic model, the noninstantaneous receipt model, and the model with shortages. To demonstrate the capabilities of this program, we will use our basic EOQ example, for which the solution output summary is shown in [Exhibit 16.1](#).

Exhibit 16.1.

[\[View full size image\]](#)

Inventory Results			
I-75 Carpet Discount Store Solution			
Parameter	Value		Parameter
Demand rate(D)	10,000		Optimal order quantity (Q^*)
Setup/Ordering cost(S)	150		Maximum Inventory Level (I_{max})
Holding cost(H)	.75		Average inventory
Unit cost	0		Orders per period(year)
			Annual Setup cost
			Annual Holding cost
			Unit costs (PD)
			Total Cost

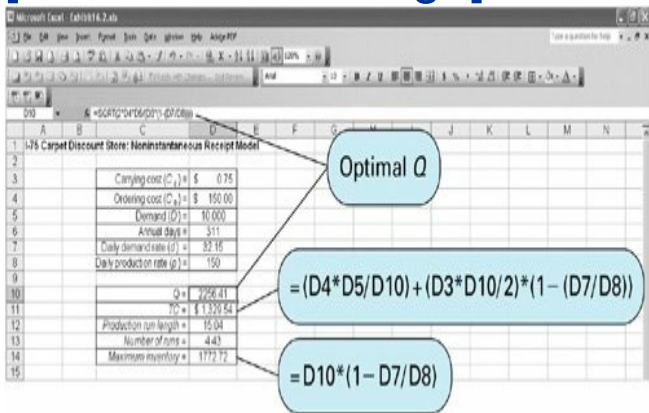
EOQ Analysis with Excel and Excel QM

[Exhibit 16.2](#) shows an Excel spreadsheet set up to perform EOQ analysis for our noninstantaneous receipt model example. The parameters of the model have been input in cells **D3:D8**, and all the formulas for optimal Q , total cost, and so on have been embedded in cells **D10:D14**. Notice that the formula for computing optimal Q in cell

D10 is shown on the formula bar at the top of the screen.

Exhibit 16.2.

[\[View full size image\]](#)



In [Chapter 1](#) we introduced Excel QM, a set of spreadsheet macros that we have also used in several other chapters. Excel QM includes a set of spreadsheet macros for "Inventory" that includes EOQ analysis. After Excel QM is activated, the "Excel QM" menu is accessed by clicking on "QM" on the menu bar at the top of the spreadsheet. Clicking on "Inventory" from this menu results in a Spreadsheet Initialization window, in which

you enter the problem title and the form of holding (or carrying) cost. Clicking on "OK" will result in the spreadsheet shown in [Exhibit 16.3](#). Initially, this spreadsheet will have example values in the data cells **B8:B13**. Thus, the first step in using this macro is to type in cells **B8:B13** the data for the noninstantaneous receipt model for our I-75 Carpet Discount Store problem. The model results are computed automatically in cells **B16:B26** from formulas already embedded in the

spreadsheet.

[Page 747]

Exhibit 16.3.

[\[View full size image\]](#)

Click on "QM" to access "Inventory" macro.

Input model parameters in cells B8:B13.

Quantity Discounts

It is often possible for a customer to receive a price discount on an item if a predetermined number of units is ordered. For example, occasionally in the back of a magazine you might see an advertisement for a firm that will produce a coffee mug (or hat) with a company or organizational logo on it, and the price will be \$5 per mug if you purchase 100, \$4 per mug

if you purchase 200, or \$3 per mug if you purchase 500 or more. Many manufacturing companies receive price discounts for ordering materials and supplies in high volume, and retail stores receive price discounts for ordering merchandise in large quantities.

The basic EOQ model can be used to determine the optimal order size with quantity discounts; however, the application of the model is slightly altered. The total inventory cost function must

now include the purchase price for the order, as follows:

$$TC = C_o \frac{D}{Q} + C_c \frac{Q}{2} + PD$$

where

P = per unit price of the item

D = annual demand

Determining whether an order size with a discount is more cost-effective than optimal Q .

Purchase price was not considered as part of our basic EOQ formulation earlier because it had no real impact on the optimal order size. PD in the foregoing formula is a constant value that would not alter the basic shape of the total cost curve (i.e., the minimum point on the cost curve would still be at the same location, corresponding to the same value of Q). Thus, the

optimal order size will be the same, no matter what the purchase price. However, when a discount price is available, it is associated with a specific order size that may be different from the optimal order size, and the customer must evaluate the trade-off between possibly higher carrying costs with the discount quantity versus EOQ cost. As a result, the purchase price does influence the order size decision when a discount is available.

Quantity discounts can be

evaluated using the basic EOQ model under two different scenarios with constant carrying costs and with carrying costs as a percentage of the purchase price. It is not uncommon for carrying costs to be determined as a percentage of purchase price, although it was not considered as such in our previous basic EOQ model. Carrying cost very well could have been a percentage of purchase price, but it was reflected as a constant value, C_c , in the basic EOQ model because the purchase price was

not part of the EOQ formula. However, in the case of a quantity discount, carrying cost will vary with the change in price if it is computed as a percentage of purchase price.

Quantity discounts are evaluated with constant C_c and as a percentage of price.

Quantity Discounts with

Constant Carrying Costs

In the EOQ cost model with constant carrying costs, the optimal order size, Q_{opt} , is the same, regardless of the discount price. Although total cost decreases with each discount in price because ordering and carrying cost are constant, the optimal order size, Q_{opt} , does not change. The total cost with Q_{opt} must be compared with any lower total cost with a discount price to see which is the minimum.

The following example will illustrate the evaluation of an EOQ model with a quantity discount when the carrying cost is a constant value. Comptek Computers wants to reduce a large stock of personal computers it is discontinuing. It has offered the University Bookstore at Tech a quantity discount pricing schedule if the store will purchase the personal computers in volume, as follows:

Quantity

Price

149	\$1,400
-----	---------

5089	1,100
------	-------

90+	900
-----	-----

The annual carrying cost for the bookstore for a computer is \$190, the ordering cost is \$2,500, and annual demand for this particular model is estimated to be 200 units. The bookstore wants to determine whether it should take

advantage of this discount or order the basic EOQ order size.

First, determine both the optimal order size and the total cost by using the basic EOQ model:

$$C_o = \$2,500$$

$$C_c = \$190 \text{ per unit}$$

$$D = 200$$

$$\begin{aligned} Q_{\text{opt}} &= \sqrt{\frac{2C_o D}{C_c}} \\ &= \sqrt{\frac{2(2,500)(200)}{190}} \end{aligned}$$

$$Q_{\text{opt}} = 72.5$$

This order size is eligible for the first discount of \$1,100; therefore, this price is used to compute total cost, as follows:

$$\begin{aligned} TC &= \frac{C_o D}{Q_{\text{opt}}} + C_c \frac{Q_{\text{opt}}}{2} + PD \\ &= \frac{(2,500)(200)}{(72.5)} + (190) \frac{(72.5)}{2} + (1,100)(200) \end{aligned}$$

$$TC_{\min} = \$233,784$$

Because there is a discount for an order size larger than 72.5, this total cost of \$233,784 must be compared with total cost with an order size of 90 and a price of \$900, as follows:

$$\begin{aligned}
 TC &= \frac{C_o D}{Q} + C_c \frac{Q}{2} + PD \\
 &= \frac{(2,500)(200)}{(90)} + \frac{(190)(90)}{2} + (900)(200) \\
 &= \$194,105
 \end{aligned}$$

Because this total cost is lower (\$194,105 < \$233,784), the maximum discount price should be taken and 90 units ordered.

Quantity Discounts with Constant Carrying Costs as a Percentage of Price

The difference between the model in the previous section

and the quantity discount model with carrying cost as a percentage of price is that there is a different optimal order size, Q_{opt} , for each price discount. This requires that the optimal order size with a discount be determined a little differently from the case for a constant carrying cost.

The optimal order size and total cost are determined by using the basic EOQ model for the case with no quantity discount. This total cost value is then compared with the various

discount quantity order sizes to determine the minimum cost order size. However, once this minimum cost order size is determined, it must be compared with the EOQ-determined order size for the specific discount price because the EOQ order size, Q_{opt} , will change for every discount level.

Reconsider our previous example, except now assume that the annual carrying cost for a computer at the University Bookstore is 15% of

the purchase price. Using the same discount pricing schedule, determine the optimal order size.

The annual carrying cost is determined as follows:

<i>Quantity</i>	<i>Price</i>	<i>Carrying Cost</i>
049	\$1,400	$1,400(.15) = \$210$
5089	1,100	$1,100(.15) = 165$
90+	900	$900(.15) = 135$

$$C_o = \$2,500$$

$$D = 200 \text{ computers per year}$$

First, compute the optimal order size for the purchase price without a discount and with $C_c = \$210$, as follows:

$$\begin{aligned} Q_{\text{opt}} &= \sqrt{\frac{2C_o D}{C_c}} \\ &= \sqrt{\frac{2(2,500)(200)}{210}} \\ Q_{\text{opt}} &= 69 \end{aligned}$$

Because this order size exceeds 49 units, it is not feasible for this price, and a lower total cost will automatically be achieved with the first price discount of \$1,100. However, the optimal order size will be different for this price discount because carrying cost is no longer constant. Thus, the new order size is computed as follows:

$$Q_{\text{opt}} \sqrt{\frac{2(2,500)(200)}{165}} = 77.8$$

This order size is the true optimum for this price discount instead of the 50 units required to receive the discount price; thus, it will result in the minimum total cost, computed as follows:

$$\begin{aligned} TC &= \frac{C_o D}{Q} + C_c \frac{Q}{2} + PD \\ &= \frac{(2,500)(200)}{77.8} + 165 \frac{(77.8)}{2} + (1,100)(200) \\ &= \$232,845 \end{aligned}$$



Management Science

Application: Quantity

Discount Orders at Mars

Mars is one of the world's largest privately owned companies, with over \$14 billion in annual sales. It has grown from making and selling buttercream candies door-to-door to a global business spanning 100 countries that includes food, pet care, beverage vending, and electronic payment systems. It produces such well-known products as Mars candies, M&M's, Snickers, and Uncle Ben's rice.



Mars relies on a small number of suppliers for each of the huge number of materials it uses in its

products. One way that Mars purchases materials from its suppliers is through online electronic auctions, in which Mars buyers negotiate bids for orders from suppliers. The most important purchases are those of high value and large volume, for which the suppliers provide quantity discounts. The suppliers provide a pricing schedule that includes quantity ranges associated with each price level. Such quantity-discount auctions are tailored (by online brokers) for industries in which volume discounts are common, such as bulk chemicals and agricultural commodities.

A Mars buyer selects the bids that minimize total purchasing costs, subject to several rules: There must be a minimum and maximum number of suppliers so that Mars is not dependent on too few suppliers nor loses quality control by having too

many; there must be a maximum amount purchased from each supplier to limit the influence of any one supplier; and a minimum amount must be ordered to avoid economically inefficient orders (i.e., less than a full tuckload).

Source: G. Hohner, J. Rich, E. Ng, A. Davenport, J. Kalagnanam, H. Lee, and C. An, "Combinatorial and Quantity-Discount Procurement Auctions Benefit Mars, Incorporated and Its Suppliers," *Interfaces* 33, no. 1 (January/February 2003): 2335.

This cost must still be compared with the total cost for lowest discount price (\$900) and order quantity (90 units), computed as follows:

$$TC = \frac{(2,500)(200)}{90} + \frac{(135)(90)}{2} + (900)(200) \\ = \$191,630$$

Because this total cost is lower (\$191,630 < \$232,845), the maximum discount price should be taken and 90 units ordered. However, as before, we still must check to see whether there is a new optimal order size for this discount that will

result in an even lower cost.
The optimal order size with C_c
= \$135 is computed as follows:

$$Q_{\text{opt}} = \sqrt{\frac{2(2,500)(200)}{135}} = 86.1$$

Because this order size is less than the 90 units required to receive the discount, it is not feasible; thus, the optimal order size is 90 units.

Quantity Discount Model Solution with QM for Windows

QM for Windows has the capability to perform EOQ analysis with quantity discounts when carrying costs are constant. [Exhibit 16.4](#) shows the solution summary for our University Bookstore example.

Exhibit 16.4.

[\[View full size image\]](#)

Inventory Results									
University Bookstore Example Solution									
Parameter	Value					Parameter	Value		
Demand rate(D)	200	xxxxxxx	xxxxxxx			Optimal order quantity (Q^*)	90		
Setup/Ordering cost(S)	2,500	xxxxxxx	xxxxxxx			Maximum Inventory Level	90		
Holding cost(H)	190	xxxxxxx	xxxxxxx			Average inventory	45		
						Orders per period(year)	2.2222		
	From	To	Price			Annual Setup cost	5,555.556		
1	1	49	1,400			Annual Holding cost	8,550		
2	50	89	1,100						
3	90	999,999	900			Unit costs (PD)	180,000		
						Total Cost	194,105.6		

Quantity Discount Model Solution with Excel

It is also possible to use Excel

to solve the quantity discount model with constant carrying costs. [Exhibit 16.5](#) shows the Excel solution screen for the University Bookstore example. Notice that the selection of the appropriate order size, Q , that results in the minimum total cost for each discount range is determined by the formulas embedded in cells E8, E9, and E10. For example, the formula for the first quantity discount range, 149, is embedded in cell E8 and shown on the formula bar at the top of the screen, **=IF(D8>=B8,D8,B8)**. This

means that if the discount order size in cell D8 (i.e., $Q = 72.55$) is greater than or equal to the quantity in cell B8 (i.e., 1), the quantity in cell D8 (72.55) is selected; otherwise, the amount in cell B8 is selected. The formulas in cells E9 and E10 are constructed similarly. The result is that the order quantity for the final discount range, $Q = 90$, is selected.

Exhibit 16.5.

[\[View full size image\]](#)

Microsoft Excel - Exhibit 16.5.xls

File Edit View Insert Format Tools Data Window Help

Type equation for help

Formulas > =IF(D10>=B10,D10,B10)

University Bookstore: Quantity Discount Model

3		Carrying cost (C_c) =	\$	190	
4		Ordering cost (C_o) =	\$	2,500	
5		Demand (D) =		200	
7	Quantity	Price	Q	Discount Q	Total Cost
8	1	1,400	72.55	72.55	\$ 293,784.05
9	50	1,100	72.65	72.55	\$ 233,784.05
10	90	900	72.55	90.00	\$ 194,105.56
11					Optimal

$$=(D4*D5/E10)+(D3*E10/2)+C10*D5$$

$$=IF(D10 \geq B10, D10, B10)$$

Reorder Point

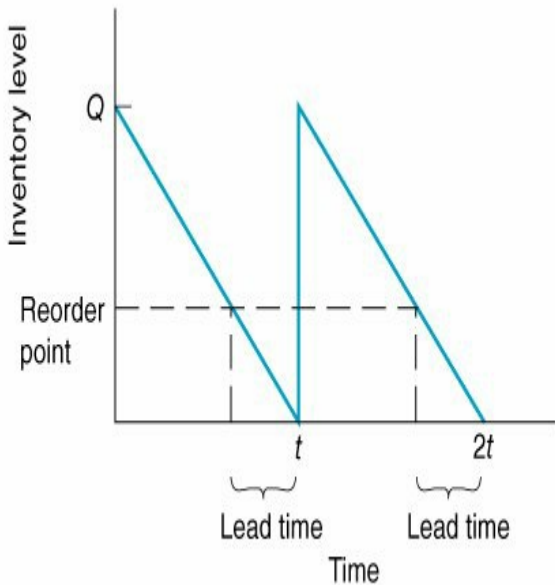
In our presentation of the basic EOQ model in the previous section, we addressed one of the two primary questions related to inventory management: *How much should be ordered?* In this section we will discuss the other aspect of inventory management: *When to order?* The determinant of when to order in a continuous inventory system is the **reorder point**,

the inventory level at which a new order is placed.

The concept of lead time is illustrated graphically in [Figure 16.9](#). Notice that the order must be made prior to the time when the level of inventory falls to zero. Because demand is consuming the inventory while the order is being shipped, the order must be made while there is enough inventory in stock to meet demand during the lead-time period. This level of inventory is referred to as the *reorder point* and is so designated in

[Figure 16.9.](#)

Figure 16.9. Reorder point and lead time



The reorder point for our basic EOQ model with constant demand and a constant lead time to receive an order is relatively straightforward. It is simply equal to the amount demanded during the lead-time period, computed using the following formula:

$$R = dL$$

where

d = demand rate per time period (i.e., daily)

L = lead time

Consider the I-75 Carpet Discount Store example described previously. The store is open 311 days per year. If annual demand is 10,000 yards of Super Shag carpet and the lead time to receive an order is 10 days, the reorder point for carpet is determined as follows:

$$\begin{aligned} R &= dL \\ &= \left(\frac{10,000}{311} \right) (10) \\ &= 321.54 \text{ yd.} \end{aligned}$$

Thus, when the inventory level

falls to approximately 321 yards of carpet, a new order is placed. Notice that the reorder point is not related to the optimal order quantity or any of the inventory costs.

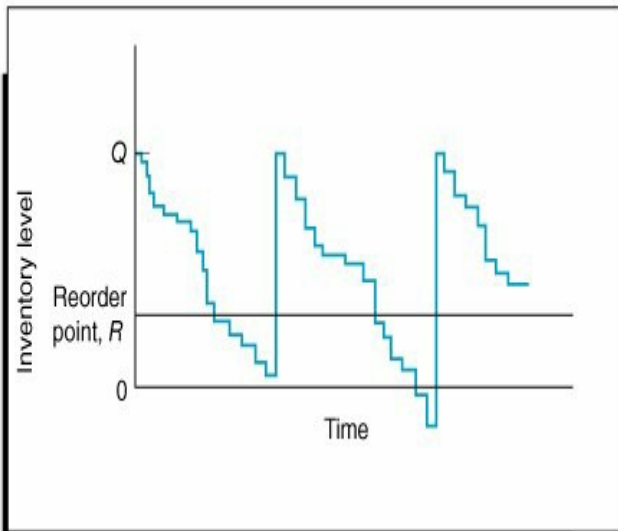
Safety Stocks

In our previous example for determining the reorder point, an order is made when the inventory level reaches the reorder point. During the lead time, the remaining inventory in stock is depleted as a

constant demand rate, such that the new order quantity arrives at exactly the same moment as the inventory level reaches zero in [Figure 16.9](#). Realistically, however, demand and to a lesser extent lead time is uncertain. The inventory level might be depleted at a slower or faster rate during lead time. This is depicted in [Figure 16.10](#) for uncertain demand and a constant lead time.

Figure 16.10.

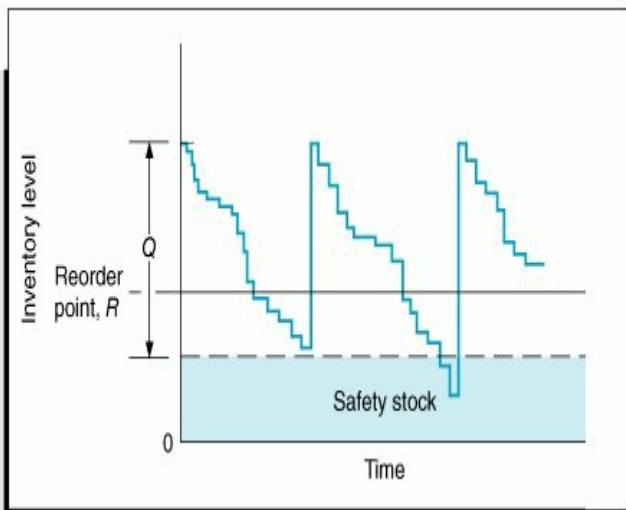
Inventory model with uncertain demand



Notice in the second order cycle that a stockout occurs when demand exceeds the available inventory in stock. As a hedge against stockouts when demand is uncertain, a safety (or buffer) stock of inventory is frequently added to the demand during lead time. The addition of a safety stock is shown in [Figure 16.11](#).

Figure 16.11.

Inventory model with safety stock



Determining Safety Stock By Using Service Levels

There are several approaches to determining the amount of the safety stock needed. One of the most popular methods is to establish a safety stock that will meet a specified **service level**. The service level is the probability that the amount of inventory on hand during the lead time is sufficient to meet expected demand (i.e., the probability that a stockout will not occur). The word *service* is used because the higher the probability that inventory will be on hand, the more likely that customer demand will be met (i.e., the customer can be served). For example, a service level of 90% means that there is a .90 probability that demand will be met during the lead time period and

a .10 probability that a stockout will occur. The specification of the service level is typically a policy decision based on a number of factors, including costs for the "extra" safety stock and present and future lost sales if customer demand cannot be met.

*The **service level** is the probability that the inventory available during the lead time will meet demand.*

Reorder Point with Variable Demand

To compute the reorder point with a safety stock that will meet a specific service level,

we will assume that the individual demands during each day of lead time are uncertain and independent and can be described by a normal probability distribution. The average demand for the lead-time period is the sum of the average daily demands for the days of the lead time, which is also the product of the average daily demand multiplied by the lead time. Likewise, the variance of the distribution is the sum of the daily variances for the number of days in the lead-time period. Using these parameters, the reorder point to meet a specific service level can be computed as follows:

$$R = \bar{d}L + Z\sigma_d\sqrt{L}$$

where

$\bar{d}$ = average daily demand

L = lead time

σ_d = the standard deviation of daily demand

Z = number of standard deviations corresponding to the service level probability

$Z\sigma_d\sqrt{L}$ = safety stock

[Page 755]

The term $\sigma_d\sqrt{L}$ in this formula for reorder point is the square root of the sum of the

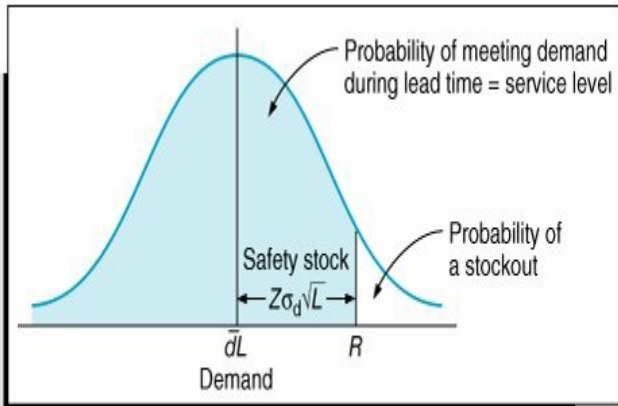
daily variances during lead time:

$$\text{variance} = (\text{daily variances}) \times (\text{number of days of lead time}) = \sigma_d^2 L$$

$$\text{standard deviation} = \sqrt{\sigma_d^2 L} = \sigma_d \sqrt{L}$$

The reorder point relative to the service level is shown in [Figure 16.12](#). The service level is the shaded area, or probability, to the left of the reorder point, R .

Figure 16.12. Reorder point for a service level



The I-75 Carpet Discount Store sells Super Shag carpet. The average daily customer demand for the carpet stocked by the store is normally distributed, with a mean daily demand of 30 yards and a standard deviation of 5 yards of carpet per day. The lead time for receiving a new order of carpet is 10 days. The store wants a

reorder point and safety stock for a service level of 95%, with the probability of a stockout equal to 5%:

$\square = 30$ yd. per day, $L = 10$ days, $\sigma_d = 5$ yd. per day

For a 95% service level, the value of Z (from [Table A.1](#) in [Appendix A](#)) is 1.65. The reorder point is computed as follows:

$$R = \bar{d}L + Z\sigma_d\sqrt{L} = 30(10) + (1.65)(5)(\sqrt{10}) = 300 + 26.1 = 326.1 \text{ yd.}$$

The safety stock is the second term in the reorder point formula:

$$\text{safety stock} = Z\sigma_d\sqrt{L} = (1.65)(5)(\sqrt{10}) = 26.1 \text{ yd.}$$

Determining the Reorder Point Using Excel

Excel can be used to determine the reorder point with variable demand. [Exhibit 16.6](#) shows an Excel spreadsheet set up to compute the reorder point for our I-75 Carpet Discount Store example. Notice that the formula for computing the reorder point in cell E7 is shown on the formula bar at the top of the spreadsheet.

[Page 756]

Exhibit 16.6.

[\[View full size image\]](#)

Microsoft Excel - 1440416.xls

File Edit View Insert Format Tools Data Window Help

Type a question for help

Formulas: =E7*5+(1.65*5*SQRT(10))

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	
1	Reorder Point with Variable Demand for i-75 Discount Carpet Store																
2																	
3				Average daily demand =	50												
4				Standard deviation =	5												
5				Lead time =	10												
6																	
7				R =	325.00												
8																	

Reorder Point with Variable Lead Time

In the model in the previous section for determining the reorder point, we assumed a variable demand rate and a constant lead time. In the case where demand is constant and the lead time varies, we can use a similar formula, as follows:

$$R = d\boxed{} + Zd\sigma_L$$

where

d = constant daily demand

$\boxed{}$ = average lead time

σ_L = standard deviation of lead time

$d\sigma_L$ = standard deviation of demand
during lead time

$Zd\sigma_L$ = safety stock

For our previous example of the I-75 Carpet Discount Store, we will now assume that daily demand for Super Shag carpet is a constant 30 yards. Lead time is normally distributed, with a mean of 10 days and a standard deviation of 3 days. The reorder point and safety stock corresponding to a 95% service level are computed as follows:

$$d = 30 \text{ yd. per day}$$

$$\square = 10 \text{ days}$$

$$\sigma_L = 3 \text{ days}$$

$$Z = 1.65 \text{ for a 95\% service level}$$

$$R = d\bar{L} + Zd\sigma_L = (30)(10) + (1.65)(30)(3) = 300 + 148.5 = 448.5 \text{ yd.}$$

Reorder Point with Variable Demand and Lead Time

The final reorder point case we will consider is the case in which both demand and lead time are variables. The reorder point formula for this model is as follows.

$$R = \bar{d} \bar{L} + Z \sqrt{\sigma_d^2 \bar{L} + \sigma_L^2 \bar{d}^2}$$

where

☐ = average daily demand

☐ = average lead time

$$\sqrt{\sigma_d^2 \bar{L} + \sigma_L^2 \bar{d}^2}$$

$$Z \sqrt{\sigma_d^2 \bar{L} + \sigma_L^2 \bar{d}^2}$$

[Page 757]

Management Science

Application: Establishing Inventory Safety Stocks at Kellogg's

Kellogg's is the world's largest cereal producer and a leading maker of convenience foods, with worldwide sales in 1999 of almost \$7 billion. The company started with a single product, Kellogg's Corn Flakes, in 1906, and over the years has developed a product line of other cereals, including Rice Krispies and Corn Pops, as well as convenience foods, such as Pop-Tarts and Nutri-Grain cereal bars. Kellogg's operates 5 plants in the United States and Canada and 7 distribution centers, and it contracts with 15 co-packers

to produce or pack some Kellogg's products. Kellogg's must coordinate the production, packaging, inventory, and distribution of roughly 80 cereal products alone at these various facilities.

For more than a decade, Kellogg's has been using a model called the Kellogg Planning System (KPS) to plan its weekly production, inventory, and distribution decisions. The data used in the model are subject to much uncertainty, and the greatest uncertainty is in product demand. Demand in the first few weeks of a planning horizon is based on customer orders and is fairly accurate; however, demand in the third and fourth weeks may be significantly different from marketing forecasts. However, Kellogg's primary goal is to meet customer demand, and in order to achieve this goal,

Kellogg's employs safety stocks as a buffer against uncertain demand. The safety stock for a product at a specific production facility in week t is the sum of demands for weeks t and $t + 1$. However, for a product that is being promoted in an advertising campaign, the safety stock is the sum of forecasted demand for a 4-week horizon or longer. KPS has saved Kellogg's many millions of dollars since the mid-1990s. The tactical version of KPS recently helped the company consolidate production capacity, with estimated projected savings of almost \$40 million.



Source: G. Brown, J. Keegan, B. Vigus, and K. Wood, "The Kellogg Company Optimizes Production, Inventory, and Distribution," *Interfaces* 31, no. 6 (November/December 2001): 115.

Again we will consider the I-75 Carpet Discount Store example, used previously. In this case, daily demand is normally distributed, with a mean of 30 yards and a standard deviation of 5 yards. Lead time is also assumed to be normally distributed, with a mean of 10 days and a standard deviation of 3 days. The reorder point and safety stock for a 95% service level are computed as

follows:

$$\bar{d} = 30 \text{ yd./day}$$

$$\sigma_d = 5 \text{ yd./day}$$

$$\bar{L} = 10 \text{ days}$$

$$\sigma_L = 3 \text{ days}$$

$$Z = 1.65 \text{ for a 95\% service level}$$

$$\begin{aligned} R &= \bar{d} \bar{L} + Z \sqrt{\sigma_d^2 \bar{L} + \sigma_L^2 \bar{d}^2} \\ &= (30)(10) + (1.65) \sqrt{(5)^2 (10) + (3)^2 (30)^2} \\ &= 300 + 150.8 \\ R &= 450.8 \text{ yd.} \end{aligned}$$

[Page 758]

Thus, the reorder point is 450.8 yards, with a safety stock of 150.8 yards. Notice

that this reorder point encompasses the largest safety stock of our three reorder point examples, which would be anticipated, given the increased variability resulting from variable demand and lead time.

Order Quantity for a Periodic Inventory System

Previously we defined a continuous, or fixed order quantity, inventory system as one in which the order quantity was constant and the time between orders varied. So far, this type of inventory system has been the primary focus of our discussion. The less common periodic, or *fixed time*

period, **inventory system** is one in which the time between orders is constant and the order size varies. A drugstore is one example of a business that sometimes uses a fixed-period inventory system. Drugstores stock a number of personal hygiene and health-related products, such as shampoo, toothpaste, soap, bandages, cough medicine, and aspirin. Normally, the vendors that provide these items to the store will make periodic visits for example, every few weeks or every month and

count the stock of inventory on hand for their products. If the inventory is exhausted or at some predetermined reorder point, a new order will be placed for an amount that will bring the inventory level back up to the desired level. The drugstore managers will generally not monitor the inventory level between vendor visits but instead rely on the vendor to take inventory at the time of the scheduled visit.

***A periodic
inventory system***

uses variable order sizes at fixed time intervals.

A limitation of this type of inventory system is that inventory can be exhausted early in the time period between visits, resulting in a stockout that will not be remedied until the next scheduled order. Alternatively, in a fixed order quantity system, when inventory

reaches a reorder point, an order is made that minimizes the time during which a stockout might exist. As a result of this drawback, a larger safety stock is normally required for the fixed-interval system.

A periodic inventory system normally requires a larger safety stock.

Order Quantity with Variable Demand

If the demand rate and lead time are constant, then the fixed-period model will have a fixed order quantity that will be made at specified time intervals, which is the same as the fixed quantity (EOQ) model under similar conditions.

However, as we have already explained, the fixed-period model reacts significantly differently from the fixed order quantity model when demand is

a variable.

The order size for a fixed-period model, given variable daily demand that is normally distributed, is determined by the following formula:

$$Q = \bar{d}(t_b + L) + Z\sigma_d\sqrt{t_b + L} - I$$

where

$\bar{d}$ = average demand rate

t_b = the fixed time between orders

L = lead time

σ_d = standard deviation of demand

$Z\sigma_d\sqrt{t_b + L}$ = safety stock

I = inventory in stock

The first term in the preceding formula, $d(t_b + L)$, is the average demand during the order cycle time plus the lead time. It reflects the amount of

inventory that will be needed to protect against shortages during the entire time from this order to the next and the lead time, until the order is received. The second term, $Z\sigma_d\sqrt{t_b + L}$, is the safety stock for a specific service level, determined in much the same way as previously described for a reorder point. The final term, I , is the amount of inventory on hand when the inventory level is checked and an order is made. We will demonstrate the computation of Q with an example.

The Corner Drug Store stocks a popular brand of sunscreen. The average demand for the sunscreen is 6 bottles per day, with a standard deviation of 1.2 bottles. A vendor for the sunscreen producer checks the drugstore stock every 60 days, and during a particular visit the drugstore had 8 bottles in stock. The lead time to receive an order is 5 days. The order size for this order period that will enable the drugstore to maintain a 95% service level is

computed as follows:

$\bar{d} = 6$ bottles per day

$\sigma_d = 1.2$ bottles

$t_b = 60$ days

$L = 5$ days

$I = 8$ bottles

$Z = 1.65$ for 95% service level

$$\begin{aligned} Q &= \bar{d}(t_b + L) + Z\sigma_d\sqrt{t_b + L} - I \\ &= (6)(60 + 5) + (1.65)(1.2)\sqrt{60 + 5} - 8 \\ &= 398 \text{ bottles} \end{aligned}$$

Determining the Order Quantity for the Fixed-Period Model with Excel

[Exhibit 16.7](#) shows an Excel

spreadsheet set up to compute the order quantity for the fixed-period model with variable demand for our Corner Drug Store example. Notice that the formula for the order quantity in cell D10 is shown on the formula bar at the top of the spreadsheet.

Exhibit 16.7.

[\[View full size image\]](#)

[illegible]

Summary

In this chapter the classical economic order quantity model has been presented. The basic form of the EOQ model we discussed included simplifying assumptions regarding order receipt, no shortages, and constant demand known with certainty. By relaxing some of these assumptions, we were able to create increasingly complex but realistic models. These EOQ variations included

the reorder point model, the noninstantaneous receipt model, the model with shortages, and models with safety stocks. The techniques for inventory analysis presented in this chapter are not widely used to analyze other types of problems. Conversely, however, many of the techniques presented in this text are used for inventory analysis (in addition to the methods presented in these chapters). The wide use of management science techniques for inventory

analysis attests to the importance of inventory to all types of organizations.

[Page 760]

References

Buchan, J., and Koenigsberg, E.
*Scientific Inventory
Management*. Upper Saddle
River, NJ: Prentice Hall, 1963.

Buffa, E. S., and Miller, J. G.
*Production-Inventory Systems:
Planning and Control*, 3rd ed.
Homewood, IL: Irwin, 1979.

Churchman, C. W., Ackoff, R. L.,
and Arnoff, E. L. *Introduction to
Operations Research*. New
York: John Wiley & Sons, 1957.

Hadley, G., and Whitin, T. M.
Analysis of Inventory Systems.
Upper Saddle River, NJ:

Prentice Hall, 1963.

Johnson, L. A., and Montgomery, D. C. *Operations Research in Production Planning, Scheduling and Inventory Control*. New York: John Wiley & Sons, 1974.

Magee, J. F., and Boodman, D. M. *Production Planning and Inventory Control*, 2nd ed. New York: McGraw-Hill, 1967.

Starr, M. K., and Miller, D. W. *Inventory Control: Theory and Practice*. Upper Saddle River, NJ: Prentice Hall, 1962.

Wagner, H. M. *Statistical Management of Inventory Systems*. New York: John Wiley & Sons, 1962.

Whitin, T. M. *The Theory of Inventory Management*. Princeton, NJ: Princeton University Press, 1957.

Example Problem Solutions

The following example will demonstrate EOQ analysis for the classical model and the model with shortages and back ordering.

Problem Statement

Electronic Village stocks and sells a particular brand of personal computer. It costs the

store \$450 each time it places an order with the manufacturer for the personal computers. The annual cost of carrying the PCs in inventory is \$170. The store manager estimates the annual demand for the PCs will be 1,200 units.

- 1.** Determine the optimal order quantity and the total minimum inventory cost.
- 2.** Assume that shortages are allowed and that the shortage cost is \$600 per unit per year. Compute the optimal order quantity and

the total minimum
inventory cost.

Solution

**(part a): Determine
the Optimal Order
Quantity**

Step 1.

$D = 1,200$ personal computers

$C_c = \$170$

$C_o = \$450$

$$Q = \sqrt{\frac{2C_o D}{C_c}}$$

$$= \sqrt{\frac{2(450)(1,200)}{170}}$$

$= 79.7$ personal computers

$$\text{total cost} = C_c \frac{Q}{2} + C_o \frac{D}{Q}$$

$$= 170 \left(\frac{79.7}{2} \right) + 450 \left(\frac{1,200}{79.7} \right)$$

$$= \$13,549.91$$

[Page 761]

**(part b): Compute
the EOQ with**

Shortages

Step 2.

$$C_s = \$600$$

$$Q = \sqrt{\frac{2C_o D}{C_c} \left(\frac{C_s + C_c}{C_s} \right)}$$

$$= \sqrt{\frac{2(450)(1,200)}{170} \left(\frac{600 + 170}{600} \right)}$$

$$= 90.3 \text{ personal computers}$$

$$S = Q \left(\frac{C_c}{C_c + C_s} \right)$$

$$= 90.3 \left(\frac{170}{170 + 600} \right)$$

$$= 19.9 \text{ personal computers}$$

$$\text{total cost} = \frac{C_s S^2}{2Q} + C_c \frac{(Q - S)^2}{2Q} + \frac{C_o D}{Q}$$

$$= \frac{(600)(19.9)^2}{2(90.3)} + 170 \frac{(90.3 - 19.9)^2}{2(90.3)} + 450 \left(\frac{1,200}{90.3} \right)$$

$$= \$1,315.65 + 4,665.27 + 5,980.07$$

$$= \$11,960.98$$

Problem Statement

A computer products store stocks color graphics monitors, and the daily demand is normally distributed, with a mean of 1.6 monitors and a standard deviation of 0.4 monitor. The lead time to receive an order from the manufacturer is 15 days. Determine the reorder point that will achieve a 98% service level.

Solution

Identify Parameters

$\square = 1.6$ monitors per day

$L = 15$ days

Step 1.

$\sigma_d = 0.4$ monitors per day

$Z = 2.05$ (for a 98% service level)

Solve for R

Step 2.

$$R = \bar{d}L + Z\sigma_d\sqrt{L} = (1.6)(15) + (2.05)(0.4)\sqrt{15} = 24 + 3.18 = 27.18 \text{ monitors}$$

Problems

Hayes Electronics stocks and sells a particular brand of personal computer. It costs the firm \$450 each time it places an order with the manufacturer for the personal computers. The cost of carrying one PC in inventory for a year is \$170. The store manager estimates that total annual demand for the computers will be 1,200 units, with a constant demand rate throughout the year. Orders are received within minutes after placement from a local warehouse maintained by the manufacturer. The store policy is

- 1.** never to have stockouts of the PCs. The store is open for business every day of the year except Christmas Day. Determine the following.
-

[Page 762]

- 1.** The optimal order quantity per order
- 2.** The minimum total annual inventory costs
- 3.** The optimal number of orders per year
- 4.** The optimal time between orders (in working days)

Hayes Electronics in [Problem 1](#) assumed with certainty that the ordering cost is \$450 per order

and the inventory carrying cost is \$170 per unit per year. However, the inventory model parameters are frequently only estimates that are subject to some degree of uncertainty. Consider four cases of variation in the model parameters: (a) Both ordering cost and carrying cost are 10% less than originally estimated, (b) both ordering cost and carrying cost are 10% higher than originally estimated, (c) ordering cost is 10% higher and carrying cost is 10% lower than originally estimated, and (d) ordering cost is 10% lower and carrying cost is 10% higher than originally estimated. Determine the optimal order quantity and total inventory cost for each of the four cases. Prepare a table with values from all four cases and compare the sensitivity of the model

solution to changes in parameter values.

A firm is faced with the attractive situation in which it can obtain immediate delivery of an item it stocks for retail sale. The firm has therefore not bothered to order the item in any systematic way. Recently, however, profits have been squeezed due to increasing competitive pressures, and the firm has retained a management consultant to study its inventory management. The consultant has determined that the various costs associated with making an order for the item stocked are approximately \$30 per order. She has also determined that the costs of carrying the item in inventory

amount to approximately \$20 per unit per year (primarily direct storage costs and forgone profit on investment in inventory).

3. Demand for the item is reasonably constant over time, and the forecast is for 19,200 units per year. When an order is placed for the item, the entire order is immediately delivered to the firm by the supplier. The firm operates 6 days a week plus a few Sundays, or approximately 320 days per year. Determine the following.

- 1.** The optimal order quantity per order
- 2.** The total annual inventory costs
- 3.** The optimal number of orders to place per year

4. The number of operating days between orders, based on the optimal number of orders.

The Western Jeans Company purchases denim from Cumberland Textile Mills. The Western Jeans Company uses 35,000 yards of denim per year to make jeans. The cost of ordering denim from the textile company is \$500 per order.

4. It costs Western \$0.35 per yard annually to hold a yard of denim in inventory. Determine the optimal number of yards of denim the Western Jeans Company should order, the minimum total annual inventory cost, the optimal number of orders per year, and the optimal time between orders.

The Metropolitan Book Company purchases paper from the Atlantic Paper Company. Metropolitan produces magazines and paperbacks that require 1,215,000 pounds of paper per year. The cost per order for the company is \$1,200; the cost of holding 1 pound of paper in inventory is \$0.08 per year. Determine the following.

5.

- 1.** The economic order quantity
- 2.** The minimum total annual cost
- 3.** The optimal number of orders per year
- 4.** The optimal time between orders

- The Simple Simon Bakery produces fruit pies for freezing and subsequent sale. The bakery, which operates 5 days per week, 52 weeks per year, can produce pies at the rate of 64 pies per day. The bakery sets up the pie production operation and produces until a predetermined number (Q) of pies has been produced. When not producing pies, the bakery uses its personnel and facilities for producing other bakery items. The setup cost for a production run of fruit pies is \$500. The cost of holding frozen pies in storage is \$5 per pie per year. The annual
6. demand for frozen fruit pies, which is constant over time, is 5,000 pies. Determine the following.

- 1.** The optimal production run quantity (Q)
- 2.** The total annual inventory costs
- 3.** The optimal number of production runs per year
- 4.** The optimal cycle time (time between run starts)
- 5.** The run length, in working days

The Pedal Pusher Bicycle Shop operates 7 days per week, closing only on Christmas Day. The shop pays \$300 for a particular bicycle purchased from the manufacturer. The annual holding cost per bicycle is estimated to be 25% of the dollar value of inventory. The shop

sells an average of 25 bikes per week. Frequently, the dealer does **7.** not have a bike in stock when a customer purchases it, and the bike is back ordered. The dealer estimates his shortage cost per unit back ordered, on an annual basis, to be \$250 due to lost future sales (and profits). The ordering cost for each order is \$100. Determine the optimal order quantity and shortage level and the total minimum cost.

The Petroco Company uses a highly toxic chemical in one of its manufacturing processes. It must have the product delivered by special cargo trucks designed for safe shipment of chemicals. As such, ordering (and delivery) costs are relatively high, at \$2,600 per

- order. The chemical product is packaged in 1-gallon plastic containers. The cost of holding the chemical in storage is \$50 per gallon per year. The annual demand for the chemical, which is constant over time, is 2,000 gallons per year. The lead time from time of order placement until receipt is 10 days. The company operates 310 working days per year. Compute the optimal order quantity, the total minimum inventory cost, and the reorder point.
- 8.**

The Big Buy Supermarket stocks Munchies Cereal. Demand for Munchies is 4,000 boxes per year (365 days). It costs the store \$60 per order of Munchies, and it costs \$0.80 per box per year to keep the cereal in stock. Once an order

for Munchies is placed, it takes 4 **9.** days to receive the order from a food distributor. Determine the following.

- 1.** The optimal order size
- 2.** The minimum total annual inventory cost
- 3.** The reorder point

The Wood Valley Dairy makes cheese to supply to stores in its area. The dairy can make 250 pounds of cheese per day, and the demand at area stores is 180 pounds per day. Each time the dairy makes cheese, it costs \$125 **10.** to set up the production process. The annual cost of carrying a pound of cheese in a refrigerated storage area is \$12. Determine the

optimal order size and the minimum total annual inventory cost.

The Rainwater Brewery produces Rainwater Light Beer, which it stores in barrels in its warehouse and supplies to its distributors on demand. The demand for Rainwater is 1,500 barrels of beer per day. The brewery can produce 2,000 barrels of Rainwater per day.

11. It costs \$6,500 to set up a production run for Rainwater. Once it is brewed, the beer is stored in a refrigerated warehouse at an annual cost of \$50 per barrel. Determine the economic order quantity and the minimum total annual inventory cost.

The purchasing manager for the Atlantic Steel Company must determine a policy for ordering coal to operate 12 converters. Each converter requires exactly 5 tons of coal per day to operate, and the firm operates 360 days per year. The purchasing manager has determined that the ordering cost is \$80 per order and the cost of holding coal is 20% of the average dollar value of inventory held. The purchasing manager has negotiated a contract to obtain the coal for \$12 per ton for the coming year.

12.

[Page 764]

- 1.** Determine the optimal quantity of coal to receive in each order.

- 2.** Determine the total inventory-related costs associated with the optimal ordering policy (do not include the cost of the coal).
- 3.** If 5 days of lead time are required to receive an order of coal, how much coal should be on hand when an order is placed?

The Pacific Lumber Company and Mill processes 10,000 logs annually, operating 250 days per year. Immediately upon receiving an order, the logging company's supplier begins delivery to the lumber mill, at a rate of 60 logs per day. The lumber mill has determined that the ordering cost is \$1,600 per order and the cost of carrying logs in inventory before

they are processed is \$15 per log on an annual basis. Determine the

13. following.

- 1.** The optimal order size
- 2.** The total inventory cost associated with the optimal order quantity
- 3.** The number of operating days between orders
- 4.** The number of operating days required to receive an order

The Roadking Tire Store sells a brand of tires called the Roadrunner. The annual demand from the store's customers for Roadrunner tires is 3,700. The cost

- to order tires from the tire manufacturer is \$420 per order.
- 14.** The annual carrying cost is \$1.75 per tire. The store allows shortages, and the annual shortage cost per tire is \$4. Determine the optimal order size, maximum shortage level, and minimum total annual inventory cost.

- The Laurel Creek Lawn Shop sells Fastgro Fertilizer. The annual demand for the fertilizer is 270,000 pounds. The cost to order the fertilizer from the Fastgro Company is \$105 per order. The annual carrying cost is \$0.25 per pound. The store operates with shortages, and the annual shortage cost is \$0.70 per pound. Compute the optimal order
- 15.**

size, minimum total annual inventory cost, and maximum shortage level.

Videoworld is a discount store that sells color televisions. The annual demand for color television sets is 400. The cost per order from the manufacturer is \$650. The carrying cost is \$45 per set each year. The store has an inventory policy that allows shortages. The shortage

16. cost per set is estimated at \$60. Determine the following.

- 1.** The optimal order size
- 2.** The maximum shortage level
- 3.** The minimum total annual inventory cost

The University Bookstore at Tech stocks the required textbook for Management Science 2405. The demand for this text is 1,200 copies per year. The cost of placing an order is \$350, and the annual carrying cost is \$2.75 per book. If a student requests the book and it is not in stock, the student will likely go to the privately owned Tech Bookstore. It is likely that the student will not buy books at the University Bookstore in the future; thus the shortage cost to the University Bookstore is estimated to be \$45 per book. Determine the optimal order size, the maximum shortage level, and the total inventory cost.

17.

The A-to-Z Office Supply Company is open from 8:00 A.M. to 6:00

P.M., and it receives 200 calls per day for delivery orders. It costs A-to-Z \$20 to send out its trucks to make deliveries. The company estimates that each minute a customer spends waiting for an order costs A-to-Z \$0.20 in lost sales.

18.

- 1.** How frequently should A-to-Z send out its delivery trucks each day? Indicate the total daily cost of deliveries.
- 2.** If a truck could carry only six orders, how often would deliveries be made, and what would be the cost?

The Union Street Microbrewery makes 1220 Union beer, which it bottles and sells in its adjoining restaurant and by the case. It

costs \$1,700 to set up, brew, and bottle a batch of the beer. The annual cost to store the beer in inventory is \$1.25 per bottle. The annual demand for the beer is 18,000 bottles, and the brewery has the capacity to produce 30,000 bottles annually.

19.

[Page 765]

- 1.** Determine the optimal order quantity, the total annual inventory cost, the number of production runs per year, and the maximum inventory level.
- 2.** If the microbrewery has only enough storage space to hold a maximum of 2,500 bottles of beer in inventory, how will that affect total inventory costs?

20. Eurotronics is a European manufacturer of electronic components. During the course of a year, it requires container cargo space on ships leaving Hamburg bound for the United States, Mexico, South America, and Canada. Annually, the company needs 160,000 cubic feet of cargo space. The cost of reserving cargo space is \$7,000, and the cost of holding cargo space is \$0.80/ft³. Determine how much storage space Eurotronics should optimally order, the total cost, and how many times per year it should place orders to reserve space.

The Summer Outdoor Furniture Company produces wooden lawn

chairs. The annual demand from its store customers is 17,400 chairs per year. The transport and handling costs are \$2,600 each time a shipment of chairs is delivered to stores from its warehouse. The annual carrying cost is \$3.75 per chair.

21.

- 1.** Determine the optimal order quantity and minimum total annual cost.
- 2.** The company is thinking about relocating its warehouse closer to its customers, which would reduce transport and handling costs to \$1,900 per order but increase carrying costs to \$4.50 per chair per year. Should the company relocate based on inventory costs?

The Spruce Creek Vegetable Farm produces organically grown greenhouse tomatoes that are sold to area grocery stores. The annual demand for Spruce Creek's tomatoes is 270,000 pounds. The farm is able to produce 305,000 pounds annually. The cost to transport the tomatoes from the farm to the stores is \$620 per load. The annual carrying cost is \$0.12 per pound.

22.

- 1.** Compute the optimal order size, the maximum inventory level, and the total minimum cost.
- 2.** If Spruce Creek can increase production capacity to 360,000 tomatoes per year, will it reduce total inventory cost?

The Uptown Kiln is an importer of ceramics from overseas. It has arranged to purchase a particular type of ceramic pottery from a Korean artisan. The artisan makes the pottery in 120-unit batches and will ship only that exact number of units. The

- 23.** transportation and handling cost of a shipment is \$7,600 (not including the unit cost). The Uptown Kiln estimates its annual demand to be 900 units. What storage and handling cost per unit does it need to achieve in order to minimize its inventory cost?

The I-75 Carpet Discount Store has an annual demand of 10,000

- yards of Super Shag carpet. The annual carrying cost for a yard of this carpet is \$0.75, and the ordering cost is \$150. The carpet manufacturer normally charges the store \$8 per yard for the carpet; however, the manufacturer has offered a discount price of \$6.50 per yard if the store will order 5,000 yards. How much should the store order, and what will be the total annual inventory cost for that order quantity?
- 24.**

- The Fifth Quarter Bar buys Old World draft beer by the barrel from a local distributor. The bar has an annual demand of 900 barrels, which it purchases at a price of \$205 per barrel. The annual carrying cost is 12% of the price, and the cost per order is \$160.
- 25.**

The distributor has offered the bar a reduced price of \$190 per barrel if it will order a minimum of 300 barrels. Should the bar take the discount?

[Page 766]

The bookstore at State University purchases from a vendor sweatshirts emblazoned with the school name and logo. The vendor sells the sweatshirts to the store for \$38 apiece. The cost to the bookstore for placing an order is \$120, and the carrying cost is 25% of the average annual inventory value. The bookstore manager estimates that 1,700 sweatshirts will be sold during the year. The vendor has offered the bookstore the following volume

discount schedule:

26.	Order Size	Discount %
	1299	0
	300499	2
	500799	4
	800+	5

The bookstore manager wants to determine the bookstore's optimal order quantity, given the foregoing quantity discount information.

Determine the optimal order quantity of sweatshirts and total annual cost in [Problem 26](#) if the carrying cost is a constant \$8 per sweatshirt per year.

The office manager for the Gotham Life Insurance Company orders letterhead stationery from an office products firm in boxes of 500 sheets. The company uses 6,500 boxes per year. Annual carrying costs are \$3 per box, and ordering costs are \$28. The following discount price schedule is provided by the office supply company:

**Order Quantity
(boxes)**

Price per Box

28. 200999	\$16
-------------------	------

1,0002,999	14
------------	----

3,0005,999	13
------------	----

6,000+	12
--------	----

Determine the optimal order quantity and the total annual inventory cost.

Determine the optimal order quantity and total annual inventory

29. cost for boxes of stationery in [Problem 28](#) if the carrying cost is 20% of the price of a box of stationery.

The 23,000-seat City Coliseum houses the local professional ice hockey, basketball, indoor soccer, and arena football teams, as well as various trade shows, wrestling and boxing matches, tractor pulls, and circuses. Coliseum vending annually sells large quantities of soft drinks and beer in plastic cups, with the name of the coliseum and the various team logos on them. The local container cup manufacturer that supplies the cups in boxes of 100 has offered coliseum management the following discount price schedule for cups:

	Order Quantity (boxes)	Price per Box
30.	2,000	\$47
	6,999	
	7,000	43
	11,999	
	12,000	41
	19,999	
	20,000+	38

[Page 767]

The annual demand for cups is 2.3 million, the annual carrying cost per box of cups is \$1.90, and the

ordering cost is \$320. Determine the optimal order quantity and total annual inventory cost.

Community Hospital orders latex sanitary gloves from a hospital supply firm. The hospital expects to use 40,000 pairs of gloves per year. The cost to order and to have the gloves delivered is \$180. The annual carrying cost is \$0.18 per pair of gloves. The hospital supply firm offers the following quantity discount pricing schedule:

Quantity	Price
09,999	\$0.34
10,00019,999	0.32

31.

20,00029,999	0.30
--------------	------

30,00039,999	0.28
--------------	------

40,00049,000	0.26
--------------	------

50,000+	0.24
---------	------

Determine the order size for the hospital.

Tracy McCoy is the office administrator for the department of management science at Tech.

The faculty uses a lot of printer paper, and although Tracy is constantly reordering, paper frequently runs out. She orders the paper from the university central stores. Several faculty members have determined that the lead time

- 32.** to receive an order is normally distributed, with a mean of 2 days and a standard deviation of 0.5 day. The faculty has also determined that daily demand for the paper is normally distributed, with a mean of 2 packages and a standard deviation of 0.8 package. What reorder point should Tracy use in order not to run out 99% of the time?

Determine the optimal order quantity and total annual inventory

- 33.** cost for cups in [Problem 30](#) if the

carrying cost is 5% of the price of a box of cups.

- The amount of denim used daily by the Western Jeans Company in its manufacturing process to make jeans is normally distributed, with an average of 3,000 yards of denim and a standard deviation of 600 yards. The lead time required
- 34.** to receive an order of denim from the textile mill is a constant 6 days. Determine the safety stock and reorder point if the Western Jeans Company wants to limit the probability of a stockout and work stoppage to 5%.

- In [Problem 34](#), what level of
- 35.** service would a safety stock of 2,000 yards provide?

The Atlantic Paper Company produces paper from wood pulp ordered from a lumber products firm. The paper company's daily demand for wood pulp is a constant 8,000 pounds. Lead time

36. is normally distributed, with an average of 7 days and a standard deviation of 1.6 days. Determine the reorder point if the paper company wants to limit the probability of a stockout and work stoppage to 2%.

The Uptown Bar and Grill serves Rainwater draft beer to its customers. The daily demand for beer is normally distributed, with an average of 18 gallons and a standard deviation of 4 gallons. The lead time required to receive

37. an order of beer from the local distributor is normally distributed, with a mean of 3 days and a standard deviation of 0.8 day. Determine the safety stock and reorder point if the restaurant wants to maintain a 90% service level. What would be the increase in the safety stock if a 95% service level were desired?

In [Problem 37](#), the manager of the Uptown Bar and Grill has negotiated with the beer distributor for the lead time to receive orders to be a constant 3 days. What effect does this have on the reorder point developed in [Problem 37](#) for a 90% service level?

The daily demand for Sunlight paint at the Rainbow Paint Store in East Ridge is normally distributed, with a mean of 26 gallons and a standard deviation of 10 gallons. The lead time for receiving an order of paint from the Sunlight distributor is 9 days. Because this is the only paint store in East Ridge, the manager is interested in maintaining only a

39. 75% service level. What reorder point should be used to meet this service level? The manager subsequently has learned that a new paint store will open soon in East Ridge, which has prompted her to increase the service level to 95%. What reorder point will maintain this service level?

PM Computers assembles personal computers from generic

- components. It purchases its color monitors from a manufacturer in Taiwan; thus, there is a long and uncertain lead time for receiving orders. Lead time is normally distributed, with a mean of 25 days and a standard deviation of 10 days. Daily demand is also normally distributed, with a mean of 2.5 monitors and a standard deviation of 1.2 monitors. Determine the safety stock and reorder point corresponding to a 90% service level.
- 40.**

- PM Computers in [Problem 40](#) is considering purchasing monitors from an American manufacturer that would guarantee a lead time of 8 days, instead of the Taiwanese company. Determine the new reorder point, given this lead time,
- 41.**

and identify the factors that would enter into the decision to change manufacturers.

- The Corner Drug Store fills prescriptions for a popular children's antibiotic, amoxicillin. The daily demand for amoxicillin is normally distributed, with a mean of 200 ounces and a standard deviation of 80 ounces. The vendor for the pharmaceutical firm that supplies the drug calls the
- 42.** drugstore pharmacist every 30 days to check the inventory of amoxicillin. During a call, the druggist indicated that the store had 60 ounces of the antibiotic in stock. The lead time to receive an order is 4 days. Determine the order size that will enable the drugstore to maintain a 95%

service level.

The Fast Service Food Mart stocks frozen pizzas in a refrigerated display case. The average daily demand for the pizzas is normally distributed, with a mean of 8 pizzas and a standard deviation of 2.5 pizzas. A vendor for a packaged food distributor checks the market's inventory of frozen foods every 10 days, and during a particular visit, there were no pizzas in stock. The lead time to receive an order is 3 days.

- 43.** Determine the order size for this order period that will result in a 99% service level. During the vendor's following visit, there were 5 frozen pizzas in stock. What is the order size for the next order period?

The Impanema Restaurant stocks a red Brazilian table wine it purchases from a wine merchant in a nearby city. The daily demand for the wine at the restaurant is normally distributed, with a mean of 18 bottles and a standard deviation of 4 bottles. The wine merchant sends a representative

- 44.** to check the restaurant's wine cellar every 30 days, and during a recent visit, there were 25 bottles in stock. The lead time to receive an order is 2 days. The restaurant manager has requested an order size that will enable him to limit the probability of a stockout to 2%. Determine the order size.

The concession stand at the

Blacksburg High School stadium sells slices of pizza during soccer games. Concession stand sales are a primary source of revenue for high school athletic programs, so the athletic director wants to sell as much food as possible.

However, any pizza not sold is given away to the players, coaches, and referees, or it is thrown away. The athletic director wants to determine a reorder point that will meet, not exceed, the demand for pizza. Pizza sales are normally distributed, with a mean of 6 pizzas per hour and a

45.

standard deviation of 2.5 pizzas. The pizzas are ordered from Pizza Town restaurant, and the mean delivery time is 30 minutes, with a standard deviation of 8 minutes.

- 1.** Currently, the concession stand places an order when it has 1 pizza left. What level of service does this result in?
- 2.** What should the reorder point be to have a 98% service level?

Case Problem

THE NORTHWOODS GENERAL STORE

The Northwoods General Store in Vermont sells a variety of outdoor clothing items and equipment and several food products at its modern but rustic-looking retail store. Its food products include salmon and maple syrup. The store also runs a lucrative catalog operation. One of its most popular products is maple

syrup, which is sold in metal half-gallon cans with a picture of the store on the front.

Maple syrup was one of the first products the store produced and sold, and it continues to do so. Setting up the syrup-making equipment to produce a batch of syrup costs \$450. Storing the syrup for sales throughout the year is a tricky process because the syrup must be kept in a temperature-controlled facility. The annual cost of carrying a gallon of the syrup is \$15. Based on past sales data, the

store has forecasted a demand of 7,500 gallons of maple syrup for the coming year. The store can produce approximately 100 gallons of syrup per day during the maple syrup season, which runs from November through February.

Because of the short season when the store can actually get sap out of trees, it obviously must produce enough during this 4-month season to meet demand for the whole year. Specifically, store management would like a production and inventory schedule that

minimizes costs and indicates when during the year they need to start operating the syrup-making facility full time on a daily basis to meet demand for the remaining 8 months.

Develop a syrup production and inventory schedule for the Northwoods General Store.

Case Problem

THE TEXANS STADIUM STORE

The Fort Worth Texans have won three Super Bowls in the past 5 years, including two in a row the past 2 years. As a result, sportswear such as hats, sweatshirts, sweatpants, and jackets with the Texans logo are particularly popular in Texas. The Texans operate a stadium store outside the football stadium where they

play. It is near a busy highway, so the store has heavy customer traffic throughout the year, not just on game days. In addition, the stadium holds high school or college football and soccer games almost every week in the fall, and it holds baseball games in the spring and summer. The most popular single item the stadium store sells is a blue and silver baseball cap with the Texans logo embroidered on it in a special and very attractive manner. The cap has an elastic headband inside it, which

automatically conforms to different head sizes. However, the store has had a difficult time keeping the cap in stock, especially during the time between the placement and receipt of an order. Often customers come to the store just for the hat; when it is not in stock, customers are visibly upset, and the store management believes they tend to go to competing stores to purchase their Texans clothing. To rectify this problem, the store manager, Jenny Jones, would like to

develop an inventory control policy that would ensure that customers would be able to purchase the cap 99% of the time they asked for it. Jenny has accumulated the following demand data for the cap for a 30-week period:

Week Demand Week Demand Week Demand

1 38 11 28 21

2 51 12 41 22

3 25 13 37 23

4	60	14	44	24
5	35	15	45	25
6	42	16	56	26
7	29	17	62	27
8	46	18	53	28
9	55	19	46	29
10	19	20	41	30

(Demand includes actual sales plus a record of the times a cap has been requested but not available and an estimate of the number of times a customer wanted a cap when it was not available but did not ask for it.)

The store purchases the hats from a small manufacturing company in Jamaica. The shipments from Jamaica are somewhat erratic, with a lead time anywhere between 10

days and 1 month. The following lead time data (in days) were accumulated during approximately a 1-year period:

[Page 770]

<i>Order</i>	<i>Lead Time</i>	<i>Order</i>	<i>Lead Time</i>
1	12	11	14
2	16	12	16
3	25	13	23

4	18	14	18
---	----	----	----

5	10	15	21
---	----	----	----

6	30	16	19
---	----	----	----

7	24	17	17
---	----	----	----

8	19	18	16
---	----	----	----

9	17	19	22
---	----	----	----

10	15	20	18
----	----	----	----

In the past, Jenny has placed an order whenever the stock got down to 150 caps. To what level of service does this reorder point correspond? What would the reorder point and safety stock need to be to attain the desired service level? Discuss how Jenny might determine the order size for caps and what additional, if any, information would be needed to determine the order size.

Case Problem

THE A-TO-Z OFFICE SUPPLY COMPANY

Christine Yamaguchi is the manager of the A-to-Z Office Supply Company in Charlotte. The company attempts to gain an advantage over its competitors by providing quality customer service, which includes prompt delivery of orders by truck or van and always being able to meet customer demand from its

stock. In order to achieve this degree of customer service, A-to-Z must stock a large volume of items on a daily basis at a central warehouse and at three retail stores in the city and suburbs. Christine maintains these inventory levels by borrowing cash on a daily basis from the First Piedmont Bank. She estimates that for the coming fiscal year, the company's demand for cash to pay for inventory will be \$17,000 per day for 305 working days. Any money she borrows during the year must

be repaid with interest by the end of the year. The annual interest rate currently charged by the bank is 9%. Any time Christine takes out a loan to purchase inventory, the bank charges the company a loan origination fee of \$1,200 plus 2 G points (2.25% of the amount borrowed).

Christine often uses EOQ analysis to determine optimal amounts of inventory to order for different office supplies. Now she is wondering if she can use the same type of analysis to determine an

optimal borrowing policy. Determine the amount of the loan Christine should secure from the bank, the total annual cost of the company's borrowing policy, and the number of loans the company should obtain during the year. Also determine the level of cash on hand at which the company should apply for a new loan, given that it takes 15 days for a loan to be processed by the bank.

Suppose the bank offers Christine a discount, as follows: On any loan amount equal to

or greater than \$500,000, the bank will lower the number of points charged on the loan origination fee from 2.25% to 2.00%. What would the company's optimal loan amount be?

Case Problem

DIAMANT FOODS COMPANY

Diamant Foods Company produces a variety of food products, including a line of candies. One of its most popular candy items is Divine Diamonds, a bag of a dozen individually wrapped, diamond-shaped candies made primarily from a blend of dark and milk chocolates, macadamia nuts, and a blend of heavy cream

fillings. The item is relatively expensive, so Diamant Foods produces it only for its eastern market, encompassing urban areas such as New York, Atlanta, Philadelphia, and Boston. The item is not sold in grocery or discount stores but mainly in specialty shops and specialty groceries, candy stores, and department stores. Diamant Foods supplies the candy to a single food distributor, which has several warehouses on the East Coast. The candy is shipped in cases of 60 bags of the candy per case.

Diamonds sell well, despite the fact that they are expensive, at \$9.85 per bag (wholesale). Diamant uses high-quality, fresh ingredients and does not store large stocks of the candy in inventory for very long periods of time.

Diamant's distributor believes that demand for the candy follows a seasonal pattern. It has collected demand data (i.e., cases sold) for Diamonds from its warehouses and the stores it supplies for the past 3 years, as follows:

Demand (cases)			
Month	<i>Year 1</i>	<i>Year 2</i>	<i>Year 3</i>
January	192	212	228
February	210	223	231
March	205	216	226
April	260	252	293
May	228	235	246

June	172	220	229
July	160	209	217
August	147	231	226
September	256	263	302
October	342	370	410
November	261	260	279
December	273	277	293

The distributor must hold the candy inventory in climate-controlled warehouses and be careful in handling it. The annual carrying cost is \$116 per case. Diamonds must be shipped a long distance from the manufacturer to the distributor, and in order to keep the candy as fresh as possible, trucks must be air-conditioned, shipments must be direct, and shipments are often less than truck-load. As a result, the ordering cost is \$4,700.

Diamant Foods makes
Diamonds from three primary

ingredients it orders from different suppliers: dark and milk chocolate, macadamia nuts, and a special heavy cream filling. Except for its unique shape, a Diamond is almost like a chocolate truffle. Each Diamond weighs 1.2 ounces and requires 0.70 ounce of blended chocolates, 0.50 ounce of macadamia nuts, and 0.40 ounce of filling to produce (including spillage and waste). Diamant Foods orders chocolate, nuts, and filling from its suppliers by the pound. The annual ordering cost is \$5,700

for chocolate, and the annual carrying cost is \$0.45 per pound. The ordering cost for macadamia nuts is \$6,300, and the annual carrying cost is \$0.63 per pound. The ordering cost for filling is \$4,500, and the annual carrying cost is \$0.55 per pound.

Each of the suppliers offers the candy manufacturer a quantity discount price schedule for the ingredients, as follows:

Chocolate	Macadamia Nut
------------------	----------------------

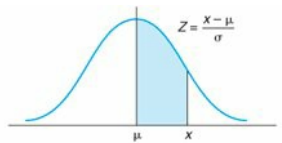
<i>Price</i>	<i>Quantity (lb.)</i>	<i>Price</i>	<i>Quantity (lb.)</i>
\$3.05	050,000	\$6.50	030,000
2.90	50,001100,000	6.25	30,00170,000
2.75	100,001150,000	5.95	70,000+
2.60	150,000+		

Determine the inventory order quantity for Diamant's distributor. Compare the optimal order quantity with a

seasonally adjusted forecast for demand. Does the order quantity seem adequate to meet the seasonal demand pattern for Diamonds (i.e., is it likely that shortages or excessive inventories will occur)? Can you identify the causes of the seasonal demand pattern for Diamonds? Determine the inventory order quantity for each of the three primary ingredients that Diamant Foods orders from its suppliers.

Appendix A. Normal and Chi-Square Tables

Table A.1. The normal table Normal curve areas

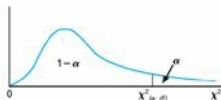


Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990

Table A.2. Chi-square table

For a particular number of degrees of freedom, an entry represents the critical value of χ^2 corresponding to a specified upper tail

area (α)



Upper Tail Areas (α)

Degrees of Freedom	.995	.99	.975	.95	.90	.75	.25	.10	.05	.025	.01	.005
1			0.001	0.004	0.016	0.102	1.323	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	0.575	2.773	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	1.213	4.108	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	1.923	5.385	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	2.675	6.626	9.236	11.071	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	3.455	7.841	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	4.255	9.037	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	5.071	10.219	13.362	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	4.168	5.899	11.389	14.684	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	4.865	6.737	12.549	15.987	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	5.578	7.584	13.701	17.275	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	6.304	8.438	14.845	18.549	21.026	23.337	26.217	28.299
13	3.565	4.107	5.009	5.892	7.042	9.299	15.984	19.812	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	7.790	10.165	17.117	21.064	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	8.547	11.037	18.245	22.307	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	9.312	11.912	19.369	23.542	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	10.085	12.792	20.489	24.769	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	10.865	13.675	21.605	25.989	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	11.651	14.562	22.718	27.204	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	12.443	15.452	23.828	28.412	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	13.240	16.344	24.935	29.615	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	14.042	17.240	26.039	30.813	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	14.848	18.137	27.141	32.007	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.848	15.659	19.037	28.241	33.196	36.415	39.364	42.980	45.559
25	10.520	11.524	13.120	14.611	16.473	19.939	29.339	34.382	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	17.292	20.843	30.435	35.563	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	18.114	21.749	31.528	36.741	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	18.939	22.657	32.620	37.916	41.337	44.461	48.278	50.993
29	13.121	14.257	16.047	17.708	19.768	23.567	33.711	39.087	42.557	45.722	49.588	52.336
30	13.787	14.954	16.791	18.493	20.599	24.478	34.800	40.256	43.773	46.979	50.892	53.672

[\[View Full Width\]](#)

Appendix B. Setting Up and Editing a Spreadsheet

One of the benefits of using Excel spreadsheets to solve management science problems is that spreadsheets provide a nice medium for visual presentation. They enable the user to customize the problem presentation in any manner or format desired. However, problems and models do not just "appear" on spreadsheets

as they do in this book without some careful editing. The purpose of this brief appendix is to review some of the steps required to set up and edit spreadsheets like the ones shown in this book.

Virtually all the spreadsheet editing functions and tools can be accessed directly from the toolbars at the top of the spreadsheet window. [Exhibit B.1](#) shows our Excel spreadsheet originally shown in [Exhibit 3.4](#) in [Chapter 3](#) with the solution to the Beaver Creek Pottery Company linear

programming example. We will describe the Excel editing features as they apply to this example spreadsheet. If you have not read [Chapter 3](#) yet, don't worry; it's not necessary to know how to solve a linear programming problem in order to understand the spreadsheet editing tools.

Titles and Headings

The title "The Beaver Creek Pottery Company" was typed in cell A1. The **bold type** for this heading was created by activating the bold type button, "**B**," on the toolbar. Note that we could have centered this heading across the top of the spreadsheet by covering row 1 with the cursor and clicking on the button with an "a" with arrows on both sides of it. *Italic type* for headings such as the

"Products" heading in cell A3 is created by using the italic button, *"I,"* on the toolbar.

Borders

You will notice that all the data and headings in [Exhibit B.1](#) are enclosed by boxes created by black line borders. These borders are created by clicking on the "Borders" button on the toolbar at the top of the spreadsheet. Clicking on this button will bring down a selection of different line locations and line intensities for borders. Activate the border you want, cover the area on

your spreadsheet where you want to include this border, and then click on the activated border that you selected. Most of the borders in [Exhibit B.1](#) were created by using a border that results in a closed box. For example, the border that surrounds the headings and numbers in cells **A3:D7** was created by covering this area with the cursor and clicking on the "Borders" button at the top of the spreadsheet.

Exhibit B.1.

[\[View full size image\]](#)

The image is a screenshot of the Microsoft Excel 2003 application window. The menu bar at the top includes File, Edit, Format, Tools, Window, and Help. The toolbar below the menu bar contains icons for various functions. The spreadsheet area shows a worksheet named 'The Beaver Creek Pottery Company'. The data is organized into two tables: 'Products' and 'Resources'. The 'Products' table has columns for Product, Hour, and Mug. The 'Resources' table has columns for Resource, Constraint, Available, and Left over. The 'Production' section shows the total hours for each product and the total profit. The formula bar at the bottom shows the formula entered in cell B12: $=C4*B10+D4*B11$.

Callouts and their descriptions:

- Bold type**: Points to the Bold button in the toolbar.
- Italic type**: Points to the Italic button in the toolbar.
- Click on the "Borders" button to create borders.**: Points to the Borders button in the toolbar.
- Increases and decreases decimal places.**: Points to the Increase and Decrease Decimal Places buttons in the toolbar.
- Centers items in columns.**: Points to the Center button in the toolbar.
- Click here to vary spreadsheet screen size.**: Points to the Zoom button in the toolbar.
- Click on "File," then select "Zoom" to reduce or increase the size of the spreadsheet screen.**: Points to the File menu.
- Click on "File" and then "Page Setup" to set up a printed spreadsheet.**: Points to the File menu.
- The formula entered in cell B12 is $=C4*B10+D4*B11$, which is also shown on the formula bar.**: Points to the formula bar.

Product	Hour	Mug
Profit per unit	40	50

Resource	Constraint	Available	Left over	
labor (hr/unit)	1	2	40	0
clay (lb/unit)	4	1	120	0

Production	
total =	36
Mugs =	8
Profit =	1080

Column Centering

Headings and numbers can be centered in the columns (or moved to the right or left side of the column) by using the "Centering" button on the toolbar. To center a column of values, such as column C in [Exhibit B.1](#), cover cells **C3:C7** with the cursor and then click on the "Centering" button.

Deleting and Inserting Rows and Columns

Rows and columns can be inserted in your spreadsheet by clicking the cursor at the row or column you want a new row or column to appear next to; then click on "Insert" at the top of the spreadsheet window and select to insert either rows or columns from the menu. Rows and columns can be deleted by clicking the cursor on the row or column you want to delete

and then selecting "Delete" from the "Edit" menu at the top of the spreadsheet window.

Decimal Places

The number of decimal places in a numeral can be reduced, increased, or eliminated by using the "Decimal" buttons on the toolbar. Notice that there is one button to increase the number of decimal places and one button to decrease the number of decimal places.

Increasing or Decreasing the Spreadsheet Area

The spreadsheet in [Exhibit B.1](#) has fourteen columns (A through N) and 13 rows, which are less than the numbers of rows and columns that are present when a new spreadsheet is activated. Notice the small window in the upper-right-hand corner with the number "130%." This indicates that the spreadsheet is 130% its normal (100%) size, thus

reducing the number of rows and columns visible on the screen. For this example the spreadsheet was blown up a little, strictly for presentation purposes, so the typed material in the spreadsheet would be easier to read. To increase or reduce the visible spreadsheet, click on "View" at the top of the window, then click on "Zoom," and this will display a window in which you can alter the size of the spreadsheet. Increasing the size above 100% will reduce the number of rows and columns, and decreasing the

spreadsheet size below 100% will increase the number of rows and columns that will appear on the spreadsheet screen.

Expanding or Reducing Column and Row Widths

Sometimes it is desirable to expand the width of a column so that you can see a longer heading or a bigger number, or to reduce the width to reflect a smaller number. To accomplish this, click and hold the left mouse button at the very top of the column between the two column letters on the line separating the column you want to expand and the column

next to it. This should result in a set of "crossed arrows," which you can move left or right to expand or reduce the width of the column by the amount you want. The same procedure can be followed to expand row widths.

Inserting an Equation or a Formula into a Cell

To insert a number or word in a cell, you normally position the cursor on the target cell and type the word or number, then press Enter or click the cursor on another cell. To type a formula or an equation into a cell, you click the cursor on the target cell and then an = sign, then your formula (for example, the formula in cell B12 in [Exhibit B.1](#)). The

formula is shown on the formula bar at the top of the spreadsheet. Be aware that Excel is very sensitive to the order of mathematical operations, so your formulas must be very clearly stated. In general, you use an * to show multiplication and a / to show division.

Printing a Spreadsheet

The spreadsheets shown in the various exhibits in this book were developed by using a "screen capture" program, which allows the entire screen to be printed just as you would see it on the screen. However, unless your computer has a screen capture program, you should not expect your spreadsheets to look exactly like the ones in this book when you print them out. Instead,

you will print your spreadsheet out by using the normal Excel and Windows print routines. To set up your spreadsheet document for printing, activate the "File" menu located in the top-left-hand corner of the window and then select "Page Setup." This will result in the window shown in [Exhibit B.2](#). This screen enables you to reduce or increase the spreadsheet size as well as position the spreadsheet on the page in portrait or landscape format. Notice the "Header/Footer" tab at the top

of this window. By clicking on this tab, you can delete the page numbers and other spreadsheet information, called headers and footers, that normally appears at the top and bottom of a printed spreadsheet, or you can customize your own header and footer. By clicking on the "Sheet" tab at the top of the "Page Setup" window, you can remove the grid lines that separate the cells on every spreadsheet. When you do so, only the borders that you have created will show up. You can

preview what your spreadsheet will look like by selecting "Print Preview" at top right or by returning to the "File" menu.

[Page 778]

Exhibit B.2.

Access the "Page Setup" window from the "File" menu.

Click here to eliminate or alter headers and footers.

Click on the "Sheet" tab to eliminate grid lines.

Page Setup [?] [X]

Page Margins Header/Footer Sheet

Orientation

 ☒ Portrait  ☐ Landscape

Scaling

☒ Adjust to: 130 % normal size

☐ Fit to: 1 page(s) wide by 1 tall

Paper size: Letter

Print quality: 300 dpi

First page number: Auto

Print...
Print Preview
Options...

OK Cancel

The editing features just described are the most basic and are provided to help you get started in setting up and solving problems by using Excel spreadsheets. More sophisticated Excel functions that can be used to solve problems are located in specific chapters where they are applied and can be located by using the book's index.

Appendix C. The Poisson and Exponential Distributions

[The Poisson Distribution](#)

[The Exponential Distribution](#)

The Poisson Distribution

The formula for a Poisson distribution is

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}$$

where

λ = average arrival rate (i.e., arrivals during a specified period of time)

x = number of arrivals during the specified time period

$$e = 2.71828$$

$x!$ = the factorial of a value, x

[i.e., $x! = x (x - 1) (x - 2) \dots (3) (2) (1)$]

As an example of this distribution, consider a service facility that has an average arrival rate of five customers per hour ($\lambda = 5$). The probability that exactly two customers will arrive at the service facility is found by letting $x = 2$ in the preceding Poisson formula:

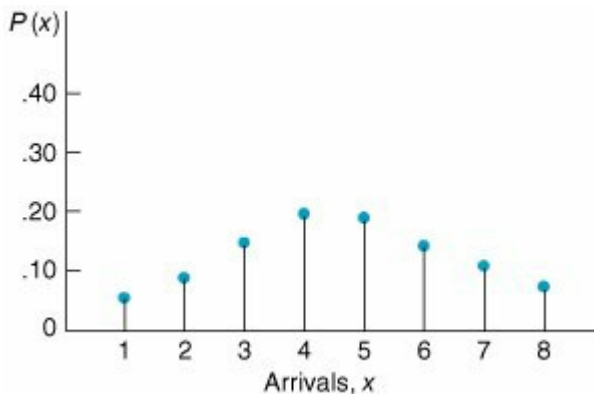
$$P(x = 2) = \frac{5^2 e^{-5}}{2!} = \frac{25(.007)}{(2)(1)} = .084$$

The value .084 is the probability of exactly two customers' arriving at the service facility.

By substituting values of x into the Poisson formula, we can develop a distribution of customer arrivals during a 1-hour period, as shown in [Figure C.1](#). However, remember that this distribution is for an arrival rate of five customers per hour. Other values of λ will result in distributions different from the

one in [Figure C.1](#).

Figure C.1. Poisson distribution for $\lambda = 5$



The Exponential Distribution

The formula for the exponential distribution is

$$f(t) = \mu e^{-\mu t}, t \geq 0$$

where

μ = average number of customers served during a specified period of time

t = service time

$$e = 2.71828$$

The *probability* that a customer will be served within a specified time period can be determined by using the exponential distribution, in the following form:

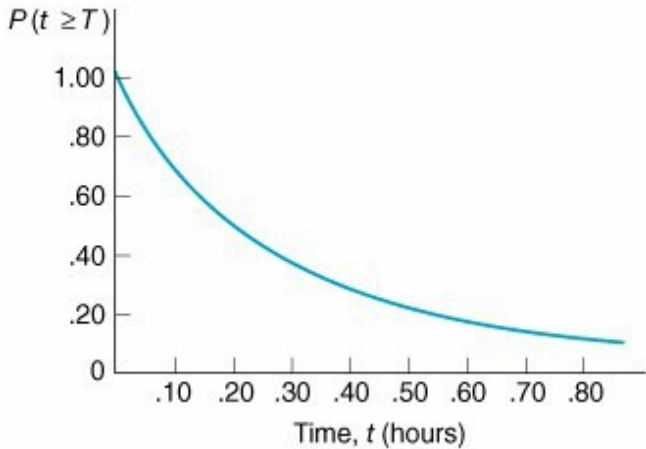
$$P(T \leq t) = 1 - e^{-\mu t}$$

If, for example, the service rate is six customers per hour, then the probability that a customer will be served within 10 minutes (.17 hour) is determined as follows:

$$\begin{aligned}P(T \leq .17) &= 1 - e^{-6(.17)} \\&= 1 - e^{-1.0} \\&= 1 - .368 \\&= .632\end{aligned}$$

Thus, the probability of a customer's being served within 10 minutes is .632. [Figure C.2](#) is the exponential probability distribution for this service rate ($\mu = 6$).

Figure C.2. Exponential distribution for $\mu = 6$



Solutions to Selected Odd- Numbered Problems

[Chapter 1](#)

[Chapter 2](#)

[Chapter 3](#)

[Chapter 4](#)

[Chapter 5](#)

[Chapter 6](#)

[Chapter 7](#)

[Chapter 8](#)

[Chapter 9](#)

[Chapter 10](#)

[Chapter 11](#)

[Chapter 12](#)

[Chapter 13](#)

[Chapter 14](#)

[Chapter 15](#)

[Chapter 16](#)

Chapter 1

1. (a) $TC = \$27,500$; $TR = \$54,000$;
 $Z = \$26,500$; (b) $v = 69.56$ tables

3. (a) $TC = \$29,100$; $TR = \$23,400$;
 $Z = \$5,700$; (b) $v = 24,705.88$
yd./month

7. (a) $v = 1,250$ dolls

9. 98.8%

11. reduces v to 2,727.3 tires

increases v to 65,789.47 lb.

13.

15. do not raise price

17. yes, v will be reduced

19. (a) executive plan; (b) 937.5 min.
per month

21. (a) $v = 26.9$ pupils; (b) $v = 106.3$
pupils; (c) $p = \$123.67$

23. $v = 209.2$ teams

25. $x = 30, y = 10, Z = \$1,400$

27. 34,500 visits

Chapter 2

1. $x_1 = 4, x_2 = 0, Z = 40$

(a) $\min. Z = .05x_1 + .03x_2$; s.t.

3. $8x_1 + 6x_2 \geq 48, x_1 + 2x_2 \geq 12,$
 $x_i \geq 0$; (b) $x_1 = 0, x_2 = 8, Z =$
 0.24

(a) $\min. Z = 3x_1 + 5x_2$; s.t. $10x_1 +$

5. $2x_2 \geq 20, 6x_1 + 6x_2 \geq 36, x_2 \geq$
 $2, x_i \geq 0$; (b) $x_1 = 4, x_2 = 2, Z =$
 22

7. No labor, 4.8 lb. wood

9. (a) $\max. Z = x_1 + 5x_2$; s.t. $5x_1 + 5x_2 \leq 25$, $2x_1 + 4x_2 \leq 16$, $x_1 \leq 5$, $x_1, x_2 \geq 0$; (b) $x_1 = 0$, $x_2 = 4$, $Z = 20$

11. $x_1 = 0$, $x_2 = 9$, $Z = 54$

13. (a) $\max. Z = 300x_1 + 400x_2$; s.t. $3x_1 + 2x_2 \leq 18$, $2x_1 + 4x_2 \leq 20$, $x_2 \leq 4$, $x_1, x_2 \geq 0$; (b) $x_1 = 4$, $x_2 = 3$, $Z = 2,400$

15. (a) maximum demand is not achieved by one bracelet; (b) \$600

17. $x_1 = 15.8, x_2 = 20.5, Z = 1,610$

A: $s_1 = 4, s_2 = 1, s_3 = 0$; B: $s_1 =$
19. $0, s_2 = 5, s_3 = 0$; C: $s_1 = 0, s_2 =$
 $6, s_3 = 1$

A: $s_1 = 0, s_2 = 0, s_3 = 8, s_4 = 0$;
B: $s_1 = 0, s_2 = 3.2, s_3 = 0, s_4 =$
21. 4.8 ; C: $s_1 = 26, s_2 = 24, s_3 = 0,$
 $s_4 = 10$

23. changes the optimal solution

25. $x_1 = 28.125, x_2 = 0, Z =$
 $\$1,671.95$; no effect

27. infeasible solution

29. $x_1 = 4, x_2 = 1, Z = 18$

31. $x_1 = 4.8, x_2 = 2.4, Z = 26.4$

33. $x_1 = 3.2, x_2 = 6, Z = 37.6$

35. no additional profit

37. (a) max. $Z = 800x_1 + 900x_2$; s.t.
 $2x_1 + 4x_2 \leq 30, 4x_1 + 2x_2 \leq 30,$
 $x_1 + x_2 \geq 9, x_i \geq 0$; (b) $x_1 = 5,$
 $x_2 = 5, Z = 8,500$

39. $x_1 = 5.3, x_2 = 4.7, Z = 806$

41. (a) 12 hr.; (b) new solution: $x_1 = 5.09$, $x_2 = 5.45$, $Z = 111.27$

43. $x_1 = 38.4$, $x_2 = 57.6$, $Z = 19.78$;
profit reduced

(a) $\min. Z = .09x_1 + .18x_2$, s.t.
 $.46x_1 + .35x_2 \leq 2,000$, $x_1 \geq 1,000$, $x_2 \geq 1,000$, $.91x_1 + .82x_2 = 3,500$, $x_1 \geq 0$, $x_2 \geq 0$; (b) 32 fewer defects

47. $x_1 = 160$, $x_2 = 106.67$, $Z = 568$

49. $x_1 = 25.71$, $x_2 = 14.29$, $Z = 14,571$

(a) $\max. Z = .18x_1 + .06x_2$, s.t. $x_1 + x_2 \leq 720,000$, $x_1/(x_1 + x_2) \leq$

51. $.65$, $.22x_1 + .05x_2 \leq 100,000$, $x_1, x_2 \geq 0$ (b) $x_1 = 376,470.59$, $x_2 = 343,526.41$, $Z = 88,376.47$

53. one more hour for Sarah would reduce the regraded exams from 10 to 9.8; another hour for Brad has no effect

55. only more Columbian would affect the solution; 1 lb of Columbian would increase sales to \$463.20; increasing the brewing capacity has no effect; extra advertising increases sales

57. infeasible

Chapter 3

Cells: B10:B12; Constraints:

B10:B12 ≥ 0 , G6 $\leq$ F6, G7 $\leq$ F7;

3. Profit: =

B10*C4+B11*D4+B12*E4; Target cell = B13.

(a and b) $\max. Z = 12x_1 + 16x_2 +$

$0s_1 + 0s_2$; s.t. $3x_1 + 2x_2 + s_1 =$

5. $500, 4x_1 + 5x_2 + s_2 = 800, x_i \geq$

$0, s_i \geq 0$

(a and b) $\infty \leq c_1, \leq 12.8, 15 \leq c_2$

7. $\leq \infty, 320 \leq q_1 \leq \infty, 0 \leq q_2 \leq$

1,250; (c) \$0, \$3.20

(a) $x_1 = 4, x_2 = 3, Z = 57$, A: $s_1 = 40, s_2 = 0$, B: $s_1 = 0, s_2 = 0$, C: $s_1 = 0, s_2 = 20$; (b) $x_1 = 2, x_2 = 4$; (c) solution point same for profit = \$15, new solution $x_1 = 0, x_2 = 5, Z = 100$ for profit = \$20

(a and b) $\max. Z = 2.25x_1 + 3.10x_2 + 0s_1 + 0s_2 + 0s_3$; s.t.

11. $5.0x_1 + 7.5x_2 + s_1 = 6,500, 3.0x_1 + 3.2x_2 + s_2 = 3,000, x_2 + s_3 = 510, x_i \geq 0, s_j \geq 0$

(a) Additional processing time, \$0.75/hr.; (b) $0 \leq c_1 \leq 2.906, 2.4 \leq c_2 \leq \infty, 6,015 \leq q_1 \leq \infty$,

13.

$$1,632 \leq q_2 \leq 3,237, 0 \leq q_3 \leq 692.308$$

- 15.** (a) $x_1 = 4, x_2 = 0, s_1 = 12, s_2 = 0, s_3 = 11, Z = 24,000$; (b) $x_1 = 1, x_2 = 3, Z = 28,500$; (c) C still optimal

- 17.** (a and b) $\max. Z = 300x_1 + 520x_2$; s.t. $x_1 + x_2 = 410, 105x_1 + 210x_2 = 52,500, x_2 = 100, x_i \geq 0$

- 19.** (a) no, max. price = \$80; (b) \$2.095

(a) $x_1 = 300, x_2 = 100, s_1 = 0 \text{ lb.}, s_2 = 15 \text{ lb.}, s_4 = 0.6 \text{ hr.}, Z = 230;$

21. (b) $c_1 = \$0.60, x_1 = 257, x_2 = 143, Z = 240;$ (c) $x_1 = 300, x_2 = 125, Z = 242.50$

(a and b) $\min. Z = 50x_1 + 70x_2;$

23. s.t. $80x_1 + 40x_2 = 3,000, 80x_1 = 1,000, 40x_2 = 800, x_i$

(a) personal interviews,

25. \$0.625/interview; (b) $25 \leq c_2 \leq \infty, 1,800 \leq q_1 \leq \infty$

(a) $x_1 = 333.3, x_2 = 166.7, s_1 = 100 \text{ gal.}, s_2 = 133.3 \text{ gal.}, s_3 =$

27. $83.3 \text{ gal.}, s_4 = 100 \text{ gal.}, Z = 1,666;$ (b) any values of c

$$\text{max. } Z = 1.20x_1 + 1.30x_2; \text{ s.t. } x_1 + x_2 \leq 95,000, .18x_1 + .30x_2 \leq 20,000, x_i \geq 0$$

29.

31. (a) 5%, \$16,111.11; (b) $x_1 = 0$, $x_2 = 66,666.7$, $Z = 86,666.67$

(a and b) $\text{min. } Z = 400x_1 + 180x_2 + 90x_3$ s.t. $x_1 \geq 200$, $x_2 \geq 300$, $x_3 \geq 100$, $4x_3 - x_1 - x_2 \leq 0$, $x_1 + x_2 + x_3 = 1,000$, $x_i \geq 0$; (c) $x_1 = 200$, $x_2 = 600$, $x_3 = 200$, $Z = 206,000$

33.

$$\text{max. } Z = 0.50x_1 + 0.75x_2; \text{ s.t. } 0.17x_1 + 0.25x_2 \leq 4,000, x_1 + x_2 \leq 18,000, x_1 \geq 8,000, x_2 \geq$$

35. $8,000, x_1, x_2 \geq 0$; (b) $\max. Z = 0.50x_1 + 0.75x_2$; $0.17x_1 + 0.25x_2 = 4,000, x_1 + x_2 = 18,000, x_1 = 8,000, x_2 = 8,000, x_i \geq 0$.

$x_1 = 8,000, x_2 = 10,000, Z = 11,500$; (a) \$375; increase, $x_1 =$

37. $8,000, x_2 = 10,500, Z = \$11,875$; $x_1 = 8,000, x_2 = 10,560, Z = \$11,920$

39. (a) purchase land; (b) not purchase land

(a) \$0.78, 360 cartons; (b) \$0;

41. (c) $x_1 = 108, x_2 = 54, x_3 = 162, Z = 249.48$, no discount

43. $x_1 = 3, x_3 = 6, Z = 3,600$; (a) more assembly hr.; (b) additional profit = \$600; (c) no effect

$x_1 = 1,000, x_2 = 800, x_3 = 200, Z = 760$; (a) increase by 100, \$38 in additional profit; (b) $x_1 = 1,000, x_2 = 1,000, Z = 770$; (c) $Z = 810$;
 $x_1 = 1,600, x_2 = 200, x_3 = 200$

47. $x_1 = 20, x_2 = 33.33, x_3 = 26.67, Z = \$703,333.4$

$x_{13} = 350, x_{21} = 158.33, x_{22} = 296.67, x_{23} = 75, x_{31} = 610, x_{42} = 240, Z = \$77,910$

51.

$$\begin{aligned} \max Z &= 12x_1 + 8x_2 + 10x_3 + 6x_4; \text{ s.t. } 2x_1 + 9x_2 + 1.3x_3 + 2.5x_4 \leq 74; \\ 2x_1 + .25x_2 + x_3 + x_4 &\leq 24; 2x_1 + 5x_3 + 2x_4 \leq 36; \\ 45x_1 + 35x_2 + 50x_3 + 16x_4 &\leq 480, x_i \geq 0. \end{aligned}$$

Chapter 4

Model must be resolved; $Z =$

- 1.** 43,310, do not implement; no; $x_1 = x_2 = x_3 = x_4 = 112.5$

No effect; \$740; $x_1 =$

- 3.** 22,363.636, $x_3 = 43,636.364$, $x_4 = 14,000$

Add slack variables for 3 warehouses $\leq$ constraints;

- 5.** coefficients in objective function \$9 for s_1 , \$6 for s_2 , \$7 for s_3 , solution does not change

$$(a) \max. Z = 190x_1 + 170x_2 + 155x_3; \text{ s.t. } 3.5x_1 + 5.2x_2 + 2.8x_3 \leq 500, 1.2x_1 + 0.8x_2 + 1.5x_3 \leq$$

7. $240, 40x_1 + 55x_2 + 20x_3 \leq 6,500, x_i \geq 0; (b) x_1 = 41.27, x_2 = 0, x_3 = 126.98, Z = 27,523.81$

$$(a) \min. Z = 4x_1 + 3x_2 + 2x_3; \text{ s.t. }$$

9. $2x_1 + 4x_2 + x_3 \geq 16, 3x_1 + 2x_2 + x_3 \geq 12, x_i \geq 0; (b) x_1 = 2, x_2 = 3, Z = \0.17

$$(a) \max. Z = 8x_1 + 10x_2 + 7x_3;$$

$$\text{s.t. } 7x_1 + 10x_2 + 5x_3 \leq 2,000,$$

$$2x_1 + 3x_2 + 2x_3 \leq 660, x_1 \leq$$

11. $200, x_2 \leq 300, x_3 \leq 150, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0; (b) x_1 = 178.57, x_3 = 150, Z = 2,478.57$

(a) $\max. Z = 7x_1 + 5x_2 + 5x_3 + 4x_4$; s.t. $2x_1 + 4x_2 + 2x_3 + 3x_4 \leq 45,000$, $x_1 + x_2 \leq 6,000$, $x_3 + x_4 \leq 7,000$, $x_1 + x_3 \leq 5,000$, $x_2 + x_4 \leq 6,000$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$; (b) $x_1 = 5,000$, $x_2 = 1,000$, $x_4 = 5,000$, $Z = 60,000$

13.

(a) $\max. Z = 1,800x_{1a} + 2,100x_{1b} + 1,600x_{1c} + 1,000x_{2a} + 700x_{2b} + 900x_{2c} + 1,400x_{3a} + 800x_{3b} + 2,200x_{3c}$; s.t. $x_{1a} + x_{1b} + x_{1c} = 30$, $x_{2a} + x_{2b} + x_{2c} = 30$, $x_{3a} + x_{3b} + x_{3c} = 30$, $x_{1a} + x_{2a} + x_{3a} \leq 40$, $x_{1b} + x_{2b} + x_{3b} \leq 60$, $x_{1c} + x_{2c} + x_{3c} \leq 50$, $x_{ij} \geq 0$; (b) $x_{1b} = 30$, $x_{2a} = 30$, $x_{3c} = 30$, $Z =$

15.

159,000

(a) $\min. Z = 1.7(x_{t1} + x_{t2} + x_{t3}) + 2.8(x_{m1} + x_{m2} + x_{m3}) + 3.25(x_{b1} + x_{b2} + x_{b3});$ s.t. $.50x_{t1} - .50x_{m1} - .50x_{b1} \leq 0;$ $-.20x_{t1} + .80x_{m1} - .20x_{b1} \geq 0,$ $-.30x_{t2} - .30x_{m2} + .70x_{b2} \geq 0,$ $-.30x_{t2} + .70x_{m2} - .30x_{b2} \geq 0,$ $.80x_{t2} - .20x_{m2} - .20x_{b2} \leq 0,$ $.50x_{t3} - .50x_{m3} - .50x_{b3} \geq 0,$ $.30x_{t3} - .70x_{m3} - .70x_{b3} \leq 0,$ $-.10x_{t3} - .10x_{m3} + .90x_{b3} \geq 0,$ $x_{t1} + x_{m1} + x_{b1} \geq 1,200,$ $x_{t2} + x_{m2} + x_{b2} \geq 900,$ $x_{t3} + x_{m3} + x_{b3} \geq 2,400;$ $x_{ij} \geq 0$ (b) $x_{t1} = 600, x_{t2} = 180, x_{t3} = 1,680,$ $x_{m1} = 600, x_{m2} = 450, x_{m3} = 480, x_{b1} = 0, x_{b2} = 270, x_{b3} = 240, Z = 10,123.50$

17.

$$\begin{aligned}
 & \text{(a) max. } Z = .02x_1 + .09x_2 + .06x_3 + .04x_4; \text{ s.t. } x_1 + x_2 + x_3 + x_4 = 4,000,000, \\
 & x_1 \leq 1,600,000, x_2 \leq 1,600,000, x_3 \leq 1,600,000, \\
 & \text{19. } x_4 \leq 1,600,000, x_2 - x_3 - x_4 \leq 0, \\
 & x_1 - x_3 \geq 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0; \\
 & \text{(b) } x_1 = 800,000, x_2 = 1,600,000, x_3 = 800,000, x_4 = 800,000, Z = 240,000
 \end{aligned}$$

$$\begin{aligned}
 & \text{(a) max. } Z = 7.8x_{11} + 7.8x_{12} + 8.2x_{13} + 7.9x_{14} + 6.7x_{21} + \\
 & 8.9x_{22} + 9.2x_{23} + 6.3x_{24} + 8.4x_{31} + 8.1x_{32} + 9.0x_{33} + \\
 & 5.8x_{34}; \text{ s.t. } 35x_{11} + 40x_{21} + 38x_{31} \leq 9,000, \\
 & 41x_{12} + 36x_{22} + 37x_{32} \leq 14,400, \\
 & 34x_{13} + 32x_{23} +
 \end{aligned}$$

21. $33x_{33} \leq 12,000, 39x_{14} + 43x_{24} + 40x_{34} \leq 15,000, x_{11} + x_{12} + x_{13} + x_{14} = 400, x_{21} + x_{22} + x_{23} + x_{24} = 570, x_{31} + x_{32} + x_{33} + x_{34} = 320, x_{ij} \geq 0;$ (b) $x_{11} = 15.385, x_{14} = 384.615, x_{22} = 400, x_{23} = 170, x_{31} = 121.212, x_{33} = 198.788, Z = 11,089.73$

(a) $\min. Z = 1.7x_{11} + 1.4x_{12} + 1.2x_{13} + 1.1x_{14} + 1.05x_{15} + 1.0x_{16} + 1.7x_{21} + 1.4x_{22} + 1.2x_{23} + 1.1x_{24} + 1.05x_{25} + 1.7x_{31} + 1.4x_{32} + 1.2x_{33} + 1.1x_{34} + 1.7x_{41} + 1.4x_{42} + 1.2x_{43} + 1.7x_{51} + 1.4x_{52} + 1.7x_{61};$ s.t. $x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} = 47,000, x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{21} + x_{22} +$

$x_{23} + x_{24} + x_{25} = 35,000$, $x_{13} +$
23. $x_{14} + x_{15} + x_{16} + x_{22} + x_{23} + x_{24}$
 $+ x_{25} + x_{31} + x_{32} + x_{33} + x_{34} =$
 $52,000$, $x_{14} + x_{15} + x_{16} + x_{23} +$
 $x_{24} + x_{25} + x_{32} + x_{33} + x_{34} + x_{41}$
 $+ x_{42} + x_{43} = 27,000$, $x_{15} + x_{16}$
 $+ x_{24} + x_{25} + x_{33} + x_{34} + x_{42} +$
 $x_{43} + x_{51} + x_{52} = 19,000$, $x_{16} +$
 $x_{25} + x_{34} + x_{43} + x_{52} + x_{61} =$
 $15,000$, (b) $x_{11} = 12,000$, $x_{13} =$
 $25,000$, $x_{14} = 8,000$, $x_{15} =$
 $2,000$, $x_{33} = 2,000$, $x_{34} =$
 $15,000$, $Z = \$80,200$; (c) $x_{16} =$
 $52,000$, $Z = \$52,000$

(a) add $x_{ss} \leq 150$, $x_{ww} \leq 300$, x_{cc}
 ≤ 250 ; $x_{nc} = 700$, $x_{nw} = 0$, $x_{sw} =$
 150 , $x_{ss} = 150$, $x_{es} = 900$, $x_{wc} =$
 250 , $x_{ww} = 300$, $x_{cc} = 250$, $x_{cw} =$

25. $250, x_{ws} = 50, Z = 20,400$; (b) changes demand constraints from $\leq 1,200$ to $= 1,000$; $x_{nc} = 400, x_{nw} = 300, x_{sw} = 150, x_{ss} = 150, x_{ec} = 50, x_{es} = 850, x_{wc} = 300, x_{ww} = 300, x_{cc} = 250, x_{cw} = 250, Z = 21,200$

(a) $\max. Z = 2x_1 + 4x_2 + 3x_3 + 7x_4$; s.t. $x_2 + x_4 \leq 300, 6x_1 +$

27. $15x_2 \leq 1,200, 5x_3 + 12x_4 \leq 2,400, x_i \geq 0$; (b) $x_1 = 200, x_3 = 480, Z = 1,840$

(a) $\max. Z = 35x_1 + 20x_2 + 58x_3$; s.t. $14x_1 + 12x_2 + 35x_3 \leq$

29. $35,000, 6x_1 + 3x_2 + 12x_3 \leq 20,000, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$;

$$(b) x_1 = 2,500, Z = 87,500$$

31. (a) $\min. Z = 15,000x_1 + 4,000x_2 + 6,000x_3$; s.t. $x_3/x_2 \geq 2/1$,
 $25,000x_1 + 10,000x_2 + 15,000x_3 \geq 100,000$, $(15,000x_1 + 3,000x_2 + 12,000x_3) / (10,000x_1 + 7,000x_2 + 3,000x_3) \geq 2/1$,
 $(15,000x_1 + 4,000x_2 + 9,000x_3) / (25,000x_1 + 10,000x_2 + 15,000x_3) \geq .30$, $x_2 \leq 7$, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$; (b) $x_2 = 2.5$, $x_3 = 5.0$, $Z = 40,000$; (c) x_4 , no effect

$$(a) \min. Z = .11x_1 + .05 \sum_1^6 y_1; \text{ s.t. } x_1 + y_1 + 20,000 - c_1 = 60,000, c_1 + y_2 + 30,000 - c_2 = 60,000 + y_1,$$

$$\begin{aligned}
 &c_2 + y_3 + 40,000 - c_3 = 80,000 + \\
 &y_2, c_3 + y_4 + 50,000 - c_4 = \\
 &30,000 + y_3, c_4 + y_5 + 80,000 - \\
 &c_5 = 30,000 + y_4, c_5 + y_6 + \\
 &\mathbf{33.} \quad 100,000 - c_6 = 20,000 + y_5, x_1 + \\
 &y_6 \leq c_6, x_1, y_i, c_i \geq 0; \text{ (b) } x_1 = \\
 &70,000, y_3 = 40,000, y_4 = \\
 &20,000, y_1 = y_2 = y_5 = y_6 = 0, c_1 \\
 &= 30,000, c_5 = 30,000, c_6 = \\
 &110,000, Z = \$10,700; \text{ (c) } x_1 = \\
 &90,000, y_3 = 20,000, c_1 = \\
 &50,000, c_2 = 20,000, c_5 = \\
 &50,000, c_6 = 130,000, Z = \\
 &\$9,100
 \end{aligned}$$

$$\begin{aligned}
 &\text{(a) max. } Z = .7x_{cr} + .6x_{br} + .4x_{pr} \\
 &+ .85x_{ar} + 1.05x_{cb} + .95x_{bb} + \\
 &.75x_{pb} + 1.20x_{ab} + 1.55x_{cm} + \\
 &1.45x_{bm} + 1.25x_{pm} + 1.70x_{am};
 \end{aligned}$$

$$\text{s.t. } x_{cr} + x_{cb} + x_{cm} \leq 200, x_{br} + x_{bb} + x_{bm} \leq 300, x_{pr} + x_{pb} + x_{pm} \leq 150, x_{ar} + x_{ab} + x_{am} \leq 400,$$

35. $.90x_{br} + .90x_{pr} - .10x_{cr} - .10x_{ar} \leq 0, .80x_{cr} - .20x_{br} - .20x_{pr} - .20x_{ar} \geq 0, .25x_{bb} + .75x_{cb} - .75x_{pb} - .75x_{ab} \geq 0, x_{am} = 0, .5x_{bm} + .5x_{pm} - .5x_{cm} - .5x_{am} \leq 0, x_{ij} \geq 0;$ (b) $x_{cm} = 125, x_{ar} = 300, x_{cr} = 75, x_{bb} = 300, x_{pm} = 125, x_{ab} = 100, Z = 1,602.50$

[Page 783]

(a) $\min. Z = 40x_1 + 65x_2 + 70x_3 + 30x_4;$ s.t. $x_1 + x_2 = 250, x_3 + x_4 = 400, x_1 + x_3 = 300, x_2 + x_4 = 350, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0;$ (b) $x_1 = 250, x_3 = 50, x_4 =$

37.

$$350, Z = 24,000$$

$$39.(a) \max.Z = 175 (7x_1); \text{ s.t.}$$

$$8x_1 + 5x_2 + 6.5x_3 \leq 3,000, x_1 +$$

$$x_2 + x_3 \leq 120, 90(7x_1) \leq$$

39. $10,000, 7x_1 - 12x_2 = 0, 12x_2 - 10x_3 = 0, 7x_1 - 10x_3 = 0, x_1 \geq 0, x_2 \geq 0, x_3 \geq 0; (b) x_1 = 15.9, x_2 = 9.3, x_3 = 11.1, Z = 19,444.44$

$$(a) \min.Z = 3x_{13} + 4x_{14} + 5x_{12} +$$

$$2x_{34} + 7x_{45} + 8x_{25}; \text{ s.t. } x_{13} + x_{14}$$

$$+ x_{12} = 5, x_{45} + x_{25} = 5, x_{13} =$$

41. $x_{34}, x_{14} + x_{34} = x_{45}, x_{12} = x_{25}, x_{ij} \geq 0; (b) x_{14} = 5, x_{45} = 5, Z = 55,000$

$$(a) \min.Z = x_1 + x_2 + x_3 + x_4 +$$

$x_5 + x_6$; s.t. $3x_1 + 2x_2 + 2x_3 + x_4 = 700$, $x_3 + 2x_4 + x_5 = 1,200$, $x_2 + x_5 + 2x_6 = 300$, $x_i \geq 0$; (b) $x_2 = 50$, $x_4 = 600$, $x_6 = 125$, $Z = 775$; (c) $\min. Z = 4x_1 + x_2 + 2x_3 + 0x_4 + 6x_5 + 5x_6$; s.t. $3x_1 + 2x_2 + 2x_3 + x_4 \geq 700$, $x_3 + 2x_4 + x_5 \geq 1,200$, $x_2 + x_5 + 2x_6 \geq 300$, $x_i \geq 0$; $x_2 = 300$, $x_4 = 600$, $Z = 300$

43.

(a) $\max. Z = 4x_1 + 8x_2 + 6x_3 + 7x_4 - 5y_1 - 6y_2 - 7y_3$; s.t. $s_0 = 2,000$, $x_1 \leq s_0$, $s_1 = s_0 - x_1 + y_1$, $s_1 \leq 10,000$, $x_2 \leq s_1$, $s_2 = s_1 - 2x_2 + y_2$, $s_2 \leq 10,000$, $x_3 \leq s_2$, $s_3 = s_2 - x_3 + y_3$, $s_3 \leq 10,000$, $x_4 \leq s_3$, $x_i \geq 0$, $y_i \geq 0$, $s_i \geq 0$; (b) $x_1 = 0$, $x_2 = 10,000$, $x_4 = 10,000$, $y_1 = 8,000$, $y_2 = 10,000$, $s_0 = 2,000$,

45.

$$s_1 = s_2 = s_3 = 10,000, Z = 50,000$$

(a) $\max. Z = 130x_{1a} + 150x_{1b} + 90x_{1c} + 275x_{2a} + 300x_{2b} + 100x_{2c} + 180x_{3a} + 225x_{3b} + 140x_{3c} + 200x_{4a} + 120x_{4b} + 160x_{4c};$ s.t. $x_{1a} + x_{1b} + x_{1c} = 1,$
 $x_{2a} + x_{2b} + x_{2c} = 1, x_{3a} + x_{3b} + x_{3c} = 1, x_{4a} + x_{4b} + x_{4c} = 1, 1 \leq$
 $x_{1a} + x_{2a} + x_{3a} + x_{4a} \leq 2, 1 \leq$
 $x_{1b} + x_{2b} + x_{3b} + x_{4b} \leq 2, 1 \leq$
 $x_{1c} + x_{2c} + x_{3c} + x_{4c} \leq 2, x_{ij} \geq 0;$
 (b) $x_{1a} = 1, x_{2b} = 1, x_{3b} = 1, x_{4c} = 1, Z = 815;$ (c) $\max. Z = 130x_{1a} + 150x_{1b} + 90x_{1c} + 275x_{2a} + 300x_{2b} + 100x_{2c} + 180x_{3a} + 225x_{3b} + 140x_{3c} + 200x_{4a} + 120x_{4b} + 160x_{4c};$ s.t. $x_{1a} + x_{1b} +$

47.

$$\begin{aligned}
& x_{1c} \leq 1, x_{2a} + x_{2b} + x_{2c} \leq 1, x_{3a} \\
& + x_{3b} + x_{3c} \leq 1, x_{4a} + x_{4b} + x_{4c} \\
& \leq 1, x_{1a} + x_{2a} + x_{3a} + x_{4a} = 1, \\
& x_{1b} + x_{2b} + x_{3b} + x_{4b} = 1, x_{1c} + \\
& x_{2c} + x_{3c} + x_{4c} = 1, x_{ij} \geq 0; x_{2a} = \\
& 1, x_{3b} = 1, x_{4c} = 1, Z = 660
\end{aligned}$$

49. Z values: A = 1.000, B = 1.000, C = 1.000; all 3 efficient

$$\begin{aligned}
& \text{(a) min. } Z = x; \text{ s.t. } 150x = 650 + \\
& y_1, 150x + y_1 = 450 + y_2, 150x + \\
& y_2 = 600 + y_3, 150x + y_3 = 500 \\
& + y_4, 150x + y_4 = 700 + y_5, 150x \\
& + y_5 = 650 + y_6, 150x + y_6 = \\
& 750 + y_7, 150x + y_7 = 900 + y_8, \\
& 150x + y_8 = 800 + y_9, 150x + y_9 \\
& = 650 + y_{10}, 150x + y_{10} = 700 + \\
& y_{11}, 150x + y_{11} \geq 500; \text{(b) } Z = x
\end{aligned}$$

51.

$$= 4.45, y_1 = 18.18, y_2 = 236.36, \\ y_3 = 304.54, y_4 = 472.72, y_5 = \\ 440.91, y_6 = 459.09, y_7 = \\ 377.27, y_8 = 145.45, y_9 = 13.63, \\ y_{10} = 31.81, y_{11} = 0$$

(a) $\max. Z = y$; s.t. $y x_1 = 0, y 2x_2 = 0, y 2 x_3 = 0, 10x_1 + 8x_2 + 6x_3 \leq 960, 9x_1 + 21x_2 + 15x_3 \leq 1,440, 2x_1 - 3x_2 - 2x_3 \leq 60, -2x_1 + 3x_2 + 2x_3 \leq 60, x_i \geq 0, y \geq 0$;

(b) $x_1 = x_2 = x_3 = y = 20$; (c) remove balancing requirement, $x_1 = x_2 = x_3 = y = 32$

53.

$$\max. Z = 850x_1 + 600x_n + 750x_s + 1,000x_w; \text{ s.t. } x_1 + x_n + x_s + x_w = 18, x_1 + x_n + x_s + x_w + y_1 + y_n$$

$+ y_s + y_w = 60, 400y_1 + 100y_n + 175y_s + 90y_w \leq 9,000, 10 \leq y_1 \leq 25, 5 \leq y_n \leq 10, 5 \leq y_s \leq 10, 5 \leq y_w \leq 10, x_1 \leq 6, x_n \leq 6, x_s \leq 6, x_w \leq 6, x_i \geq 0, y_i \geq 0; x_1 = 6, x_n = 0, x_s = 6, x_w = 6, y_1 = 14.44, y_n = 10, y_s = 7.56, y_w = 10, Z = 15,600$ (multiple optimal)

(a) $\max. Z = .85x_1 + .90x_2 - y_1 - y_2; \text{ s.t. } x_1 \leq 5,000 + 3y_1, x_2 \leq 4,000 + 5y_2, .60x_1 + .85x_2 + y_1 + y_2 \leq 16,000, x_1 \geq .3(x_1 + x_2), x_1 \leq .6(x_1 + x_2), x_1 \geq 0, x_2 \geq 0, y_1 \geq 0, y_2 \geq 0;$

57. (b) $x_1 = 5,458.128, x_2 = 12,735.63, y_1 = 152.709, y_2 = 1,747.126, Z = 14,201.64$

(a) $\min.Z = \sum x_{ij}$ (ranking) • x_{ij} , s.t. $\sum x_{ij} \leq \text{hr.}$, $\sum x_{ij} = \text{project hr.}$, $\sum x_{ij} \leq \text{budget}$; (b)

$x_{A3} = 400, x_{A4} = 50, x_{B4} = 250,$

59. $x_{B5} = 350, x_{C4} = 175, x_{C7} = 274.1, x_{C8} = 50.93, x_{D2} = 131.7, x_{D7} = 15.93, x_{E1} = 208.33, x_{E8} = 149.07, x_{F1} = 291.67, x_{F2} = 108.3, x_{F6} = 460, Z = \$12,853.33$

$x_1 = 0, x_2 = 4, x_3 = 18.4, x_4 = 6.4, x_5 = 24.8, y_1 = 72.22, y_2 =$

61. $72.44, y_3 = 64.95, y_4 = 62.34, y_5 = 52.24, y_6 = 38.9, y_7 = 28.53, y_8 = 43.35, Z = \$360,196$

$\min.Z = \sum x_{ij}$ (priority ij) s.t.

$$\sum_{j=1}^{12} x_{ij} = 1, \sum_{i=1}^{16} x_{ij} \leq \text{available slots } j,$$

- 63.** U11B: 3-5M, U11G: 3-5T, U12B: 3-5T, U12G: 3-5M, U13B: 3-5T, U13G: 3-5M, U14B: 5-7M, U14G: 3-5M, U15B: 5-7T, U15G: 3-5M, U16B: 5-7T, U16G: 5-7T, U17B: 5-7M, U17G: 5-7T, U18B: 3-5T, U18G: 5-7M, $Z = 27$

- 65.** 1-D (1 hr.), 2-E (1 hr.), 3-F (2 hr.), 3-H (6 hr.), 4-I (1 hr.), 5-J (1 hr.), A-5 (8 hr.), B-4 (5 hr.), C-6 (8 hr.), G-7 (2 hr.), $Z = 35$ hr.; 6 crews originate in Pittsburgh, 4 in Orlando

$$\begin{aligned} \text{min. } Z = & .41x_{14} + .57x_{15} + .37x_{24} \\ & + .48x_{25} + .51x_{34} + .60x_{35} + \\ & .22x_{46} + .10x_{47} + .20x_{48} + \\ & .15x_{56} + .16x_{57} + .18x_{58}; \text{ s.t. } x_{14} \end{aligned}$$

$$\begin{aligned}
 &+ x_{15} \leq 24,000, x_{24} + x_{25} \leq \\
 &18,000, x_{34} + x_{35} \leq 32,000, x_{14} \\
 &+ x_{24} + x_{34} \leq 48,000, x_{15} + x_{25}
 \end{aligned}$$

67. $+ x_{35} \leq 35,000$ $(x_{14} + x_{24} + x_{34})/2 = x_{46} + x_{47} + x_{48}$, $(x_{15} + x_{25} + x_{35})/2 = x_{56} + x_{57} + x_{58}$, $x_{46} + x_{56} = 9,000$, $x_{47} + x_{57} = 12,000$, $x_{47} + x_{58} = 15,000$, $x_{ij} \geq 0$; $x_{14} = 24,000$, $x_{24} = 18,000$, $x_{34} = 6,000$, $x_{35} = 24,000$, $x_{47} = 12,000$, $x_{48} = 12,000$, $x_{56} = 9,000$, $x_{58} = 3,000$, $Z = \$39,450$

Chapter 5

1. $x_1 = 3, x_2 = 0, Z = 15$

3. (a) $\max. Z = 50x_1 + 40x_2$; s.t. $3x_1 + 5x_2 \leq 150, 10x_1 + 4x_2 \leq 200, x_i \geq 0$ and integer; (b) $x_1 = 10, x_2 = 24, Z = 1,460$

5. (a) $\max. Z = 50x_1 + 10x_2$; s.t. $x_1 + x_2 \leq 15, 4x_1 + x_2 \leq 25, x_i \geq 0$ and integer; (b) $x_1 = 6, x_2 = 1, Z = 310$

(a) $\max. Z = 50x_1 + 40x_2$; s.t. $2x_1 + 5x_2 \leq 35$, $3x_1 + 2x_2 \leq 20$, $x_i \geq 0$ and integer; (b) $x_1 = 4$, $x_2 = 4$, $Z = 360$

9. $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $Z = 1,800$

11. $\min. Z = 81x_1 + 50x_2$; s.t. $76x_1 + 53x_2 \geq 600$, $x_1 + x_2 \leq 10$, $1.3x_1 + 4.1x_2 \leq 24$, $x_1, x_2 \geq 0$ and integer; $x_1 = 6$, $x_2 = 3$, $Z = 636$

(a) $\max. Z = 85,000x_1 + 60,000x_2 - 18,000y_1$; s.t. $x_1 + x_2 \leq 10$, $10,000x_1 + 7,000x_2 \leq 72,000$, $x_1 - 10y_1 \leq 0$, x_1 and $x_2 \geq 0$ and integer, $y_1 = 0$ or 1 ; (b) $x_1 = 0$, $x_2 = 10$, $Z = 600,000$

15.

$$\begin{aligned} \min. Z &= x_1 + x_2 + x_3 + x_4 + x_5 + x_6; \text{ s.t. } x_6 + x_1 \geq 90, x_1 + x_2 \geq 215, \\ x_2 + x_3 &\geq 250, x_3 + x_4 \geq 65, x_4 + x_5 \geq 300, x_5 + x_6 \geq 125, \\ x_i &\geq 0; x_1 = 90, x_2 = 250, x_4 = 175, x_5 = 125, Z = 640 \end{aligned}$$

17.

$$\begin{aligned} \text{(a) } \min. Z &= 25,000x_1 + 7,000x_2 + 9,000x_3; \text{ s.t. } 53,000x_1 + 30,000x_2 + 41,000x_3 \geq 200,000, \\ (32,000x_1 + 20,000x_2 + 18,000x_3) / (21,000x_1 + 10,000x_2 + 23,000x_3) &\geq 1.5, \\ (34,000x_1 + 12,000x_2 + 24,000x_3) / (53,000x_1 + 30,000x_2 + 41,000x_3) &\geq 0.60 \\ x_1 &\geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and integer; } x_1 = 4, Z = \end{aligned}$$

\$99,999.99; (b) $x_1 = 2.9275$, $x_2 = .9713$, $x_3 = .383$, $Z =$
\$83,433.65

$\max.Z = 25,000x_1 + 18,000x_2 +$
 $31,000x_3$; s.t. $x_1 + x_2 + x_3 =$
 100 , $5,000x_1 + 11,000x_2 +$

19. $7,000x_3 \leq 700,000$, $x_1 \geq 10$, x_2
 ≥ 10 , $x_3 \geq 10$, $x_1, x_2, x_3 \geq 0$ and
integer; $x_1 = 20$, $x_2 = 10$, $x_3 =$
 70 , $Z = 2,850,000$

$\max.Z = 12,100x_1 + 8,700x_2 +$
 $10,500x_3$; s.t. $360x_1 + 375x_2 +$
 $410x_3 \leq 30,000$, $x_1 + x_2 + x_3 \leq$
 67 , $14x_1 + 10x_2 + 18x_3 \leq 2,200$,

21. $x_1/x_3 \geq 2$, $x_2/x_1 \geq 1.5$, $x_1 \geq 0$,
 $x_2 \geq 0$ and integer, $x_3 \geq 0$; $x_1 =$

$$22, x_2 = 33, x_3 = 11, Z = \$668,800$$

[Page 784]

$$(a) \max. Z = 1,650x_1 + 850x_2 + 790x_3; \text{ s.t. } 6.3x_1 + 3.9x_2 + 3.1x_3 \leq 125, 17x_1 + 10x_2 + 7x_3 \leq 320,$$

23. $x_i \geq 0$ and integer; (b) $x_1 = 10, x_3 = 20, Z = 32,300$; rounded-down solution: $x_1 = 13, x_3 = 12, Z = 30,930$

$$\max. Z = 575x_1 + 120x_2 + 65x_3; \text{ s.t. } 40x_1 + 15x_2 + 4x_3 \leq 600,$$

25. $30x_1 + 18x_2 + 5x_3 \leq 480, 4x_1 + 2x_2 \leq 0, x_3 = 20y_1, x_1, x_2, x_3 \geq 0$ and integer, $y_1 = 0$ or 1 ; $x_1 = 3, x_2 = 16, x_3 = 20, y_1 = 1, Z = \$4,945$

27. (b) $x_{1C} = 1, x_{3D} = 1, x_{4B} = 1, x_{5A} = 1, Z = 83$ parts

$$\begin{aligned} \min. Z &= 120x_1 + 75x_2 + 4.5x_3; \\ \text{s.t. } 220x_1 + 140x_2 + 12x_3 &\geq 6,300, \\ 8x_1 + 8x_2 + x_3 &\leq 256, \end{aligned}$$

29. $.4x_1 + .9x_2 + .16x_3 \leq 15, x_1, x_2 \geq 0$ and integer, $x_3 \geq 0; x_1 = 28, x_2 = 0, x_3 = 11.67, Z = \$3,412.50$

31. $x_{13} = 1, x_{22} = 1, x_{32} = 1, x_{43} = 1, x_{53} = 1, x_{61} = 1, Z = \125 million

$$\begin{aligned} \max. Z &= 127x_1 + 83x_2 + 165x_3 + 96x_4 + 112x_5 + 88x_6 + 135x_7 + 141x_8 + 117x_9 + 94x_{10}; \\ \text{s.t. } x_1 + \end{aligned}$$

33. $x_3 \leq 1, x_1 + x_2 + x_4 \leq 1, x_4 + x_5 + x_6 \leq 1, x_6 + x_7 + x_8 \leq 1, x_6 + x_9 \leq 1, x_8 + x_{10} \leq 1, x_9 + x_{10} \leq 1, x_i = 0 \text{ or } 1; x_2 = 1, x_3 = 1, x_5 = 1, x_8 = 1, x_9 = 1, Z = 618,000$

35. $A(1,2,3), B(1,2,3), C(4,5,6), D(5,6,7), E(4,5,6), F(1,2,3), G(5,6,7), H(4,5,6), Z = 100 \text{ hr.}$

$\text{max. } Z = .9(3600)x_{A1} + .5(7200)x_{A2} + .9(2400)x_{B1} + .7(3600)x_{B2} + .95(3000)x_{C1} + .4(6000)x_{C2} + .95(3300)x_{D1} + .6(5400)x_{D2}; \text{ s.t. } x_{A1} + x_{A2} = 1, x_{B1} + x_{B2} = 1, x_{C1} + x_{C2} = 1, x_{D1} + x_{D2} = 1, .9x_{A1} + .5x_{A2} + .9x_{B1} + .7x_{B2} + .95x_{C1} + .4x_{C2} + .95x_{D1} + .6x_{D2} \geq 3, x_{ij} \geq 0, x_{A2} =$

37.

$1, x_{B2} = 1, x_{C1} = 1, x_{D1} = 1, Z =$
\$4,035 per month

Chapter 6

1. St. LouisChicago = 250,
RichmondChicago = 50,
RichmondAtlanta = 350, $Z = 24,000$

3. $A3 = 100, B1 = 135, B2 = 45, C2 = 130, C3 = 70, Z = 2,350$

5. $x_{11} = 70, x_{13} = 20, x_{22} = 10, x_{23} = 20, x_{32} = 100, Z = 1,240$

$$\min.Z = 14x_{A1} + 9x_{A2} + 16x_{A3} + 18x_{A4} + 11x_{b1} + 8x_{b2} + 100x_{b3} +$$

$$\begin{aligned}
 &16x_{B4} + 16x_{C1} + 12x_{C2} + 10x_{C3} + 22x_{C4}; \text{ s.t. } x_{A1} + x_{A2} + x_{A3} + x_{A4} \\
 &\leq 150, x_{b1} + x_{b2} + x_{b3} + x_{B4} \leq 210, x_{C1} + x_{C2} + x_{C3} + x_{C4} \leq 320, \\
 &x_{A1} + x_{b1} + x_{C1} = 130, x_{A2} + x_{b2} + x_{C2} = 70, x_{A3} + x_{b3} + x_{C3} = 180, \\
 &x_{A4} + x_{B4} + x_{C4} = 240, x_{ij} \geq 0; x_{A2} = 70, x_{A4} = 80, x_{b1} = 50, \\
 &x_{B4} = 160, x_{C1} = 80, x_{C3} = 180, Z = 8,260
 \end{aligned}$$

9. $A3 = 90, B1 = 30, B3 = 20, C2 = 80, D1 = 40, D2 = 20, Z = 1,590$

$$\begin{aligned}
 &1C = 5, 2C = 10, 3B = 20, 4A = 10, Z = \$195; \text{ min. } Z = 7x_{1A} + 8x_{1B} + 5x_{1C} + 6x_{2A} + 100x_{2B} + 6x_{2C} + 10x_{3A} + 4x_{3B} + 5x_{3C} + 3x_{4A} + 9x_{4B} + 100x_{4C}; \text{ s.t. } x_{1A} +
 \end{aligned}$$

11. $x_{1B} + x_{1C} \leq 5, x_{2A} + x_{2B} + x_{2C} \leq 25, x_{3A} + x_{3B} + x_{3C} \leq 20, x_{4A} + x_{4B} + x_{4C} \leq 25, x_{1A} + x_{2A} + x_{3A} + x_{4A} = 10, x_{1B} + x_{2B} + x_{3B} + x_{4B} = 20, x_{1C} + x_{2C} + x_{3C} + x_{4C} = 15, x_{ij} \geq 0$

13. $A_2 = 1,800, A_4 = 950, A_6 = 750, B_1 = 1,600, B_3 = 1,500, B_5 = 1,250, B_6 = 650, Z = 3,292.50$

15. $1B = 250, 1D = 170, 2A = 520, 2C = 90, 3C = 130, 3D = 210, Z = 21,930$

17. $1B = 250, 1D = 170, 2A = 520, 2C = 90, 3C = 130, 3D = 210, 4C = 180, Z = \$26,430;$
 transportation cost = \$21,930,

shortage cost = \$4,500

19. $1B = 60, 2A = 45, 2B = 25, 2C = 35, 3B = 5, Z = 1,605$

21. $NA = 250, SB = 300, SC = 40, EA = 150, EC = 160, WD = 210, CB = 100, CD = 190, Z = 20,700$
(multiple optimal)

23. $A3 = 8, A4 = 18, B3 = 13, B5 = 27, D3 = 5, D6 = 35, E1 = 25, E2 = 15, E3 = 4, Z = 1,528$ (multiple optimal)

25. 17 cases from Albany; \$51

27. $1C = 2, 1E = 5, 2C = 10, 3E = 5, 4D = 8, 5A = 9, 6B = 6, Z =$

1,275 hr.

R_J Jan = 300, O_J Jan = 110, R_F FEB = 300, O_F Feb = 20, O_F March = 120, R_M March = 180, R_M April = 120, O_M March = 200, R_A April = 300, O_A April = 200, R_M May = 300, O_M May = 130, R_J June = 300, O_J June = 80, $Z = 3,010,040$

Increasing supply at Sacramento, Jacksonville, and Ocala has little effect; increasing supply at San Antonio and Montgomery reduces cost to \$242,500

33. Total cost = \$1,198,500

$$x_{14} = 42, x_{15} = 13, x_{26} = 63, x_{35} = 37, x_{48} = 42, x_{59} = 50, x_{67} = 60, x_{68} = 3, Z = 77,362$$

$$(a) x_{14} = 72, x_{25} = 105, x_{34} = 83, x_{46} = 75, x_{47} = 80, x_{56} = 15, x_{58} = 90, Z = 10,043,000; (b) x_{14} = 72, x_{25} = 105, x_{34} = 48, x_{35} = 35, x_{46} = 40, x_{47} = 80, x_{56} = 50, x_{58} = 90, Z = 10,043,000$$

$$x_{37} = 2.1, x_{15} = 5.2, x_{26} = 6.3, x_{59} = 5.2, x_{68} = 3.7, x_{69} = 2.6, Z = \$27.12 \text{ million}$$

$$x_{14} = 85, x_{15} = 40, x_{25} = 70, x_{20} = 15, x_{27} = 125, x_{36} = 85, x_{410} = 85, x_{59} = 55, x_{510} = 55, x_{68} =$$

41. $100, x_{78} = 70, x_{79} = 55, x_{811} = 85, x_{813} = 35, x_{814} = 50, x_{912} = 40, x_{913} = 70, x_{1012} = 20, x_{1015} = 120, Z = \$1,179,400$

43. $11, 24, 32, 53, Z = 78$

45. $1B, 2D, 3C, 4A, Z = 32$

47. $1B, 2E, 3A, 4C, 5D, 6F, Z = 36$

49. $1C, 2F, 3E, 4A, 5D, 6B, Z = 85$
defects

51. $1, 4, \text{ and } 7 \text{Columbia; } 2, 6, \text{ and } 8 \text{Atlanta; } 3, 5, \text{ and } 9 \text{Nashville; } Z = 985$

A1'sparents' brunch; Bon
Apetitpost game party; Bon
Apetitlettermen's dinner;

53. Divinebooster club;
Epicureancontributors' dinner;
Universityalumni brunch; $Z =$
\$103,800

Anniebackstroke;

55. Debbiebreaststroke; Erinfreestyle;
Faybutterfly; $Z = 10.61$ min.

57. A2, C1, D1, E3, F3, G2, H1, $Z =$
\$1,070

ANJ, BPA, CNY, DFL, EGA, FFL,

59. GVA, $Z = 498$; success rate =
71.1%

$$x_{A2} = 1, x_{J2} = 1, x_{B1} = 1, x_{K1} = 1,$$

$$x_{C1} = 1, x_{L3} = 1, x_{D2} = 1, x_{E1} =$$

61. $x_{F1} = 1, x_{G2} = 1, x_{H3} = 1, Z =$
104

Chapter 7

1. $12 = 21, 125 = 46, 13 = 17, 134 = 29, 1346 = 39, 1347 = 38$

3. $12 = 85, 13 = 53, 14 = 88, 135 = 114, 137 = 170, 1376 = 194$

5. $12 = 25, 123 = 60, 124 = 43, 15 = 48, 16 = 50, 17 = 32, 178 = 72, 179 = 61, 1510 = 72$

7. $12 = 7, 13 = 10, 14 = 8, 135 = 13, 1358 = 21, 146 = 13, 1469 = 20, 147 = 16, 146910 = 29$

$$1761012 = 18$$

9.

11. $1478 = 42$ miles

13. $131114 = 13$ days

Kotzebue: $1281115 = 10$ hr.;

15. Nome: $1281213 = 9$ hr.; Stebbins:
 $12710 = 8.5$ hr.

17. 147 ; \$64,500

[Page 785]

19. $13 = 4.1, 14 = 4.8, 23 = 3.6, 48 = 5.5, 56 = 2.1, 67 = 2.8, 78 = 2.7, 79 = 2.7, 910 = 4.6; 32,900$ ft.

21. 134658, 32, 67; 320 yd.

23. 14263, 675, 78; 1,160 yd.

25. 125689, 243, 47; 22 miles

27. 12, 1463, 6985, 911, 9107,
1013, 101214; 1,086 ft.

29. 12 = 6, 14 = 5, 13 = 5, 25 = 6,
24 = 0, 34 = 0, 36 = 5, 45 = 0,
46 = 5, 56 = 6, maximum flow =
16

31. 12 = 15, 14 = 20, 13 = 12, 25 =
11, 24 = 4, 34 = 3, 36 = 9, 46 =
27, 56 = 11, maximum flow = 47

33. $12 = 7, 14 = 7, 13 = 4, 26 = 6,$
 $24 = 1, 34 = 3, 35 = 1, 46 = 0,$
 $48 = 4, 47 = 7, 57 = 1, 68 = 6,$
 $78 = 8, \text{maximum} = 18 \text{ flights}$

35. $12 = 4, 13 = 4, 14 = 4, 23 = 0,$
 $25 = 4, 34 = 0, 35 = 2, 36 = 2,$
 $46 = 4, 56 = 0, 57 = 6, 68 = 6,$
 $78 = 0, 79 = 6, 89 = 6; 12,000$
cars

37. $12 = 6, 13 = 2, 14 = 8, 25 = 0,$
 $26 = 6, 35 = 0, 36 = 2, 45 = 8,$
 $57 = 3, 58 = 4, 59 = 1, 68 = 2,$
 $69 = 6, 710 = 3, 810 = 6, 910 =$
 $7; 16$

$12 = 70, 13 = 50, 14 = 60, 15 =$

39.

70, 26 = 25, 29 = 45, 36 = 30,
37 = 20, 47 = 35, 48 = 25, 58 =
40, 511 = 30, 610 = 65, 612 = 5,
710 = 40, 76 = 15, 813 = 60,
814 = 5, 912 = 45, 1012 = 55,
1013 = 50, 1114 = 30, 1213 =
60, 1215 = 45, 1315 = 170,
1415 = 35; 250

GdanskGalveston = 125,
HamburgJacksonville = 110,
HamburgNew Orleans = 95,
HamburgGalveston = 5,
LisbonNorfolk = 85, NorfolkKC =
75, NorfolkDallas = 10,

41.

JacksonvilleFR = 70,
JacksonvilleKC = 40, NOFR = 95,
GalvestonDallas = 130, FRDenver
= 105, FRPittsburg = 60,
KCCleveland = 65, KCNashville =
50, DallasTucson = 85,
DallasCleveland = 55

Chapter 8

1. $134 = 10$

3. $1368 = 23$

5. $1: ES = 0, EF = 7, LS = 2, LF = 9, S = 2; 2: ES = 0, EF = 10, LS = 0, LF = 10, S = 0; 3: ES = 7, EF = 13, LS = 9, LF = 15, S = 2; 4: ES = 10, EF = 15, LS = 10, LF = 15, S = 0; 5: ES = 10, EF = 14, LS = 14, LF = 18, S = 4; 6: ES = 15, EF = 18, LS = 15, LF = 18, S = 0; 7: ES = 18, EF = 20, LS = 18, LF = 20, S = 0; 2467 = 20$

1: $ES = 0, EF = 10, LS = 0, LF = 10, S = 0$; 2: $ES = 0, EF = 7, LS = 5, LF = 12, S = 5$; 4: $ES = 10, EF = 14, LS = 14, LF = 18, S = 4$; 3: $ES = 10, EF = 25, LS = 10, LF = 25, S = 0$; 5: $ES = 7, EF = 13, LS = 12, LF = 18, S = 5$; 6: $ES = 7, EF = 19, LS = 13, LF = 25, S = 6$; 7: $ES = 14, EF = 21, LS = 18, LF = 25, S = 4$; 8: $ES = 25, EF = 34, LS = 25, LF = 34, S = 0$; 138 = 34

9. 2410131417 = 78 wk.

11. 159 = 40.33, s = 5.95

13. (e) 2691112; (f) $\mu = 46$ mo., s = 5 mo.

15. $P(x \geq 50) = .2119$

17. $\text{cdfgjkor} = 160.83 \text{ min.}; P(x \leq 180) = .9875$

19. $P(x \leq 15) = .0643$

21. $\text{adjno}, \mu = 42.3 \text{ wk.}, s = 4.10 \text{ wk.}, 47.59 \text{ wk.}$

23. $\text{acdf} = 14 \text{ days}$

25. $\text{aefgjop} = 91.67 \text{ days}, s = 3.308, P(x \leq 101) = .9976$

27. $\text{ahlmnosw} = 126.67 \text{ days}; P(x \leq 150) = .9979$

29. (c) $\min.Z = x_6$; s.t. $x_2 x_1 \geq 16$, $x_3 x_1 \geq 14$, $x_4 2x_2 \geq 8$, $x_5 2x_2 \geq 5$, $x_5 x_3 \geq 4$, $x_6 x_3 \geq 6$, $x_6 x_4 \geq 10$, $x_6 x_5 \geq 15$, $x_i x_j \geq 0$

31. $\min.Z = x_9$; s.t. $x_2 x_1 \geq 8$, $x_3 x_1 \geq 6$, $x_4 x_1 \geq 3$, $x_5 2x_2 \geq 0$, $x_6 2x_2 \geq 5$, $x_5 x_3 \geq 3$, $x_5 x_4 \geq 4$, $x_7 x_5 \geq 7$, $x_7 x_8 \geq 0$, $x_8 x_5 \geq 4$, $x_8 x_4 \geq 2$, $x_9 x_6 \geq 4$, $x_9 x_7 \geq 9$, $x_i x_j \geq 0$; $x_1 = 0$, $x_2 = 9$, $x_3 = 6$, $x_4 = 3$, $x_5 = 9$, $x_6 = 14$, $x_7 = 16$, $x_8 = 16$, $x_9 = 25$

33. adgk; crash cost = \$5,100; total network cost = \$33,900

Chapter 9

$$\begin{aligned} \text{min. } & P_1 d_1, P_2 d_2, P_3 d_1^+, P_4 d_3^+, \\ \text{s.t. } & 5x_1 + 2x_2 + 4x_3 + d_1 - d_1^+ = \\ \mathbf{1.} \quad & 240, 3x_1 + 5x_2 + -x_3 + d_2 - d_2^+ \\ & = 500, 4x_1 + 6x_2 + 3x_3 + d_3 - \\ & d_3^+ = 400 \end{aligned}$$

$$\begin{aligned} \text{min. } & P_1 d_1, P_2 d_2, P_3 d_3^+, 3P_4 d_4 + \\ & 6P_4 d_5 + P_4 d_6 + 2P_4 d_7, \text{ s.t.} \\ & 80,000x_1 + 24,000x_2 + 15,000x_3 \\ & + 40,000x_4 + d_1 = 600,000, \\ & 1,500x_1 + 3,000x_2 + 500x_3 + \\ \mathbf{3.} \quad & 1,000x_4 + d_2 - d_2^+ = 20,000, 4x_1 \end{aligned}$$

$$\begin{aligned}
 &+ 8x_2 + 3x_3 + 5x_4 + d_3 - d_3^+ = \\
 &50, x_1 + d_4 - d_4^+ = 7, x_2 + d_5 - \\
 &d_5^+ = 10, x_3 + d_6 - d_6^+ = 8, x_4 + \\
 &d_7 - d_7^+ = 12
 \end{aligned}$$

$$\begin{aligned}
 &(\text{a}) \min. P_1 d_2^+, P_2 d_7^+, 4P_3 d_4 + \\
 &2P_3 d_5 + P_3 d_6, P_4 d_1, P_5 d_3; \text{ s.t. } x_1 + \\
 &x_2 + x_3 + d_1 = 2,000, 800x_1 + \\
 &1,500x_2 + 500x_3 + d_2 - d_2^+ = \\
 &20,000, 10x_1 + 12x_2 + 18x_3 + d_3
 \end{aligned}$$

$$\begin{aligned}
 \text{5. } &- d_3^+ = 800, x_1 + d_4 - d_4^+ = 800, \\
 &x_2 + d_5 - d_5^+ = 900, x_3 + d_6 - d_6^+ \\
 &= 1,100, d_3^+ + d_7 - d_7^+ = 100; \\
 &(\text{b}) x_1 = 25, d_1 = 1,975, d_3 = \\
 &550, d_4 = 775, d_5 = 900, d_6 = \\
 &1,100, d_7 = 650
 \end{aligned}$$

7. $x_1 = 20, d_2 = 10, d_3 = 50$

9. $x_1 = 15, d_1^+ = 12, d_2^+ = 10, d_4 = 6$; not satisfied

(a) min. $Z = P_1 d_1, P_2 d_4^+, 3P_3 d_2 + 2P_3 d_3, P_4 d_1^+$; s.t. $x_1 + x_2 + d_1 -$

11. $d_1^+ = 80, x_1 + d_2 = 60, x_2 + d_3 = 35, d_1^+ + d_4 d_4^+ = 10$; (b) $x_1 = 60, x_2 = 30, d_1^+ = 10, d_3 = 5$

(a) min. $P_1 d_1, P_2 d_2^+, P_3 d_3, P_4 d_4$; s.t. $x_1 + d_1 - d_1^+ = 30, 20x_1 +$

13. $40x_2 + 150x_3 + d_2 - d_2^+ = 1,200, x_2 + x_3 + d_3 - d_3^+ = 20, x_3 + d_4 - d_4^+ = 6$; (b) $x_1 = 30, x_2 = 15, d_3 = 5, d_4 = 6$

(b) x_1 (Sunday) = 0, x_2 (Monday) = 7, x_3 (Tuesday) = 0, x_4

15. (Wednesday) = 27, x_5 (Thursday) = 0, x_6 (Friday) = 25, x_7 (Saturday) = 1

17. A = .1653, B = .3346, C = .5001

19. Abbott = .3147, Bates = .1600, Cook = .5253

21. Global = .2030, Blue Chip = .5060, Bond = .2910

23. Explorer = .4542, Trooper = .2605, Passport = .2852

25. Cheraton = .3393, Milton = .1929,
Harriott = .4677

27. Robin = .4431, Terry = .1902,
Kelly = .3667

31. MB = .3454, DB = .4570, FL =
.1977

33. DSS = .6415, OM = .3585

35. (a) renovate = .6436, new =
.3564; (b) consistent

37. Memorial = .4633, Country =
.3203, General = .2163

39. A = .2843, B = .3109, C = .4049

43. $S(1) = 77.50, S(2) = 80.80, S(3) = 82.05$; site 3

45. $S(\text{South}) = 73.80, S(\text{West A}) = 74.50, S(\text{West B}) = 67.25, S(\text{East}) = 76.75$; East site

47. $S(C) = 76, S(E) = 75, S(D) = 74, S(B) = 70, S(A) = 69$

[Page 786]

49. $S(\text{Roanoke}) = 84.74, S(\text{Richmond}) = 83.79, S(\text{Greensboro}) = 82.61, S(\text{Charlotte}) = 81.10, S(\text{Bristol}) = 79.02, S(\text{Knoxville}) = 76.93$

Chapter 10

1. $p = \$186.67$, $v = 176$ chairs, $Z = \$18,313.33$

3. $p = \$40.50$, $v = 28,269.6$, $Z = \$542,641.95$

5. $p = \$25.55$, $v = 78.63$, $Z = -\$1,198$

7. $p = \$35.16$, $Z = \$10,257.57$

9. $x_1 = 10$, $x_2 = 45$, $Z = \$212.50$

11. $x_1 = 0.09, x_3 = 0.497, x_4 = 0.413$; total return = 0.136

13. Lagrange multiplier = 0.27

15.
$$\begin{aligned} \max Z &= (5.8 - .06p_h + .005p_l) p_h \\ &+ (3.0 - .11p_1 + .008p_h) p_1, \text{ s.t.} \\ 5.8 - .06p_h + .005p_1 &\leq 2.5, 3.0 - .11p_1 + .008p_h \leq 2.5, p_l, p_h \geq 0; \\ p_h &= \$56.40, p_l = \$16.70, Z = \$167.94 \end{aligned}$$

17. $x = 151.2, y = 484.08, Z = 52,684.29$

19. $x_A = .250, x_B = .750, Z = .0382$,
total return = 0.11

$$x_1 = 5, x_2 = 3, x_3 = 8, x_4 = 3, x_5$$

21. $= 2, x_6 = 5, x_7 = 5, x_8 = 4, Z =$
 $\$4,193,450.37$

Chapter 11

1. (a) subjective, (b) subjective, (c) both, (d) subjective, (e) objective, (f) subjective, (g) objective (h) objective

3. (a) 019, .066; 2029, .233; 3039, .266; 4049, .266; 50 +, .166; (b) yes, only one of the events can take place each winter

5. $P(r \geq 5) = .0433$ rejected, $P(r < 5) = .9567$ accepted

7.

$$P(r = 4) = .0367$$

9. $P(r \geq 4) = .0024$

(a and b) $P(WA) = .28, P(LA) =$
11. $.12, P(WO) = .36, P(LO) = .24;$
(c) $P(O|W) = .563$

(a and b) $P(GA) = .198, P(DA) =$
13. $.002, P(GB) = .392, P(DB) = .008,$
 $P(GC) = .368, P(DC) = .032; (c)$
 $P(B|D) = .19$

15. $P(A|D) = .667$

17. $E(\text{bbl.}) = 7.95$

19. $E(\text{lead time}) = 2.2 \text{ days}, \sigma = 0.87$

21. $E(\text{building}) = \$38,000, \sigma = 26,381, E(\text{bonds}) = 34,000, s = 4,899$; probably bonds, less variation

23. $E(\text{time between breakdowns}) = 2.75 \text{ hr.}$

25. $P(x \geq 3.5) = .0668$

27. $P(x \geq 900) = .1587$

29. $P(x \geq 6,000) = .0475$

31. $P(x \geq 670) = .7422$

33. $P(x \geq 1,200) = .1056, 65.5$

35. $P(x \geq 200,000) = .9699$

37. $P(x \leq 27,000) = .1894$

39. $P(x \geq 75) = .4364$

41. not normal

Chapter 12

1. (a) lease land; (b) savings certificate

3. (a) bellhop; (b) management; (c) bellhop

5. (a) corn; (b) soybeans; (c) corn; (d) soybeans; (e) corn

7. (a) build shopping center; (b) lease equipment; (c) build shopping center; (d) lease equipment; (e) build shopping center

(a) Army vs. Navy; (b) Alabama vs. Auburn; (c) Alabama vs.

9. Auburn

11. (a) pass; (b) off tackle or option;
(c) toss sweep

13. (a) office park; (b) office building;
(c) office park or shopping center;
(d) office building

15. press

17. EV (shoveler) = \$99.6

19. EV (operate) same as leasing;
conservative decision is to lease

21. bond fund

23. Taiwan, $EVPI = \$0.47$ million

25. compact car dealership

27. (b) Stock 16 cases; (c) stock 16 cases; (d) \$5.80

29. (b) Stock 26 dozen; (c) stock 26 dozen; (d) \$3.10

31. O'Neil

33. A, $EV(A) = \$23,300$

35. make bid, $EV = \$142$

37. produce widget, $EV = \$69,966$

39. hire Roper, $EVSI = \$5.01M$

41. Jay should settle; $EV = 600,000$

43. grower B, $EV = 39,380$

45. $EVSI = \$234,000$, $EVPI = \$350,000$

47. oil change, $EV = \$62.80$

49. $EV = \$3.75$ million, $EVSI = \$670,000$

51. risk averters

Chapter 13

(a) multiple server, FCFS, finite or infinite;

(b) multiple server, FCFS, infinite;

(c) multiple server, FCFS, infinite;

(d) single or multiple, appt., finite;

(e) single server, FCFS or appt., finite;

1.

(f) single server, FCFS, finite;

(g) multiple server, FCFS, infinite;

(h) single or multiple, FCFS,

infinite;

(i) single server, FCFS or appt.,
finite

3. (a) false; (b) true

Customers served according to a
5. schedule or alphabetically, or at
random

$P_0 = .33, P_3 = .099, L = 2, L_q =$
7. $1.33, W = .125, W_q = .083, U =$
 $.67$

9. $L_q = .9, W = .25, W_q = .15$; an
infinite queue would form

11. $\lambda = 9$

13. yes, install the second window

(a) $L_q = 9.09$, $W = 24$ min., $W_q =$

15. 21.6 min; (b) $W = 4$ min., the state should install scales

17. $W_q = .56$ min., add an advisor

19. (a) $P_0 = .20$; (b) $L_q = 1$; (c) $W_q = 1.25$ days; (d) $P_3 = .1825$

21. $L_q = 6.34$, $W_q = 18.39$ min

23. $L_q = .083$, $W_q = .083$ min

$$W = 23.67 \text{ hr.}$$

25.

[Page 787]

27. (a) 33%; (b) $W_q = 1.13 \text{ min}$, $L_q = .756$; (c) $P_0 = .061$

29. $W_q = .007 \text{ hr.}$; no

31. current service = \$312; new service = \$212.27; new service is better

33. (a) no, $W_q = 8 \text{ min.}$; (b) yes, $W_q = 0.11 \text{ min.}$; (c) yes, $W_q = .53 \text{ min.}$

35. hire clerks, 5

37. $P_w = .29976$, three salespeople
sufficient

39. 1.14 hr.

41. $W_q = 10.27$ min., $P_0 = .423$

43. $L_q = 2.25$, $L = 12.25$, $W_q = .321$
wk., $W = 1.75$ wk., $U = .833$

45. add extra employees

47. 79 A.M. = 5, 9noon = 2, noon2 P.M.
= 6, 25 P.M. = 4

49. adequate; $W_q = 19.1$ min.

51. (a) $W_q = 2.33$ hr., $P_0 = .24$; (b)
 $W_q = .89$ hr., $P_0 = .51$

53. $L = 4.06$, $L_q = 1.86$, $W_q = 1.01$
min

55. $L_q = 132.75$ passengers

Chapter 14

(a and b) $\mu \approx 3.48$, $EV = 3.65$,
the results differ because not
1. enough simulations were done in
part a; (c) approximately 21 calls;
no; repeat simulations to get
enough observations

3. (a and b) $\mu \approx 2.95$

5. sun visors

7. (a) $\mu \approx \$256$

(a) average time between arrivals
 $\approx$ 4.3 days, average waiting time
 $\approx$ 6.25 days, average number of
9. tankers waiting $\approx$ 1.16; (b)
system has not reached steady
state

total yardage $\approx$ 155 yd.; the
11. sportswriter will predict Tech will
win

[*Tribune Daily News*] = [.50 .50];
13. too few iterations to approach a
steady state

15. expansion is probably warranted

17. two of five trials (depended on
random number stream)

19. system inadequate

21. avg. Salem dates = 2.92

23. $P(\text{capacity} > \text{demand}) = .75$

avg. maintenance cost =

25. \$3,594.73; $P(\text{cost} \leq \$3,000) = .435$

27. avg. rating = 2.91; $P(x \geq 3.0) = .531$

29. (a) $P(1,2) = .974$, $P(1,3) = .959$,
 $P(1,4) = .981$, $P(2,3) = .911$,
 $P(2,4) = .980$, $P(3,4) = .653$; (b)
(1,4) and (2,4).

32. (a) 700 rooms; (b) 690 rooms

Chapter 15

(a) Apr = 8.67, May = 8.33, Jun = 8.33, Jul = 9.00, Aug = 9.67, Sep = 11.00, Oct = 11.00, Nov = 11.00, Dec = 12.00, Jan = 13.33;

- 1.** (b) Jun = 8.20, Jul = 8.80, Aug = 9.40, Sep = 9.60, Oct = 10.40, Nov = 11.00, Dec = 11.40, Jan = 12.60; (c) $MAD(3) = 1.89$, $MAD(5) = 2.43$

(a) $F_4 = 116.00$, $F_5 = 121.33$, $F_6 = 118.00$, $F_7 = 143.67$, $F_8 = 138.33$, $F_9 = 141.67$, $F_{10} = 135.00$, $F_{11} = 156.67$, $F_{12} = 143.33$, $F_{13} = 136.67$; (b) $F_6 =$

$121.80, F_7 = 134.80, F_8 =$
 $125.80, F_9 = 137.20, F_{10} =$
3. $143.00, F_{11} = 149.00, F_{12} =$
 $137.00, F_{13} = 142.00;$ (c) $F_4 =$
 $113.85, F_5 = 116.69, F_6 =$
 $125.74, F_7 = 151.77, F_8 = 132.4,$
 $F_9 = 138.55, F_{10} = 142.35, F_{11} =$
 $160.0, F_{12} = 136.69, F_{13} =$
 $130.20;$ (d) 3 - qtr $MA: E = 32.0,$
 5 - qtr $MA: E = 36.4,$ weighted
 $MA: E = 28.09$

(a) $F_4 = 400.000, F_5 = 406.67, F_6 =$
 $423.33, F_7 = 498.33, F_8 =$
 $521.67, F_9 = 571.67;$ (b) $F_2 =$
 $400.00, F_3 = 410.00, F_4 =$
5. $398.00, F_5 = 402.40, F_6 =$
 $421.92, F_7 = 452.53, F_8 =$
 $460.00, F_9 = 498.02;$ (c) 3-sem.
 $MAD = 80.33,$ exp. smooth. MAD

$$= 87.16$$

F_{11} (exp. smooth) = 68.6, F_{11} (adjusted) = 69.17, F_{11} (linear

7. trend) = 70.22; exp. smooth: $E = 14.75$, $MAD = 1.89$; adjusted: $E = 10.73$, $MAD = 1.72$; linear trend: $MAD = 1.09$

(a) $F_{21} = 74.67$, $MAD = 3.12$; (b) $F_{21} = 75.875$, $MAD = 2.98$; (c)

9. $F_{21} = 74.60$, $MAD = 2.87$; (d) 3-mo. moving average and exponentially smoothed

11. $F_{13} = 631.22$, $\bar{E} = 26.30$, $E = 289.33$, biased low

13. $F_1 = 155.6, F_2 = 192.9, F_3 = 118.2, F_4 = 155.6$

$F_{04}(\text{adjusted}) = 3,313.19, F_{04}(\text{linear trend}) = 2,785.00;$

15. adjusted: $MAD = 431.71$; linear trend: $MAD = 166.25$

fall = 44.61, winter = 40.08,

17. spring = 52.29, summer = 70.34;
yes

19. $E = 86.00, \bar{E} = 8.60, MAD = 15.00, MAPD = 0.08$

21. $MAD: MA = 1.89, \text{exp. smooth} = 2.16; 3\text{-mo. moving average}$

23. $y = 0.67 + .223x$, 88.58%

25. (a) $y = 113.4 + 2.986x$, 140.43 gal.; (b) .929

(a) $y = 6,541.66 - .448x$;
27. $y(\$9,000) = 2,503.2$, $y(\$7,000) = 3,400.6$; (b) - .754

29. $b = 2.98$, the number of gallons sold per each degree increase

$y = 13.476 - 0.4741x$, $r = -0.76$,
31. $r^2 = 0.58$; yes, relationship; $y = 2.57\%$ defects

linear trend line, 3-mo. *MA* and adj.
33. exp. smooth all are relatively

accurate

exponential smoothing appears to

35. be less accurate than linear trend line

37. F_6 (adj.) = 74.19 seat occupancy,
 $\bar{E} = 1.08$, $MAD = 8.6$

39. student choice of model

(a) $y = 119.27 + 12.28x$, y (lb.) =
41. 315.67; (b) $y = 2504.18 + .147x$,
 $r^2 = .966$.

(a) $y = 13,803.07 + 470.55x$,
43. 65.15%; (b) $y = 83 - 1.68x$;
64.5%

45. (a) $y = 745.91 + 2.226x_1 + .163x_2$;
(b) $r^2 = .992$; (c) $y = 7,186.91$

(a) $y = 117.128 + .072x_1 +$
47. $19.148x_2$; (b) $r^2 = .822$; (c) $y =$
 $\$940.60$

Chapter 16

1. (a) $Q = 79.7$; (b) \$13,550; (c) 15.05 orders; (d) 24.18 days

3. (a) $Q = 240$; (b) \$4,800; (c) 80 orders; (d) 4 days

5. (a) $Q = 190,918.8$ yd.; (b) \$15,273.51; (c) 6.36 orders; (d) 57.4 days

7. $Q = 67.13$, $S = 15.49$, $TC =$ \$3,872.98

9. (a) $Q = 774.6$ boxes, $TC = \$619.68$, $R = 43.84$ boxes

11. $Q = 23,862$, $TC = \$298,276$

13. (a) $Q = 2,529.8$ logs; (b) $TC = \$12,649.11$; (c) $T_b = 63.3$ days; (d) 42.16 days

15. $Q = 17,544.2$; $S = 4,616.84$; $TC = \$3,231.84$

17. $Q = 569.32$, $S = 32.79$, $TC = \$1,475.45$

19. (a) $Q = 11,062.62$, $TC = \$5,532.14$, runs = 1.63, max. level = 4,425.7; (b) $Q = 6,250$,

$$TC = \$6,458.50$$

21. (a) $Q = 4,912.03$, $TC = \$18,420.11$; (b) $Q = 3,833.19$, $TC = \$17,249.36$; select new location

23. $C_c = \$950$

25. take discount, $Q = 300$

[Page 788]

27. $Q = 500$, $TC = \$64,424$

29. $Q = 6,000$, $TC = \$85,230.33$

31. $Q = 30,000, TC = \$14,140$

33. $Q = 20,000, TC = \$893,368$

35. 91%

37. $R(90\%) = 74.61$ gal; increase safety stock to 26.37 gal.

39. 254.4 gal.

41. $R = 24.38$

43. 120 pizzas

45. (a) 15.15%; (b) $R = 6.977$ pizzas

Glossary

A

B

C

D

E

F

G

H

I

J

L

M

N

O

P

Q

R

S

I

U

V

W

Z

A

activities

in a CPM/PERT project network, the branches reflecting project operations

activity slack

in a CPM/PERT network, the amount of time that the start of an activity can be delayed without exceeding the critical path project time

adjusted exponential smoothing

the exponential smoothing forecasting technique adjusted for trend changes and seasonal patterns

analogue simulation

an original physical system
is replaced with a
analogous physical system
that is easier to test and
manipulate

analytical

containing or pertaining to
mathematical analysis

using formulas or equations

a priori probability

one of the two types of objective probabilities; given a set of outcomes for an activity, it is the ratio of the number of desired outcomes to the total number of outcomes

arrival rate

the number of arrivals at a service facility (within a queuing system) during a specified period of time

artificial variable

a variable that is added to an $=$ or $\geq$ constraint so that initial solutions in a linear programming problem can be obtained at the origin

assignment model

a type of linear programming model similar to a transportation model, except that the supply at each source is limited to one unit and the demand at each destination is limited to one unit

average error

the cumulative error,
averaged over the number
of time periods

B

back order

a customer order that cannot be filled from existing inventory but will be filled when inventory is replenished

backward pass

a means of determining the latest event times in a CPM/PERT network

balanced transportation model

a model in which supply equals demand

basic feasible solution

any solution in linear programming that satisfies the model constraints

basic variables

in a linear programming problem, variables that have values (other than zero) at a basic feasible solution point

Bayesian analysis

a method for altering marginal probabilities, given additional information; the altered probabilities are referred to as revised or posterior probabilities

Bernoulli process

a probability experiment that has the following properties: (1) each trial

has two outcomes, (2) the probabilities remain constant, (3) the outcomes are independent, and (4) the number of trials is discrete

beta distribution

a probability distribution used in network analysis for determining activity times

binomial distribution

a probability distribution for experiments for which the Bernoulli properties hold

branch

in a network diagram, a line that represents the flow of items from one point (i.e., node) to another

branch and bound method

a solution approach whereby a total set of feasible solutions is partitioned into smaller subsets, which are then evaluated systematically; this technique is used extensively to solve integer programming problems

break-even analysis

the determination of the number of units that must be produced and sold to equate total revenue with total cost

break-even point

the volume of units that equates total revenue with total cost

C

calculus

a branch of mathematics that is concerned with the rate of change of functions; the two basic forms of calculus are differential calculus and integral calculus

calling population

the source of customers to a waiting line

carrying cost

the cost incurred by a business for holding items in inventory

causal forecasting methods

a class of mathematical techniques that relate variables to other factors that cause forecast behavior

classical optimization

the use of calculus to find optimal values for variables

classical probability

an a priori probability

coefficient of determination

a measure of the strength of the relationship between the variables in a regression equation

coefficient of optimism

a measure of a decision

maker's optimism

collectively exhaustive events

all the possible events of an experiment

concave curve

a curve shaped like an inverted bowl

conditional probability

the probability that one event will occur, given that another event has already occurred

constrained optimization model

a model that has a single objective function and one

or more constraints

constraint

a mathematical relationship that represents limited resources or minimum levels of activity in a mathematical programming model

continuous distribution

a probability distribution in which the random variables can equal an infinite number of values within an interval

continuous random variable

a variable that can take on an infinite number of values within some interval

convex curve

a curve shaped like an upright bowl

correlation

a measure of the strength of the causal relationship between the independent and dependent variables in a linear regression equation

critical path

the longest path through a CPM/PERT network; it indicates the minimum time in which a project can be completed

critical path method (CPM)

a network technique that uses deterministic activity times for project planning

and scheduling

cumulative error

a sum of the forecast errors

cumulative probability distribution

a probability distribution in which the probability of an event is added to the sum

of the probabilities of the
previously listed events

cycle

movement up or down
during a trend in a forecast

D

data

pieces of information

database

an organized collection of
numeric information

decision analysis

the analysis of decision situations in which certainty cannot be assumed

decision support system (DSS)

a computer-based information system that a

manager can use to assist
in and support decision
making

decision tree

a graphical diagram for
analyzing a decision
situation

decision variable

a variable whose value represents a potential decision on the part of the manager

degeneracy

a tie for the pivot row is broken arbitrarily and can lead to degeneracy

Delphi method

a procedure for acquiring informal judgments and opinions from knowledgeable individuals to use as a subjective forecast

dependent demand

typically component parts used internally to produce a final product

dependent events

events for which the probability of one event is affected by the probability of occurrence of other events

derivative

in calculus, a transformed form of a mathematical function that defines the slope of the function

deterministic

characterized by the assumption that there is no uncertainty

deviational variables

in a goal programming model constraint, variables that reflect the possible deviation from a goal level

differential

the derivative of a function

directed branch

a branch in a network in which flow is possible in only one direction

discrete distribution

a probability distribution that consists of values for the random variable that are countable and usually integer

dual

an alternative form of a linear programming model that contains useful information regarding the value of resources, which

form the constraints of the model

dummy activity

a branch in a CPM/PERT network that reflects a precedence relationship but does not represent any passage of time

E

earliest activity time

in a CPM/PERT network, the earliest and latest times at which an activity can be started without exceeding the critical path project time

economic forecast

a prediction of the state of the economy in the future

economic order quantity (EOQ)

the optimal order size that corresponds to total minimum inventory cost

efficiency of sample information

an indicator of how close to perfection sample information is; it is computed by dividing the expected value of sample information by the expected value of perfect information

empirical

consisting of (or based on)

data or information gained
from experiment and
observation

equal likelihood criterion

a decision-making method
in which all states of nature
are weighted equally (also
known as the LaPlace
criterion)

equilibrium point

the outcome of a game that results from a pure strategy

event

the possible result of a probability experiment

events

in a CPM/PERT project network, the nodes that reflect the beginning and termination of activities

expected monetary value

the expected (average) monetary outcome of a decision, computed by multiplying the outcomes by their probabilities of occurrence and summing these products

expected opportunity loss

the expected cost of the loss of opportunity resulting from an incorrect decision by the decision maker

expected value

an *average* value computed by multiplying each value

of a random variable by its probability of occurrence

expected value of perfect information (*EVPI*)

the value of information expressed as the amount of money that a decision maker would be willing to pay to make a better decision

expected value of sample information

the difference between the expected value of a decision situation with and without additional information

experiment

in probability, a particular action, such as tossing a coin

exponential distribution

a probability distribution
often used to define the
service times in a queuing
system

exponential smoothing

a time series forecasting
method similar to a moving
average in which more

recent data are weighted more heavily than past data

extreme points

the maximum and minimum points on a curve (also known as relative extreme points); in a linear programming problem, a protrusion in the feasible solution space

F

factorial

for a value of n , $n! = n(n-1)(n-2) \dots (2)(1)$

feasible solution

a solution that does not violate any of the

restrictions or constraints
in a model

feedback

decision results that are fed
back into a management
information system to be
used as data

finite calling population

the source of customers for a queuing system is limited to a finite number

finite queue

a queue with a limited (maximum) size

fixed costs

costs that are independent

of the volume of units
produced

forecast

a prediction of what will
occur in the future

forecast error

the difference between
actual and forecasted

demand

forecast reliability

a measure of how closely a forecast reflects reality

forecast time frame

how far in the future a forecast projects

forward pass

a method for determining earliest activity times in a CPM/PERT network

frequency distribution

the organization of events into classes, which shows the frequency with which the events occur

functional relationship

an equation that relates a dependent variable to one or more independent variables

G

Gantt chart

a graph or bar chart with a bar for each activity in a project, showing the passage of time.

goal constraint

a constraint in a goal programming model that contains deviational variables

goal programming

a linear programming technique that considers more than one objective in the model

goals

the alternative objectives
in a goal programming
model

H

Hurwicz criterion

a method for making a decision in a decision analysis problem; the decision is a compromise between total optimism and total pessimism

I

identity matrix

a matrix containing ones along the diagonal and zeros elsewhere

implementation

the use of model results

implicit enumeration

a method for solving integer programming problems in which obviously infeasible solutions are eliminated and the remaining solutions are systematically evaluated to see which one is best

independent demand

final product demanded by an external customer

independent events

events for which the probability of occurrence of one event does not affect the probability of occurrence of the other events

inequality

a mathematical relationship containing a $\geq$ or $\leq$ sign

infeasible problem

a linear programming problem with no feasible solution area and thus no solution

instantaneous receipt

the assumption that once inventory level reaches zero, an order is received after the passage of an infinitely small amount of time

integer programming

a form of linear

programming that
generates only integer
solution values for the
model variables

inventory

a stock of items kept on
hand by an organization to
use to meet customer
demand

inventory analysis

the analysis of the problems of inventory planning and control, with the objective of minimizing inventory-related costs

J

joint probability

the probability of several events occurring jointly in an experiment

L

LaPlace criterion

a decision-making method in which all states of nature are weighted equally (more commonly known as the equal likelihood criterion)

latest activity time

in a CPM/PERT network, the latest time at which an activity can be started without exceeding the critical path project time

linear programming

a management science technique used to determine the optimal way to achieve an objective, subject to restrictions, in cases in which all the mathematical relationships

are linear

linear (simple) regression

a form of regression that reflects the relationship of two variables

linear trend line

a forecast that uses the linear regression equation

to relate demand to time

long-range forecast

a forecast that typically encompasses a period of time longer than 1 or 2 years

M

management science

the application of mathematical techniques and scientific principles to management problems to help managers make better decisions

marginal probability

the probability of a single event occurring

maximal flow problem

a network problem in which the objective is to maximize the total amount of flow from a source to a destination

maximax criterion

a method for making a decision in a decision analysis problem; the decision will result in the maximum of the maximum payoffs

maximin criterion

a method for making a decision in a decision analysis problem; the decision will result in the

maximum of the minimum
payoffs

maximization problem

a linear programming
problem in which an
objective, such as profit, is
maximized

mean absolute deviation (*MAD*)

a measure of the difference
between a forecast and
what actually occurs

mean absolute percent deviation (*MAPD*)

the absolute forecast error,
measured as a percentage
of demand

mean squared error (*MSE*)

the average of the squared
forecast errors

medium-range forecast

a forecast that
encompasses anywhere
from 1 month to 1 year

minimal spanning tree problem

a network problem in which the objective is to connect all the nodes in a network so that the total branch lengths are minimized

minimax regret criterion

a method for making a decision in a decision analysis problem; the decision will minimize the maximum regret

minimization problem

a linear programming problem in which an objective, such as cost, is minimized

minimum cell cost method

a method for determining the initial solution to a transportation model

mixed constraint problem

a linear programming problem with a mixture of $\leq$, $=$, and $\geq$ constraints

mixed integer model

an integer linear programming model that can generate a solution with both integer and

noninteger values

model

an abstract (mathematical)
representation of a problem

Monte Carlo process

a technique used in
simulation for selecting
numbers randomly from a

probability distribution

most likely time

one of three time estimates used in a beta distribution to determine an activity time; it is the time that would occur most frequently if the activity were repeated many times

moving average

a time series forecasting method that involves dividing values of a forecast variable by a sequence of time periods

multiple optimum solutions

alternative solutions to a linear programming problem, all of which achieve the same objective function value

multiple regression

a form of regression that reflects the relationship among more than two variables

multiple-server queuing system

a system in which a single waiting line feeds into two

or more servers in parallel

mutually exclusive events

in a probability experiment,
events that can occur only
one at a time

N

network

an arrangement of paths connected at various points (drawn as a diagram), through which an item (or items) moves from one point to another

network flow models

a model that represents the flow of items through a system

node

in a network diagram, a point that represents a junction or an intersection; it is represented by a circle

noninstantaneous receipt

the gradual receipt of inventory over time

nonlinear programming

a form of mathematical programming in which the objective function or constraints (or both) are nonlinear functions

normal distribution

a continuous probability distribution that has the shape of a bell

O

objective function

a mathematical relationship that represents the objective of a problem solution

objective probability

the relative frequency with which a specific outcome in an experiment has been observed to occur in the long run

operating characteristics

average values for characteristics that describe a waiting line

opportunity cost table

a table derived in the solution to an assignment problem

optimal solution

the one best solution to a problem

optimistic time

one of three time estimates used in a beta distribution to determine an activity time; it is the shortest possible time it would take to complete an activity if everything went right

order cycle

the time period during which a maximum inventory level is depleted and a new order is received to bring inventory back to

its maximum level

ordering cost

the cost a business incurs
when it makes an order to
replenish its inventory

P

parameter

a constant value that is generally a coefficient of a variable in a mathematical equation

payoff table

a table used to show the payoffs that can result from decisions under various states of nature

penalty cost

the penalty (or regret) suffered by a decision maker when a wrong decision is made

periodic inventory system

a system in which an order is placed for a variable amount at fixed time intervals

[Page 792]

permanent set

in a shortest route network problem, a set of nodes to which the shortest route from the start node has

been determined

pessimistic time

one of three time estimates used in a beta distribution to determine an activity time; it is the longest possible time it would take to complete an activity if everything went wrong

pivot column

the column corresponding to the entering variable

Poisson distribution

a probability distribution that describes the occurrence of a relatively rare event in a fixed period of time; often used to define arrivals at a service facility in a queuing system

political/social forecast

a prediction of political and social changes that may occur in the future

population mean

the mean of an entire set of data being analyzed

posterior probability

the altered marginal probability of an event based on additional information

precedence relationship

the relationship exhibited by events that must occur in sequence; such events can be represented by a

CPM/PERT network

primal

the original form of a linear programming model

priority

the importance of a goal relative to other goals in a goal programming model

probabilistic techniques

management science
techniques that take into
account uncertain
information and give
probabilistic solutions

probability distribution

a distribution showing the
probability of occurrence of

all events in an experiment

probability tree

a diagram showing the probabilities of the various outcomes of an experiment

production lot size model

an inventory model for a business that produces its

own inventory at a gradual rate (also known as the noninstantaneous receipt model)

prohibited route

in a transportation model, a route (i.e., variable) to which no allocation can be made

project crashing

a method for reducing the duration of a CPM/PERT project network by reducing one or more critical path activities and incurring a cost

project evaluation and review technique (PERT)

a network technique, designed for project planning and scheduling,

that uses probabilistic
activity times

pseudorandom numbers

random numbers generated
by a mathematical process
rather than by a physical
process

Q

qualitative forecast methods

nonquantitative, subjective forecasts based on judgment, opinion, experience, and expert opinion

quantitative forecast

methods

forecasts derived from a mathematical formula

quantity discount model

an inventory model in which a discount is received for large orders

queue

a waiting line

queue discipline

the order in which
customers waiting in line
are served

queuing analysis

the probabilistic analysis of
waiting lines

R

random number table

a table containing random numbers derived from some artificial process, such as a computer program

random numbers

numbers that are equally likely to be drawn from a large population of numbers

random variable

a variable that can be assigned numeric values reflecting the outcomes of an event; because these values occur in no particular order, they are said to be random

random variations

movements in a forecast that are not predictable and follow no pattern

regression

a statistical technique for measuring the relationship of one variable to one or more other variables; this

method is used extensively
in forecasting

regression equation

an equation derived from
historical data that is used
to forecast

regret

a value representing the

regret the decision maker suffers when a wrong decision is made

relative frequency probability

another name for an objective (a posteriori) probability; it represents the relative frequency with which a specific outcome has been observed to occur in the long run

reorder lead time

the time between the placement of an order and its receipt into inventory

reorder point

the inventory level at which an order is placed

rim requirements

the supply and demand values along the outer row and column of a transportation tableau

risk averter

a person who avoids taking risks

risk taker

a person who takes risks in the hope of achieving a large return

Row operations

are used to solve simultaneous equations where equations are multiplied by constants and added or subtracted from

each other

run

a sequence of sample values that displays the same tendency in a control chart

S

safety stock

a buffer of extra inventory used as protection against a stockout (i.e., running out of inventory)

sample mean

the mean of a subset of the population data

satisfactory solution

in a goal programming model, a solution that satisfies the goals in the best way possible

sample

a portion of the items
produced used for
inspection

scatter diagram

a diagram used in
forecasting that shows
historical data points

scientific method

a method for solving problems that includes the following steps: (1) observation, (2) problem definition, (3) model construction, (4) model solution, and (5) implementation

search techniques

methods for searching through the solutions generated by a simulation model to find the best one

seasonal factor

a numeric value that is multiplied by a normal forecast to get a seasonally adjusted forecast

seasonal pattern

in a forecast, a movement that occurs periodically and is repetitive

seed value

a number selected arbitrarily from a range of numbers to begin a stream of random numbers generated by a computerized random number generator

sensitivity analysis

the analysis of changes in the parameters of a linear programming problem

sequential decision tree

a decision tree that analyzes a series of sequential decisions

service level

the percentage of orders a business is able to fill from inventory in stock during the reorder period

service rate

the average number of customers that can be served from a queue in a specified period of time

shadow price

the price one would be willing to pay to obtain one more unit of a resource in a linear programming problem

shared slack

in a CPM/PERT network, slack that is shared among several adjacent activities

shortest route problem

a network problem in which the objective is to determine the shortest distance between an originating point and several destination points

short-range forecast

a forecast of the immediate future that is concerned

with daily operations

simple regression

a form of regression that reflects the relationship of two variables

[Page 793]

simplex method

a tabular approach to

solving linear programming problems

simplex tableau

the table in which the steps of the simplex method are conducted; each tableau represents a solution

simulated time

the representation of real time in a simulation model

simulation

the replication of a real system with a mathematical model that can be analyzed with a computer

simulation language

a computer programming language developed specifically for performing simulation

single-server waiting line

a waiting line that contains only one service facility at which customers can be served

slack variable

a variable representing unused resources that is added to a $\leq$ inequality constraint to make the constraint an equation

slope

the rate of change in a linear mathematical function

smoothing constant

a weighting factor used in the exponential smoothing forecasting technique

standard deviation

a measure of dispersion around the mean of a probability distribution

states of nature

in a decision situation, the possible events that may occur in the future

steady state

a constant value achieved by a system after an extended period of time

steady-state probability

a constant probability that a system will end up in a particular state after a large number of transition periods

stockout

running out of inventory

subjective probability

a probability that is based on personal experience, knowledge of a situation, or intuition rather than on a priori or a posteriori evidence

substitution method

a method for solving nonlinear programming

problems that contain only one equality constraint; the constraint is solved for one variable in terms of another and is substituted into the objective function

surplus variable

a variable that reflects the excess above a minimum resource requirement level; it is subtracted from a $\geq$ inequality constraint in a linear programming

problem

system

a set or an arrangement of related items that forms an organic whole

T

technological forecast

a prediction of what types of technology may be available in the future

time series methods

statistical forecasting

techniques that are based solely on historical data accumulated over a period of time

total revenue

the volume of units produced multiplied by price per unit

transportation model

a type of linear programming problem in which a product is to be transported from a number of sources to a number of destinations at the minimum cost

trend

a long-term movement of an item being forecast

U

unbalanced transportation model

a transportation model in which supply exceeds demand or demand exceeds supply

unbounded problem

a linear programming problem in which there is no completely closed-in feasible solution area and therefore the objective function can increase infinitely

unconstrained optimization model

a model with a single objective function and no constraints

undirected branch

a branch in a network that allows flow in both directions

utils

the units in which utility is measured

utility

a numeric measure of the satisfaction a person derives from money

utilization factor

the probability that a server in a queuing system will be busy

V

validation

the process of making sure model solution results are correct (valid)

variable

within a model, a

mathematical symbol that
can take on different values

variable costs

costs that are determined
on a per-unit basis

variance

a measure of how much the
values in a probability

distribution vary from the mean

Venn diagram

a pictorial representation of mutually exclusive or nonmutually exclusive events

W

weighted moving average

a time series forecasting method in which the most recent data are weighted

what-if? analysis

a form of interactive

decision analysis in which a computer is used to determine the results of making various changes in a model

work breakdown structure

a structure that breaks down a project into its major subcomponents, components, activities, and tasks

Z

zeroone integer model

an integer programming model that can have solution values of only zero or one

Photo Credits

Chapter 1: page 1: Terry Donnelly/Image Bank/Getty Images; page 6: Kim Kulish, CORBIS-NY; page 19: George Hall, CORBIS-NY; page 22: Jean Higgins, Unicorn Stock Photos.

Chapter 2: page 35: Peter Yates, Corbis/Bettmann; page 46: Courtesy Nutrition Coordinating Center, University of Minnesota; page 51: Peter Beck, CORBIS-NY.

Chapter 3: page 76: GE
Plastics; page 82: Welch Foods
Inc.

Chapter 4: page 120: U.S.
Department of Agriculture;
page 124: Joseph Sohm,
Corbis/Bettmann; page 137:
Texaco, Inc.; page 141: William
Taufic, Corbis/Stock Market.

Chapter 5: page 184: Tim
Beddow, Photo Researchers,
Inc.; page 190: Rick Fatica,
Ohio University; page 194: Tim
Carlson, Stock Boston; page
200: Bryan Mitchell, Getty
Images, Inc.-Liaison; page 202:

Thomas S. England/Bloomberg News, Landov LLC.

Chapter 6: page [226](#): Owaki-Kulla, CORBIS-NY; page [238](#): AP World Wide Photos.

Chapter 7: page [282](#): CORBIS-NY; page [288](#): Yellow Roadway Corporation.

Chapter 8: page [328](#): Jerry Edmanson, AGE Fotostock America, Inc.; page [344](#): AP World Wide Photos; page [356](#): AP Wide World Photos.

Chapter 9: page [398](#): Richard

B. Levine/Frances M. Roberts;
page [405](#): AP World Wide
Photos; page [411](#): Gary I.
Rothstein, Landov LLC; page
[414](#): Getty Images.

[Chapter 10](#): page [466](#): David
Ximeno Tejada, Getty Images
Inc.-Stone Allstock.

[Chapter 11](#): page [481](#): Library
of Congress; page [492](#): U.S.
Coast Guard Academy.

[Chapter 12](#): page [519](#): Photo
courtesy of DuPont Kevlar®;
page [535](#): Lester Lefkowitz,
Corbis/Stock Market; page

[540](#): Courtesy of Oglethorpe Power Corporation; page [542](#): American Airlines; page [546](#): AP World Wide Photos.

[Chapter 13](#): page [584](#): AP World Wide Photos; page [590](#): Corbis RF.

[Chapter 14](#): page [627](#): American Red Cross; page [646](#): Bank of Boulder.

[Chapter 15](#): page [674](#): Mark Peterson, Corbis/SABA Press Photos, Inc.; page [675](#): David Blumenfeld, Getty Images, Inc.: Liaison; page [680](#): AP

World Wide Photos; page [684](#):
Allen Tannenbaum; page [688](#):
Carol Halebian Photography;
page [696](#): Courtesy of Vermont
Gas Systems.

[Chapter 16](#): page [746](#): Dell,
Inc.; page [751](#): Will
Burgess/Reuters, Landov LLC;
page [758](#): Richard B.
Levine/Frances M. Roberts.

Module A. The Simplex Solution Method

[Page A-2]

The *simplex method*, is a general mathematical solution technique for solving linear programming problems. In the simplex method, the model is put into the form of a table, and then a number of mathematical steps are performed on the table. These

mathematical steps in effect replicate the process in graphical analysis of moving from one extreme point on the solution boundary to another. However, unlike the graphical method, in which we could simply search through *all* the solution points to find the best one, the simplex method moves from one *better* solution to another until the best one is found, and then it stops.

The manual solution of a linear programming model using the simplex method can be a lengthy and tedious process.

Years ago, manual application of the simplex method was the only means for solving a linear programming problem. Now computer solution is certainly preferred. However, knowledge of the simplex method can greatly enhance one's understanding of linear programming. Computer software programs like QM for Windows or Excel spreadsheets provide solutions to linear programming problems, but they do not convey an in-depth understanding of how those solutions are derived. To a

certain extent, graphical analysis provides an understanding of the solution process, and knowledge of the simplex method further expands on that understanding. In fact, computer solutions are usually derived using the simplex method. As a result, much of the terminology and notation used in computer software comes from the simplex method. Thus, for those students of management science who desire a more in-depth knowledge of linear programming, it is beneficial to

study the simplex solution method as provided here.

Converting the Model into Standard Form

The first step in solving a linear programming model manually with the simplex method is to convert the model into standard form. At the Beaver Creek Pottery Company Native American artisans produce bowls (x_1) and mugs (x_2) from labor and clay. The linear programming model is formulated as

maximize $Z = \$40x_1 + 50x_2$ (profit)

subject to

$$x_1 + 2x_2 \leq 40 \text{ (labor, hr)}$$

$$4x_1 + 3x_2 \leq 120 \text{ (clay, lb)}$$

$$x_1, x_2 \geq 0$$

We convert this model into *standard form* by adding slack variables to each constraint as follows.

$$\text{maximize } Z = 40x_1 + 50x_2 + 0s_1 + 0s_2$$

subject to

$$x_1 + 2x_2 + s_1 = 40$$

$$4x_1 + 3x_2 + s_2 = 120$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The slack variables, s_1 and s_2 , represent the amount of unused labor and clay, respectively. For example, if no bowls and mugs are produced, and $x_1 = 0$ and $x_2 = 0$, then the solution to the problem is

$$x_1 + 2x_2 + s_1 = 40$$

$$0 + 2(0) + s_1 = 40$$

$$s_1 = \begin{array}{l} 40 \text{ hr of} \\ \text{labor} \end{array}$$

and

$$4x_1 + 3x_2 + s_2 = 120$$

$$\begin{array}{l} 4(0) + 3(0) \\ + s_2 \end{array} = 120$$

$$s_2 = 120 \text{ lb of clay}$$

Slack variables are added to $\leq$ constraints and represent unused resources.

In other words, when we start the problem and nothing is

being produced, all the resources are unused. Since unused resources contribute nothing to profit, the profit is zero.

$$Z = \$40x_1 + 50x_2 + 0s_1 + 0s_2$$

$$= 40(0) + 50(0) + 0(40) + 0(120)$$

$$Z = \$0$$

It is at this point that we begin to apply the simplex method. The model is in the required form, with the inequality constraints converted to equations for solution with the simplex method.

The Solution of Simultaneous Equations

Once both model constraints have been transformed into equations, the equations should be solved simultaneously to determine the values of the

variables at every possible solution point. However, notice that our example problem has *two* equations and *four* unknowns (i.e., two decision variables and two slack variables), a situation that makes direct simultaneous solution impossible. The simplex method alleviates this problem by assigning some of the variables a value of zero. The number of variables assigned values of zero is $n \ m$, where n equals the number of variables and m equals the number of constraints

(excluding the nonnegativity constraints). For this model, $n = 4$ variables and $m = 2$ constraints; therefore, two of the variables are assigned a value of zero (i.e., $4 - 2 = 2$).

For example, letting $x_1 = 0$ and $s_1 = 0$ results in the following set of equations.

$$x_1 + 2x_2 + s_1 = 40$$

$$4x_1 + 3x_2 + s_2 = 120$$

and

$$0 + 2x_2 + 0 = 40$$

$$0 + 3x_2 + s_2 = 120$$

First, solve for x_2 in the first equation:

$$2x_2 = 40$$

$$x_2 = 20$$

[Page A-4]

Then, solve for s_2 in the second equation:

$$3x_2 + s_2 = 120$$

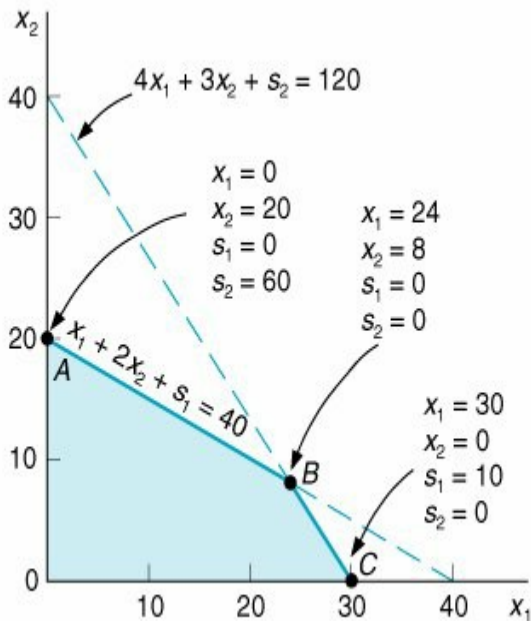
$$3(20) + s_2 = 120$$

$$s_2 = 60$$

This solution corresponds with point A in [Figure A-1](#). The graph in [Figure A-1](#) shows that at point A, $x_1 = 0$, $x_2 = 20$, $s_1 = 0$, and $s_2 = 60$, the exact solution obtained by solving simultaneous equations. This solution is referred to as a *basic feasible solution*. A feasible solution is any solution that satisfies the constraints. A *basic feasible solution* satisfies the constraints and contains as many variables with

nonnegative values as there are model constraints that is, m variables with nonnegative values and $n - m$ values set equal to zero. Typically, the m variables have positive nonzero solution values; however, when one of the m variables equals zero, the basic feasible solution is said to be *degenerate*. (The topic of *degeneracy* will be discussed at a later point in this module.)

Figure A-1. Solutions at points A, B, and C



*A **basic feasible solution** satisfies the model constraints and has the same number of variables with non-negative values as there are constraints.*

Consider a second example where $x_2 = 0$ and $s_2 = 0$. These values result in the following set of equations.

$$x_1 + 2x_2 + s_1 = 40$$

$$4x_1 + 3x_2 + s_2 = 120$$

and

$$x_1 + 0 + s_1 = 40$$

$$4x_1 + 0 + 0 = 120$$

Solve for x_1 :

$$4x_1 = 120$$

$$x_1 = 30$$

Then solve for s_1 :

$$30 + s_1 = 40$$

$$s_1 = 10$$

This basic feasible solution corresponds to point C in [Figure A-1](#), where $x_1 = 30$, $x_2 = 0$, $s_1 = 10$, and $s_2 = 0$.

[Page A-5]

Finally, consider an example where $s_1 = 0$ and $s_2 = 0$. These values result in the following set of equations.

$$x_1 + 2x_2 + s_1 = 40$$

$$4x_1 + 3x_2 + s_2 = 120$$

and

$$x_1 + 2x_2 + 0 = 40$$

$$4x_1 + 3x_2 + 0 = 120$$

These equations can be solved using row operations. In row operations, the equations can be multiplied by constant values and then added or subtracted from each other without changing the values of the decision variables. First, multiply the top equation by 4 to get

$$4x_1 + 8x_2 = 160$$

and then subtract the second equation:

$$\begin{array}{r}
 4x_1 + 8x_2 = 160 \\
 -4x_1 - 3x_2 = -120 \\
 \hline
 5x_2 = 40 \\
 x_2 = 8
 \end{array}$$

Next, substitute this value of x_2 into either one of the constraints.

$$x_1 + 2(8) = 40$$

$$x_1 = 24$$

Row operations are

used to solve simultaneous equations where equations are multiplied by constants and added or subtracted from each other.

This solution corresponds to point B on the graph, where $x_1 = 24$, $x_2 = 8$, $s_1 = 0$, and $s_2 = 0$, which is the optimal solution point.

All three of these example solutions meet our definition of *basic feasible solutions*.

However, two specific questions are raised by the identification of these solutions.

- 1.** In each example, how was it known which variables to set equal to zero?
- 2.** How is the optimal solution identified?

The answers to both of these questions can be found by using the simplex method. The simplex method is a set of

mathematical steps that determines at each step which variables should equal zero and when an optimal solution has been reached.

The Simplex Method

The steps of the simplex method are carried out within the framework of a table, or *tableau*. The tableau organizes the model into a form that makes applying the mathematical steps easier. The Beaver Creek Pottery Company example will be used again to demonstrate the simplex tableau and method.

maximize $Z = \$40x_1 + 50x_2 + 0s_1 + 0s_2$

subject to

$$\begin{aligned}x_1 + 2x_2 + s_1 &= 40 \text{ hr} \\4x_1 + 3x_2 + s_2 &= 120 \text{ lb} \\x_1, x_2, s_1, s_2 &\geq 0\end{aligned}$$

*The **simplex method** is a set of mathematical steps for solving a linear programming problem carried out in a table called a **simplex tableau**.*

The initial simplex tableau for this model, with the various column and row headings, is shown in [Table A-1](#).

Table A-1. The Simplex Tableau

c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
-------	-----------------	----------	-------	-------	-------	-------

z_j

$c_j z_j$

The first step in filling in [Table A-1](#) is to record the model variables along the second row from the top. The two decision variables are listed first, in order of their subscript magnitude, followed by the slack variables, also listed in order of their subscript magnitude. This step produces the row with x_1 , x_2 , s_1 , and s_2 in [Table A-1](#).

The next step is to determine a basic feasible solution. In other words, which two variables will form the basic feasible solution and which will be assigned a value of zero? Instead of arbitrarily selecting a point (as we did with points A , B , and C in the previous section), the simplex method selects the origin as the initial basic feasible solution because the values of the decision variables at the origin are always known in all linear programming problems. At that point $x_1 = 0$ and $x_2 = 0$; thus, the variables

in the basic feasible solution
are s_1 and s_2 .

$$x_1 + 2x_2 + s_1 = 40$$

$$0 + 2(0) + s_1 = 40$$

$$s_1 = 40 \text{ hr}$$

and

$$4x_1 + 3x_2 + s_2 = 120$$

$$4(0) + 3(0) + s_2 = 120$$

$$s_2 = 120 \text{ lb}$$

*The **basic feasible solution** in the initial simplex tableau is the origin where all decision variables equal zero.*

In other words, at the origin, where there is no production, all resources are slack, or unused. The variables s_1 and s_2 , which form the initial basic feasible solution, are listed in [Table A-2](#) under the column "Basic Variables," and their respective values, 40 and 120, are listed under the column "Quantity."

Table A-2. The Basic Feasible Solution

Basic Variables Quantity

C_j		x_1	x_2	s_1	s_2
	s_1			40	
	s_2			120	
Z_j					
$C_j - Z_j$					

At the initial basic

feasible solution at the origin, only slack variables have a value greater than zero.

[Page A-7]

The initial simplex tableau always begins with the solution at the origin, where x_1 and x_2 equal zero. Thus, the basic variables at the origin are the slack variables, s_1 and s_2 . Since

the quantity values in the initial solution always appear as the right-hand-side values of the constraint equations, they can be read directly from the original constraint equations.

The quantity column values are the solution values for the variables in the basic feasible solution.

The top two rows and bottom two rows are standard for all tableaus; however, the number of middle rows is equivalent to the number of constraints in the model. For example, this problem has two constraints; therefore, it has two middle rows corresponding to s_1 and s_2 . (Recall that n variables minus m constraints equals the number of variables in the problem with values of zero. This also means that the number of basic variables with values other than zero will be equal to m constraints.)

The number of rows in a tableau is equal to the number of constraints plus four.

Similarly, the three columns on the left side of the tableau are standard, and the remaining columns are equivalent to the number of variables. Since there are four variables in this model, there are four columns on the right of the tableau, corresponding to x_1 , x_2 , s_1 , and

s_2 .

The number of columns in a tableau is equal to the number of variables (including slacks, etc.) plus three.

The next step is to fill in the c_j values, which are the objective function coefficients, representing the *contribution to profit* (or cost) for each variable

x_j or s_j in the objective function. Across the top row the c_j values 40, 50, 0, and 0 are inserted for each variable in the model, as shown in [Table A-3](#).

Table A-3. The Simplex Tableau with c_j Values

			40	50	0	0
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2
0	s_1	40				
0	s_2	120				

z_j $c_j z_j$

The c_j values are the contribution to profit (or cost) for each variable.

The values for c_j on the left side of the tableau are the contributions to profit of only those variables in the basic feasible solution, in this case s_1 and s_2 . These values are inserted at this location in the tableau so that they can be used later to compute the values in the z_j row.

The columns under each variable (i.e., x_1 , x_2 , s_1 , and s_2) are filled in with the coefficients of the decision variables and slack variables in the model constraint equations.

The s_1 row represents the first model constraint; thus, the coefficient for x_1 is 1, the coefficient for x_2 is 2, the coefficient for s_1 is 1, and the coefficient for s_2 is 0. The values in the s_2 row are the second constraint equation coefficients, 4, 3, 0, and 1, as shown in [Table A-4](#).

Table A-4. The Simplex Tableau with Model Constraint Coefficients

Basic Variables	Quantity	40	50	0	0

C_j			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	Z_j					
	$C_j Z_j$					

[Page A-8]

This completes the process of

filling in the initial simplex tableau. The remaining values in the z_j and $c_j z_j$ rows, as well as subsequent tableau values, are computed mathematically using simplex formulas.

The following list summarizes the steps of the simplex method (for a maximization model) that have been presented so far.

- 1.** First, transform all inequalities to equations by adding slack variables.
- 2.** Develop a simplex tableau

with the number of columns equaling the number of variables plus three, and the number of rows equaling the number of constraints plus four.

- 3.** Set up table headings that list the model decision variables and slack variables.
- 4.** Insert the initial basic feasible solution, which are the slack variables and their quantity values.
- 5.** Assign c_j values for the

model variables in the top row and the basic feasible solution variables on the left side.

6. Insert the model constraint coefficients into the body of the table.

Computing the z_j and $c_j - z_j$ Rows

So far the simplex tableau has been set up using values taken directly from the model. From this point on the values are

determined by computation. First, the values in the z_j row are computed by multiplying each c_j column value (on the left side) by each column value under Quantity, x_1 , x_2 , s_1 , and s_2 and then summing each of these sets of values. The z_j values are shown in [Table A-5](#).

Table A-5. The Simplex Tableau with z_j Row Values

		40	50	0	0
	Basic Variables	Quantity			
c_j		x_1	x_2	s_1	s_2

0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	c_j					

The z_j row values are computed by multiplying the c_j column values by the variable column

values and summing.

For example, the value in the z_j row under the quantity column is found as follows.

$$\begin{array}{rcl} c_j & \text{Quantity} & \\ 0 \times 40 & = & 0 \\ 0 \times 120 & = & \underline{0} \\ z_q & = & 0 \end{array}$$

The value in the z_j row under the x_1 column is found similarly.

$$c_j \quad x_1$$

$$0 \times 1 = 0$$

$$0 \times 4 = \underline{0}$$

$$z_j = 0$$

All of the other z_j row values *for this tableau* will be zero when they are computed using this formula.

The simplex method works by moving from one solution (extreme) point to an adjacent point until it locates the best solution.

Now the $c_j z_j$ row is computed by subtracting the z_j row values from the c_j (top) row values. For example, in the x_1 column the $c_j z_j$ row value is computed as $40 - 0 = 40$. This value as well as other $c_j z_j$ values are shown in [Table A-6](#), which is the complete initial simplex tableau with all values filled in. This tableau represents the solution at the origin, where $x_1 = 0$, $x_2 = 0$, $s_1 = 40$, and $s_2 = 120$. The profit represented by

this solution (i.e., the Z value) is given in the z_j row under the quantity column0 in [Table A-6](#). This solution is obviously not optimal because no profit is being made. Thus, we want to move to a solution point that will give a *better* solution. In other words, we want to produce either some bowls (x_1) or some mugs (x_2). One of the *nonbasic* variables (i.e., variables not in the present basic feasible solution) will enter the solution and become basic.

Table A-6. The Complete Initial Simplex Tableau

C_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	Z_j	0	0	0	0	0
	$C_j - Z_j$		40	50	0	0

The Entering Nonbasic Variable

As an example, suppose the pottery company decides to produce some bowls. With this decision x_1 will become a basic variable. For every unit of x_1 (i.e., each bowl) produced, profit will be increased by \$40 because that is the profit contribution of a bowl. However, when a bowl (x_1) is

produced, some previously unused resources will be used. For example, if

$$x_1 = 1$$

then

$$x_1 + 2x_2 + s_1 = 40 \text{ hr of labor}$$

$$1 + 2(0) + s_1 = 40$$

$$s_1 = 39 \text{ hr of labor}$$

and

$$4x_1 + 3x_2 + s_2 = 120 \text{ lb of clay}$$

$$4(1) + 3(0) + s_2 = 120$$

$$s_2 = 116 \text{ lb of clay}$$

In the labor constraint we see

that, with the production of one bowl, the amount of slack, or unused, labor is *decreased* by *1 hour*. In the clay constraint the amount of slack is *decreased* by *4 pounds*. Substituting these increases (for x_1) and decreases (for slack) into the objective function gives

c_j	z_j
$z = 40(1) + 50(0) +$	$0(1) + 0(4)$

$$Z = \$40$$

The variable with the largest positive $c_j - z_j$ value is the entering variable.

The first part of this objective function relationship represents the values in the c_j row; the second part represents the values in the z_j row. The

function expresses the fact that to produce some bowls, we must give up some of the profit already earned from the items they replace. In this case the production of bowls replaced only slack, so no profit was lost. In general, the $c_j z_j$ row values represent the net *increase per unit of entering a nonbasic variable into the basic solution*. Naturally, we want to make as much money as possible, because the objective is to maximize profit. Therefore, we enter the variable that will give the

greatest net increase in profit per unit. From [Table A-7](#), we select variable x_2 as the entering basic variable because it has the greatest net increase in profit per unit, \$50 the highest positive value in the c_j z_j row.

[Page A-10]

Table A-7. Selection of the Entering Basic Variable

			40	50	0	0
c_j	Basic Variables	Quantity				

			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0
	$c_j z_j$		40	50	0	0

The x_2 column, highlighted in [Table A-7](#), is referred to as the *pivot column*. (The operations

used to solve simultaneous equations are often referred to in mathematical terminology as *pivot operations*.)

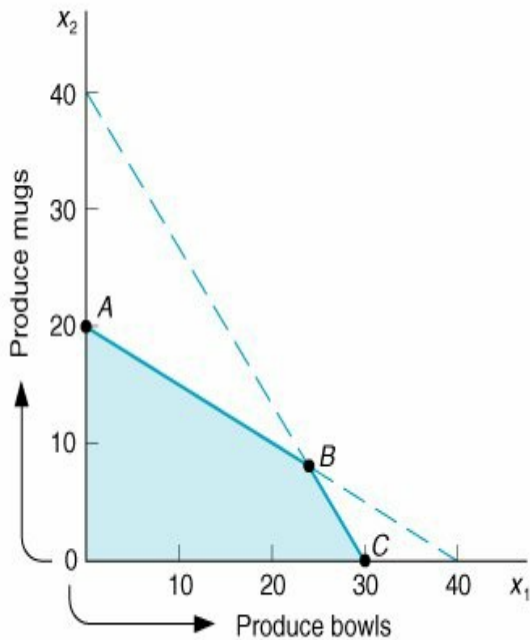
The **pivot column** is the column corresponding to the entering variable.

The selection of the entering basic variable is also demonstrated by the graph in [Figure A-2](#). At the origin

nothing is produced. In the simplex method we move from one solution point to an *adjacent* point (i.e., *one* variable in the basic feasible solution is replaced with a variable that was previously zero). In [Figure A-2](#) we can move along either the x_1 axis or the x_2 axis in order to seek a better solution. Because an increase in x_2 will result in a greater profit, we choose x_2 .

Figure A-2. Selection

**of which item to
produce the entering
basic variable**



The Leaving Basic Variable

Since each basic feasible solution contains only two variables with nonzero values, one of the two basic variables present, s_1 or s_2 , will have to leave the solution and become zero. Since we have decided to produce mugs (x_2), we want to produce as many as possible or, in other words, as many as our resources will allow. First, in the labor constraint we will use all the labor to make mugs

(because no bowls are to be produced, $x_1 = 0$; and because we will use all the labor possible and $s_1 =$ unused labor resources, $s_1 = 0$ also).

[Page A-11]

$$1x_1 + 2x_2 + s_1 = 40 \text{ hr}$$

$$1(0) + 2x_2 + 0 = 40$$

$$\begin{aligned} x_2 &= \frac{40 \text{ hr}}{2 \text{ hr/mug}} \\ &= 20 \text{ mugs} \end{aligned}$$

In other words, enough labor is available to produce 20 mugs. Next, perform the same analysis on the constraint for

clay.

$$4x_1 + 3x_2 + s_2 = 120 \text{ lb}$$

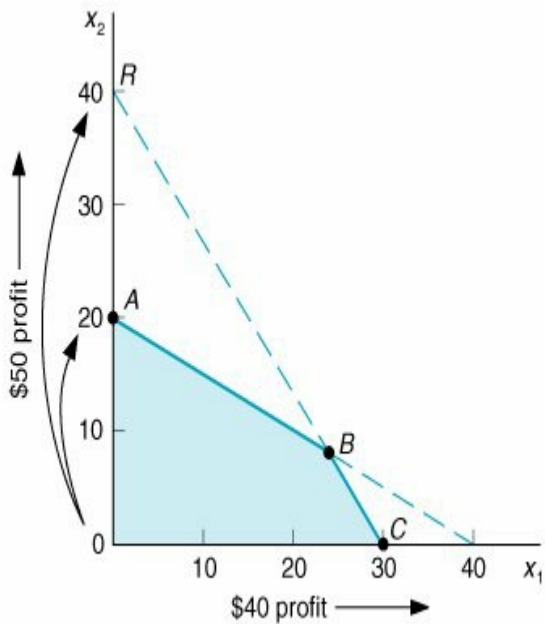
$$4(0) + 3x_2 + 0 = 120$$

$$\begin{aligned}x_2 &= \frac{120 \text{ lb}}{3 \text{ lb/mug}} \\&= 40 \text{ mugs}\end{aligned}$$

This indicates that there is enough clay to produce 40 mugs. But there is enough labor to produce only 20 mugs. We are limited to the production of only 20 mugs because we do not have enough labor to produce any more than that. This analysis is shown graphically in [Figure A-](#)

3.

**Figure A-3.
Determination of the
basic feasible
solution point**



Because we are moving out the x_2 axis, we can move from the origin to either point A or point R . We select point A because it is the *most constrained* and thus feasible, whereas point R is infeasible.

This analysis is performed in the simplex method by dividing the quantity values of the basic solution variables by the pivot column values. For this tableau,

<i>Basic Variables</i>	<i>Quantity</i>	x_2
-------------------------------	------------------------	-------------------------

s_1	40	$\div 2 = 20$, the leaving basic variable
s_2	120	$\div 3 = 40$

The leaving variable is determined by dividing the quantity values by the pivot column values and selecting the minimum possible value or zero.

The leaving basic variable is the variable that corresponds to the minimum nonnegative quotient, which in this case is 20. (Note that a value of zero would qualify as the minimum quotient and would be the choice for the leaving variable.) Therefore, s_1 is the leaving variable. (At point A in [Figure A-3](#), s_1 equals zero because all the labor is used to make the 20 mugs.) The s_1 row,

highlighted in [Table A-8](#), is also referred to as the *pivot row*.

Table A-8. Pivot Column, Pivot Row, and Pivot Number

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
0	s_1	40	1	2	1	0
0	s_2	120	4	3	0	1
	z_j	0	0	0	0	0

C_j Z_j

40

50

0

0

The **pivot row** is the row corresponding to the leaving variable.

The value of 2 at the intersection of the pivot row and the pivot column is called the *pivot number*. The pivot number, row, and column are

all instrumental in developing the next tableau. We are now ready to proceed to the second simplex tableau and a *better* solution.

*The **pivot number** is the number at the intersection of the pivot column and row.*

Developing a New

Tableau

[Table A-9](#) shows the second simplex tableau with the new basic feasible solution variables of x_2 and s_2 and their corresponding c_j values.

Table A-9. The Basic Variables and c_j Values for the Second Simplex Tableau

c_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2

50 x_2

0 s_2

z_j

$c_j z_j$

The various row values in the second tableau are computed using several simplex formulas. First, the x_2 row, called the *new tableau pivot row*, is computed by dividing every

value in the pivot row of the first (old) tableau by the pivot number. The formula for these computations is

$$\text{new tableau pivot row values} = \frac{\text{old tableau pivot row values}}{\text{pivot number}}$$

Computing the new tableau pivot row values.

The new row values are shown in [Table A-10](#).

Table A-10. Computation of the New Pivot Row Values

(This item is displayed on page A-13 in the print version)

C_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
50	x_2	20	1/2	1	1/2	0
0	s_2					
	Z_j					
	$C_j - Z_j$					

To compute all remaining row values (in this case there is only one other row), another formula is used.

Computing all remaining row values.

$$\text{new tableau row values} = \text{old tableau row values} - \left(\begin{array}{cc} \text{corresponding} & \text{corresponding} \\ \text{coefficients in} & \times \\ \text{pivot column} & \text{new tableau} \\ & \text{pivot row value} \end{array} \right)$$

Thus, this formula requires the use of both the old tableau and the new one. The s_2 row values are computed in [Table A-11](#).

Table A-11.

Computation of New s_2 Row Values

Column	Old Tableau Row Value	–	Corresponding Coefficients in Pivot Column	×	New Tableau Pivot Row Value	=	New Tableau Row Value
Quantity	120	–	(3	×	20)	=	60
x_1	4	–	(3	×	1/2)	=	5/2
x_2	3	–	(3	×	1)	=	0
s_1	0	–	(3	×	1/2)	=	–3/2
s_2	1	–	(3	×	0)	=	1

These values have been inserted in the simplex tableau in [Table A-12](#).

This solution corresponds to point A in the graph of this model in [Figure A-3](#). The

solution at this point is $x_1 = 0$, $x_2 = 20$, $s_1 = 0$, $s_2 = 60$. In other words, 20 mugs are produced and 60 pounds of clay are left unused. No bowls are produced and no labor hours remain unused.

Table A-12. The Second Simplex Tableau with Row Values

C_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
50	x_2	20	1/2	1	1/2	0

0	s_2	60	$5/2$	0	$3/2$	1
---	-------	----	-------	---	-------	---

 z_j
 $c_j z_j$

The second simplex tableau is completed by computing the z_j and $c_j z_j$ row values the same way they were computed in the first tableau. The z_j row is computed by summing the products of the c_j column and

all other column values.

[Page A-14]

Time Out: For George B. Dantzig

After developing the simplex method for solving linear programming problems, George Dantzig needed a good problem to test it on. The problem he selected was the "diet problem" formulated in 1945 by Nobel economist George Stigler. This problem was to determine an adequate nutritional diet at minimum cost (which was an important military and civilian issue during World War II). Formulated as a linear programming model, the diet problem consisted of 77 unknowns and 9 equations. It took 9 clerks using hand-operated (mechanical) desk calculators 120 man-days to obtain the optimal simplex solution: a diet consisting primarily of wheat

flour, cabbage, and dried navy beans that cost \$39.69 per year (in 1939 prices). The solution developed by Stigler using his own numerical method was only 24 cents more than the optimal solution.

Column

$$\text{Quantity} \quad z_j = \frac{(50)(20) + (0)(60)}{= 1000}$$

$$x_1 \quad z_1 = \frac{(50)(1/2) + (0)(5/2)}{= 25}$$

$$x_2 \quad z_2 = (50)(1) + (0)(0) = 50$$

$$s_1 \quad z_3 = \frac{(50)(1/2) + (0)(3/2)}{= 25}$$

$$s_2 \quad z_4 = (50)(0) + (0)(1) = 0$$

The z_j row values and the $c_j z_j$ row values are added to the tableau to give the completed second simplex tableau shown in [Table A-13](#). The value of 1,000 in the z_j row is the value

of the objective function (profit) for this basic feasible solution.

Table A-13. The Completed Second Simplex Tableau

C_j	Basic Variables	Quantity	40	50	0	0
			x_1	x_2	s_1	s_2
50	x_2	20	$1/2$	1	$1/2$	0
0	s_2	60	$5/2$	0	$3/2$	1
	Z_j	1,000	25	50	25	0

c_j z_j

15

0

25

0

The computational steps that we followed to derive the second tableau in effect accomplish the same thing as *row operations* in the solution of simultaneous equations. These same steps are used to derive each subsequent tableau, called *iterations*.

Each tableau is the same as performing row operations for a

set of simultaneous equations.

The Optimal Simplex Tableau

The steps that we followed to derive the second simplex tableau are repeated to develop the third tableau. First, the pivot column or entering basic variable is determined. Because

15 in the $c_j z_j$ row represents the greatest positive net increase in profit, x_1 becomes the entering nonbasic variable. Dividing the pivot column values into the values in the quantity column indicates that s_2 is the leaving basic variable *and* corresponds to the pivot row. The pivot row, pivot column, and pivot number are indicated in [Table A-14](#).

**Table A-14. The Pivot Row,
Pivot Column, and Pivot
Number**

(This item is displayed on page A-15 in the
print version)

			40	50	0	0	
Basic Variables			Quantity	x_1	x_2	s_1	s_2
C_j							
50	x_2	20	1/2	1	1/2	0	
0	s_2	60	5/2	0	3/2	1	
	z_j	1,000	25	50	25	0	
	$C_j - z_j$		15	0	25	0	

At this point you might be wondering why the net increase in profit per bowl (x_1) is \$15 rather than the original profit of \$40. It is because the production of bowls (x_1) will require some of the resources previously used to produce mugs (x_2) only. Producing some bowls means not producing as many mugs; thus, we are giving up some of the profit gained from producing mugs to gain even more by producing bowls. This difference is the *net increase* of \$15.

The new tableau pivot row (x_1) in the third simplex tableau is computed using the same formula used previously. Thus, all old pivot row values are divided through by $5/2$, the pivot number. These values are shown in [Table A-16](#). The values for the other row (x_2) are computed as shown in [Table A-15](#).

Table A-15.

Computation of the

x_2 Row for the Third Simplex Tableau

[\[View full size image\]](#)

Column	Old Tableau Row Value	–	Corresponding Coefficients in Pivot Column	×	New Tableau Pivot Row Value	=	New Tableau Row Value
Quantity	20	–	(1/2	×	24)	=	8
x_1	1/2	–	(1/2	×	1)	=	0
x_2	1	–	(1/2	×	0)	=	1
s_1	1/2	–	(1/2	×	–3/5)	=	4/5
s_2	0	–	(1/2	×	2/5)	=	–1/5

Table A-16 The Completed

Table A-10. The Completed Third Simplex Tableau

			40	50	0	0
C_j	Basic Variables	Quantity				
			x_1	x_2	s_1	s_2
50	x_2	8	0	1	4/5	1/5
40	x_1	24	1	0	3/5	2/5
	Z_j	1,360	40	50	16	6
	$C_j - Z_j$		0	0	16	6

These new row values, as well as the new z_j row and $c_j z_j$ row, are shown in the completed third simplex tableau in [Table A-16](#).

Observing the $c_j z_j$ row to determine the entering variable, we see that a nonbasic variable would not result in a positive net increase in profit, as all values in the $c_j z_j$ row are zero or negative. This means that the optimal solution has been reached. The solution is

$$x_1 = 24 \text{ bowls}$$

$$x_2 = 8 \text{ mugs}$$

$$Z = \$1,360 \text{ profit}$$

The solution is optimal when all $c_j - z_j$ values ≤ 0 .

which corresponds to point B in [Figure A-1](#).

An additional comment should be made regarding simplex solutions in general. Although this solution resulted in *integer* values for the variables (i.e., 24 and 8), it is possible to get a fractional solution for decision variables even though the variables reflect items that should be integers, such as airplanes, television sets, bowls, and mugs. To apply the simplex method, one must

accept this limitation.

*The simplex method
does not guarantee
integer solutions.*

Summary of the Simplex Method

The simplex method demonstrated in the previous section consists of the following steps.

1. Transform the model constraint inequalities into equations.

Set up the initial tableau for the basic feasible solution at

2. the origin and compute the z_j and $c_j - z_j$ row values.

Determine the pivot column (entering nonbasic solution variable) by selecting the

3. column with the highest positive value in the $c_j - z_j$ row.

Determine the pivot row (leaving basic solution variable) by dividing the quantity column values by

4. the pivot column values and selecting the row with the

minimum nonnegative quotient.

Compute the new pivot row values using the formula

5.

$$\text{new tableau pivot row values} = \frac{\text{old tableau pivot row values}}{\text{pivot number}}$$

Compute all other row values using the formula

6.

$$\begin{matrix} \text{new tableau} \\ \text{row values} \end{matrix} = \begin{matrix} \text{old tableau} \\ \text{row values} \end{matrix} - \left(\begin{matrix} \text{corresponding} \\ \text{coefficients in} \\ \text{pivot Column} \end{matrix} \times \begin{matrix} \text{corresponding} \\ \text{new tableau} \\ \text{pivot row values} \end{matrix} \right)$$

Compute the new z_j and $c_j z_j$ rows.

Determine whether or not the new solution is optimal by checking the $c_j z_j$ row. If all $c_j z_j$ row values are zero or negative, the solution is optimal. If a positive value exists, return to step 3 and repeat the simplex steps.

Simplex Solution of a Minimization Problem

In the previous section the simplex method for solving linear programming problems was demonstrated for a maximization problem. In general, the steps of the simplex method outlined at the end of this section are used for any type of linear programming problem. However, a minimization problem requires a few changes in the normal

simplex process, which we will discuss in this section.

In addition, several exceptions to the typical linear programming problem will be presented later in this module. These include problems with mixed constraints ($=$, $\leq$, and $\geq$); problems with more than one optimal solution, no feasible solution, or an unbounded solution; problems with a tie for the pivot column; problems with a tie for the pivot row; and problems with constraints with negative quantity values. None of these kinds of

problems require changes in the simplex method. They are basically unusual results in individual simplex tableaus that the reader should know how to interpret and work with.

[Page A-17]

Standard Form of a Minimization Model

Consider the following linear programming model for a farmer purchasing fertilizer.

minimize $Z = \$6x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 16$ lb of nitrogen

$4x_1 + 3x_2 \geq 24$ lb of phosphate

where

x_1 = bags of Super-gro
fertilizer

x_2 = bags of Crop-quick
fertilizer

Z = farmer's total cost (\$) of
purchasing fertilizer

This model is transformed into
standard form by subtracting

surplus variables from the two $\geq$ constraints as follows.

$$\text{minimize } Z = 6x_1 + 3x_2 + 0s_1 + 0s_2$$

subject to

$$2x_1 + 4x_2 - s_1 = 16$$

$$4x_1 + 3x_2 - s_2 = 24$$

$$x_1, x_2, s_1, s_2 \geq 0$$

The surplus variables represent the extra amount of nitrogen and phosphate that exceeded the minimum requirements specified in the constraints.

However, the simplex method requires that the initial basic feasible solution be at the

origin, where $x_1 = 0$ and $x_2 = 0$. Testing these solution values, we have

$$2x_1 + 4x_2 + s_1 = 16$$

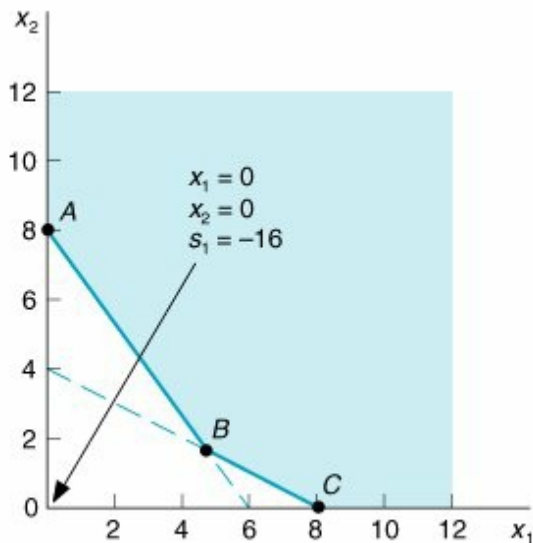
$$2(0) + 4(0) + s_1 = 16$$

$$s_1 = 16$$

The idea of "negative excess pounds of nitrogen" is illogical and violates the

nonnegativity restriction of linear programming. The reason the surplus variable does not work is shown in [Figure A-4](#). The solution at the origin is outside the feasible solution space.

Figure A-4. Graph of the fertilizer example



Transforming a model

into standard form by subtracting surplus variables will not work in the simplex method.

[Page A-18]

To alleviate this difficulty and get a solution at the origin, we add an *artificial variable* (A_1) to the constraint equation,

$$2 x_1 + 4 x_2 s_1 + A_1 = 16$$

*An **artificial variable** allows for an initial basic feasible solution at the origin, but it has no real meaning.*

The artificial variable, A_1 , does not have a meaning as a slack variable or a surplus variable does. It is inserted into the equation simply to give a positive solution at the origin; we are artificially creating a

solution.

$$2x_1 + 4x_2 + s_1 + A_1 = 16$$

$$2(0) + 4(0) + 0 + A_1 = 16$$

$$A_1 = 16$$

The artificial variable is somewhat analogous to a booster rocket its purpose is to get us off the ground; but once we get started, it has no real

use and thus is discarded. The artificial solution helps get the simplex process started, but we do not want it to end up in the optimal solution, because it has no real meaning.

When a surplus variable is subtracted and an artificial variable is added, the phosphate constraint becomes

$$4x_1 + 3x_2 - s_2 + A_2 = 24$$

The effect of surplus and artificial variables on the objective function must now be considered. Like a slack

variable, a surplus variable has no effect on the objective function in terms of increasing or decreasing cost. For example, a surplus of 24 pounds of nitrogen does not contribute to the cost of the objective function, because the cost is determined solely by the number of bags of fertilizer purchased (i.e., the values of x_1 and x_2). Thus, a coefficient of 0 is assigned to each surplus variable in the objective function.

By assigning a "cost" of \$0 to

each surplus variable, we are not prohibiting it from being in the final optimal solution. It would be quite realistic to have a final solution that showed some surplus nitrogen or phosphate. Likewise, assigning a cost of \$0 to an artificial variable in the objective function would not prohibit it from being in the final optimal solution. However, if the artificial variable appeared in the solution, it would render the final solution meaningless. Therefore, we must ensure that an artificial variable is *not* in

the final solution.

As previously noted, the presence of a particular variable in the final solution is based on its relative profit or cost. For example, if a bag of Super-gro costs \$600 instead of \$6 and Crop-quick stayed at \$3, it is doubtful that the farmer would purchase Super-gro (i.e., x_1 would not be in the solution). Thus, we can prohibit a variable from being in the final solution by assigning it a very *large cost*. Rather than assigning a dollar cost to an

artificial variable, we will assign a value of M , which represents a large positive cost (say, \$1,000,000). This operation produces the following objective function for our example:

$$\text{minimize } Z = 6x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$$

The completely transformed minimization model can now be summarized as

minimize $Z = 6x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to

$$2x_1 + 4x_1 - s_1 + A_1 = 16$$

$$4x_1 + 3x_2 - s_2 + A_2 = 24$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

Artificial variables are assigned a large cost in the objective function to eliminate them from the final solution.

The Simplex Tableau for a Minimization Problem

The initial simplex tableau for a minimization model is developed the same way as one for a maximization model, except for one small difference. Rather than computing $c_j z_j$ in the bottom row of the tableau, we compute $z_j c_j$, which represents the *net per unit decrease in cost*, and the largest positive value is selected as the entering variable and pivot column. (An

alternative would be to leave the bottom row as $c_j z_j$ and select the largest negative value as the pivot column. However, to maintain a consistent rule for selecting the pivot column, we will use $z_j c_j$.)

The $c_j z_j$ row is changed to $z_j c_j$ in the simplex tableau for a minimization problem.

The initial simplex tableau for this model is shown in [Table A-17](#). Notice that A_1 and A_2 form the initial solution at the origin, because that was the reason for inserting them in the first place to get a solution at the origin. This is not a basic feasible solution, since the origin is not in the feasible solution area, as shown in [Figure A-4](#). As indicated previously, it is an artificially created solution. However, the simplex process will move toward feasibility in subsequent tableaus. Note that across the

top the decision variables are listed first, then surplus variables, and finally artificial variables.

Table A-17. The Initial Simplex Tableau

		6	3	0	0	M	M	
c_j	Basic Variables	Quantity						
			x_1	x_2	s_1	s_2	A_1	A_2
M	A_1	16	2	4	1	0	1	0
M	A_2	24	4	3	0	1	0	1
z_j								

40M 6M 7M M M M M

z_j C_j 6M 7M M M 0 0
 6 3

Artificial variables are always included as part of the initial basic feasible solution when they exist.

In [Table A-17](#) the x_2 column was selected as the pivot column because 7M 3 is the largest positive value in the z_j c_j row. A_1 was selected as the leaving basic variable (and pivot row) because the quotient of 4 for this row was the minimum positive row value.

Once an artificial variable is selected as the leaving variable, it will never reenter the tableau, so it can be eliminated.

The second simplex tableau is developed using the simplex formulas presented earlier. It is shown in [Table A-18](#). Notice that the A_1 column has been eliminated in the second simplex tableau. Once an artificial variable leaves the basic feasible solution, it will never return because of its high cost, M . Thus, like the booster rocket, it can be eliminated from the tableau. However, artificial variables are the *only* variables that can be

treated this way.

Table A-18. The Second Simplex Tableau

		6	3	0	0	M
Basic Variables	Quantity	x_1	x_2	s_1	s_2	A
c_j						
3 x_2	4	1/2	1	1/4	0	0
$M A_2$	12	5/2	0	3/4	1	1
z_j	$12M + 12$	$5M/2 + 3/2$	3	$3/4 + M$	$3/4$	M

$z_j \ C_j$ $5M/29/2$

0

 $3/4 \ M \ 0$

+

 $3/M4$

The third simplex tableau, with x_1 replacing A_2 , is shown in [Table A-19](#). Both the A_1 and A_2 columns have been eliminated because both variables have left the solution. The x_1 row is selected as the pivot row because it corresponds to the minimum positive ratio of 16. In selecting the pivot row, the

4 value for the x_2 row was not considered because the minimum *positive* value *or zero* is selected. Selecting the x_2 row would result in a negative quantity value for s_1 in the fourth tableau, which is not feasible.

[Page A-20]

Table A-19. The Third Simplex Tableau

		6	3	0	0	
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2

3	x_2	$8/5$	0	1	$2/5$	$1/5$
6	x_1	$24/5$	1	0	$3/10$	$2/5$
	z_j	$168/5$	6	3	$3/5$	$9/5$
	$z_j - c_j$		0	0	$3/5$	$9/5$

The fourth simplex tableau, with s_1 replacing x_1 , is shown in [Table A-20](#). [Table A-20](#) is the optimal simplex tableau

because the $z_j \ c_j$ row contains no positive values. The optimal solution is

$x_1 = 0$ bags of Super-gro

$s_1 = 16$ extra lb of nitrogen

$x_2 = 8$ bags of Crop-quick

$s_2 = 0$ extra lb of phosphate

$Z = \$24$, total cost of purchasing fertilizer

Table A-20. Optimal Simplex Tableau

	Basic Variables	Quantity	6	3	0	0
--	--------------------	----------	---	---	---	---

C_j			x_1	x_2	s_1	s_2
3	x_2	8	4/3	1	0	1/3
0	s_1	16	10/3	0	1	4/3
	Z_j	24	4	3	0	1
	$Z_j - C_j$		2	0	0	1

Simplex Adjustments for

a Minimization Problem

To summarize, the adjustments necessary to apply the simplex method to a minimization problem are as follows:

- 1.** Transform all $\geq$ constraints to equations by subtracting a surplus variable and adding an artificial variable.
- 2.** Assign a c_j value of M to each artificial variable in the objective function.
- 3.** Change the c_j z_j row to z_j

C_j .

Although the fertilizer example model we just used included only $\geq$ constraints, it is possible for a minimization problem to have $\leq$ and $=$ constraints in addition to $\geq$ constraints.

Similarly, it is possible for a maximization problem to have $\geq$ and $=$ constraints in addition to $\leq$ constraints. Problems that contain a combination of different types of inequality constraints are referred to as *mixed constraint problems*.

A Mixed Constraint Problem

So far we have discussed maximization problems with all $\leq$ constraints and minimization problems with all $\geq$ constraints. However, we have yet to solve a problem with a mixture of $\leq$, $\geq$, and $=$ constraints.

Furthermore, we have not yet looked at a maximization problem with a $\geq$ constraint. The following is a maximization problem with $\leq$, $\geq$, and $=$

constraints.

A mixed constraint problem includes a combination of $\leq$, $=$, and $\geq$ constraints.

A leather shop makes custom-designed, hand-tooled briefcases and luggage. The shop makes a \$400 profit from each briefcase and a \$200 profit from each piece of luggage. (The profit for

briefcases is higher because briefcases require more hand tooling.) The shop has a contract to provide a store with exactly 30 items per month. A tannery supplies the shop with at least 80 square yards of leather per month. The shop must use at least this amount but can order more. Each briefcase requires 2 square yards of leather; each piece of luggage requires 8 square yards of leather. From past performance, the shop owners know they cannot make more than 20 briefcases per month.

They want to know the number of briefcases and pieces of luggage to produce in order to maximize profit.

This problem is formulated as

$$\text{maximize } Z = \$400x_1 + 200x_2$$

subject to

$$x_1 + x_2 = 30 \text{ contracted items}$$

$$2x_1 + 8x_2 \geq 80 \text{ yd}^2 \text{ of leather}$$

$$x_1 \leq 20 \text{ briefcases}$$

$$x_1, x_2 \geq 0$$

where x_1 = briefcases and x_2 = pieces of luggage.

The first step in the simplex

method is to transform the inequalities into equations. The first constraint for the contracted items is already an equation; therefore, it is not necessary to add a slack variable. There can be no slack in the contract with the store because exactly 30 items must be delivered. Even though this equation already appears to be in the necessary form for simplex solution, let us test it at the origin to see if it meets the starting requirements.

$$x_1 + x_2 = 30$$

$$0 + 0 = 30$$

$$0 \neq 30$$

Because zero does not equal 30, the constraint is not feasible in this form. Recall that a $\geq$ constraint did not work at the origin either in an earlier problem. Therefore, an artificial variable was added. The same thing can be done here.

$$x_1 + x_2 + A_1 = 30$$

Now at the origin, where $x_1 = 0$
and $x_2 = 0$,

$$0 + 0 + A_1 = 30$$

$$A_1 = 30$$

*An artificial variable
is added to an
equality (=)*

*constraint for
standard form.*

Any time a constraint is initially an equation, an artificial variable is added. However, the artificial variable cannot be assigned a value of M in the objective function of a maximization problem. Because the objective is to maximize profit, a positive M value would represent a large positive *profit* that would definitely end up in

the final solution. Because an artificial variable has no real meaning and is inserted into the model merely to create an initial solution at the origin, its existence in the final solution would render the solution meaningless. To prevent this from happening, we must give the artificial variable a large *cost* contribution, or M .

[Page A-22]

The constraint for leather is a $\geq$ inequality. It is converted to equation form by subtracting a

surplus variable and adding an artificial variable:

$$2x_1 + 8x_2 - s_1 + A_2 = 80$$

An artificial variable in a maximization problem is given a large cost contribution to drive it out of the problem.

As in the equality constraint, the artificial variable in this

constraint must be assigned an objective function coefficient of M .

The final constraint is a $\leq$ inequality and is transformed by adding a slack variable:

$$x_1 + s_2 = 20$$

The completely transformed linear programming problem is as follows:

$$\text{maximize } Z = 400x_1 + 200x_2 + 0s_1 + 0s_2 - MA_1 - MA_2$$

subject to

$$x_1 + x_2 + A_1 = 30$$

$$2x_1 + 8x_2 - s_1 + A_2 = 80$$

$$x_1 + s_2 = 20$$

$$x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$$

The initial simplex tableau for this model is shown in [Table A-21](#). Notice that the basic solution variables are a mix of artificial and slack variables. Note also that the third-row quotient for determining the pivot row ($20 \div 0$) is an undefined value, or ∞ . Therefore, this row would never be considered as a

candidate for the pivot row. The second, third, and optimal tableaus for this problem are shown in [Tables A-22](#), [A-23](#), and [A-24](#).

Table A-21. The Initial Simplex Tableau

		400	200	0	0	M	M	
	Basic Variables	Quantity						
c_j			x_1	x_2	s_1	s_2	A_1	A_2
M	A_1	30	1	1	0	0	1	0
M	A_2	80	2	8	1	0	0	1

0	s_2	20	1	0	0	1	0	0
z_j		$110M$	$3M$	$9M$	M	0	M	M
c_j	z_j		400	200				
			$+$	$+$	M	0	0	0
			$3M$	$9M$				

Table A-22. The Second Simplex Tableau

		400	200	0	0	M
	Basic Variables	Quantity				
c_j		x_1	x_2	s_1	s_2	A_1

M	A_1	20	$3/4$	0	$1/8$	0	1
$200x_2$		10	$1/4$	1	$1/8$	0	0
0	s_2	20	1	0	0	1	0
z_j		$2,000$	50	200	25	0	M
		$20M$	$3M/4$	$M/8$			
c_j	z_j		350		25		
			$+$	0	$+$	0	0
			$3M/4$		$M/8$		

Table A-23. The Third Simple Tableau

		400	200	0	0	
Basic Variables		Quantity				
C_j		x_1	x_2	s_1	s_2	
M	A_1	5	0	0	$1/8$	$3/4$
	$200x_2$	5	0	1	$1/8$	$1/4$
	$400x_1$	20	1	0	0	1
	z_j	9,000	400	200	25	350
		$5M$			$M/8$	$+3M/4$

c_j z_j

0

0

 25
 $+3M/4$
 $M/8$

350

Table A-24. The Optimal Simplex Tableau

		400		500		0		0	
		Basic Variables		Quantity					
c_j				x_1	x_2	s_1	s_2		
0	s_1	40		0	0	1	6		
200	x_2	10		0	1	0	1		

$400 x_1$	20	1	0	0	1
z_j	10,000	400	200	0	200
$c_j z_j$		0	0	0	200

The solution for the leather shop problem is (see [Table A-24](#)):

$x_1 = 20$ briefcases

$x_2 = 10$ pieces of luggage

$s_1 = 40$ extra yd² of leather

$Z = \$10,000$ profit per month

It is now possible to summarize a set of rules for transforming all three types of model constraints.

**Objective Function
Coefficient**

Constraint Adjustment MAXIMIZATION MIN

$\leq$

Add a slack
variable

0

=

Add an
artificial
variable

M

$\geq$

Subtract a
surplus
variable

0

and add an
artificial
variable

M

Irregular Types of Linear Programming Problems

The basic simplex solution of typical maximization and minimization problems has been shown in this module. However, there are several special types of atypical linear programming problems. Although these special cases do not occur frequently, they will be described within the simplex framework so that you can recognize them when they

arise.

For irregular problems the general simplex procedure does not always apply.

[Page A-24]

These special types include problems with more than one optimal solution, infeasible problems, problems with

unbounded solutions, problems with ties for the pivot column or ties for the pivot row, and problems with constraints with negative quantity values.

Multiple Optimal Solutions

Consider the Beaver Creek Pottery Company example with the objective function changed as follows.

$$Z = 40x_1 + 50x_2$$

to

$$Z = 40x_1 + 30x_2$$

$$\text{maximize } Z = 40x_1 + 30x_2$$

subject to

$$x_1 + 2x_2 \leq 40$$

$$4x_1 + 3x_2 \leq 120$$

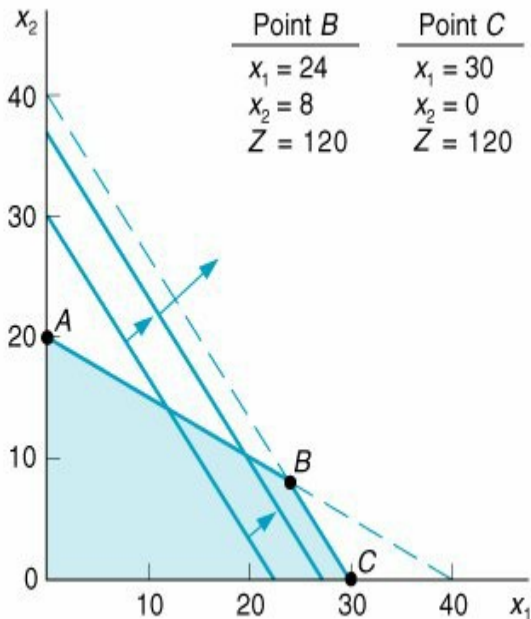
$$x_1, x_2 \geq 0$$

The graph of this model is shown in [Figure A-5](#). The slight change in the objective function makes it now *parallel* to the constraint line, $4x_1 + 3x_2 = 120$. Therefore, as the objective function edge moves outward from the origin, it

touches the whole line segment BC rather than a single extreme corner point before it leaves the feasible solution area. The endpoints of this line segment, B and C , are typically referred to as the *alternate optimal solutions*. It is understood that these points represent the endpoints of a range of optimal solutions.

Figure A-5. Graph of the Beaver Creek Pottery Company

example with multiple optimal solutions



Alternate optimal solutions have the same Z value but different variable values.

The optimal simplex tableau for this problem is shown in [Table A-25](#). This corresponds to point C in [Figure A-5](#).

Table A-25. The Optimal Simplex Tableau

(This item is displayed on page A-25 in the

				40	50	0	0
		Basic Variables	Quantity				
C_j				x_1	x_2	s_1	s_2
0	s_1		10	0	5/4	1	1/4
40	x_1		30	1	3/4	0	1/4
	Z_j		1,200	40	30	0	10
	$C_j - Z_j$			0	0	0	10

For a multiple optimal solution the $c_j z_j$ (or $z_j c_j$) value for a nonbasic variable in the final tableau equals zero.

The fact that this problem contains multiple optimal solutions can be determined from the $c_j z_j$ row. Recall that the $c_j z_j$ row values are the net increases in profit per unit for

the variable in each column. Thus, $c_j z_j$ values of zero indicate no net increase in profit *and* no net loss in profit. We would expect the basic variables, s_1 and x_1 , to have zero $c_j z_j$ values because they are part of the basic feasible solution; they are already in the solution so they cannot be entered again. However, the x_2 column has a $c_j z_j$ value of zero and it is not part of the basic feasible solution. This means that if some mugs (x_2) were produced, we would have a new

product mix but the same total profit. Thus, a multiple optimal solution is indicated by a $c_j z_j$ (or $z_j c_j$) row value of zero for a nonbasic variable.

[Page A-25]

To determine the alternate endpoint solution, let x_2 be the entering variable (pivot column) and select the pivot row as usual. This selection results in the s_1 row being the pivot row. The alternate solution corresponding to point

B in [Figure A-5](#) is shown in [Table A-26](#).

Table A-26. The Alternative Optimal Tableau

		Basic Variables		Quantity	40	50	0	0
C_j					x_1	x_2	s_1	s_2
50	x_1			8	0	1	4/5	1/5
40	x_1			24	1	0	3/5	2/5
	Z_j			1,200	40	30	0	10
	$C_j - Z_j$				0	0	0	10

An alternate optimal solution is determined by selecting the nonbasic variable with $c_j - z_j = 0$ as the entering variable.

An Infeasible Problem

Another linear programming irregularity is the case where a problem has no feasible solution area; thus, there is no basic feasible solution to the problem.

An infeasible problem does not have a feasible solution space.

An example of an infeasible problem is formulated next and depicted graphically in [Figure A-6](#).

$$\text{maximize } Z = 5x_1 + 3x_2$$

subject to

$$4x_1 + 2x_2 \leq 8$$

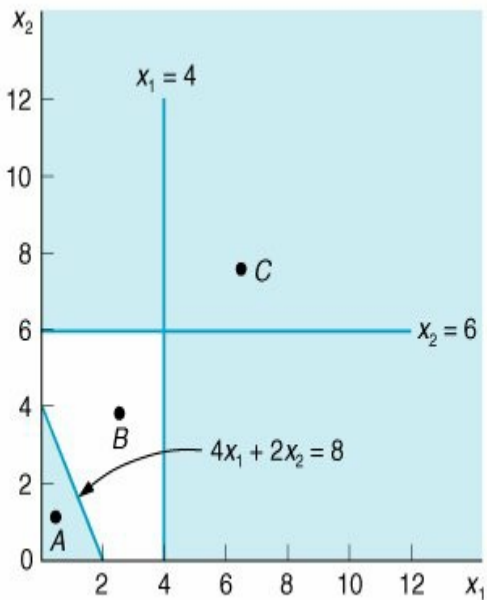
$$x_1 \geq 4$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

Figure A-6. Graph of an infeasible problem
(This item is displayed on page A-

26 in the print version)



The three constraints do not overlap to form a feasible solution area. Because no point satisfies all three constraints simultaneously, there is no solution to the problem. The final simplex tableau for this problem is shown in [Table A-27](#).

Table A-27. The Final Simplex Tableau for an Infeasible Problem

(This item is displayed on page A-26 in the print version)

	5	3	0	0	0	<i>M</i>	<i>M</i>
Basic Variables	x_1	x_2	s_1	s_2	s_3	A_1	A_2
C_j							
Quantity							

$3x_2$	4	2	1	$1/2$	0	0	0	0
MA_1	4	1	0	0	1	0	1	0
MA_2	2	2	0	$1/2$	0	1	0	1
z_j	$126M$	$6 + M$	$3 + M$	$3/2 + M/2$	M	M	M	M
$c_j z_j$		$1M$	0	$3/2 + M/2$	M	M	0	0

An infeasible problem

has an artificial variable in the final simplex tableau.

The tableau in [Table A-27](#) has all zero or negative values in the $c_j z_j$ row, indicating that it is optimal. However, the solution is $x_2 = 4$, $A_1 = 4$, and $A_2 = 2$. Because the existence of artificial variables in the final solution makes the solution meaningless, this is not a real solution. In general, any time

the $c_j z_j$ (or $z_j c_j$) row indicates that the solution is optimal but there are artificial variables in the solution, the solution is infeasible. Infeasible problems do not typically occur, but when they do they are usually a result of errors in defining the problem or in formulating the linear programming model.

[Page A-26]

An Unbounded Problem

In some problems the feasible

solution area formed by the model constraints is not closed. In these cases it is possible for the objective function to increase indefinitely without ever reaching a maximum value because it never reaches the boundary of the feasible solution area.

In an unbounded problem the objective function can increase indefinitely because the solution space is not closed.

An example of this type of problem is formulated next and shown graphically in [Figure A-7](#).

$$\text{maximize } Z = 4x_1 + 2x_2$$

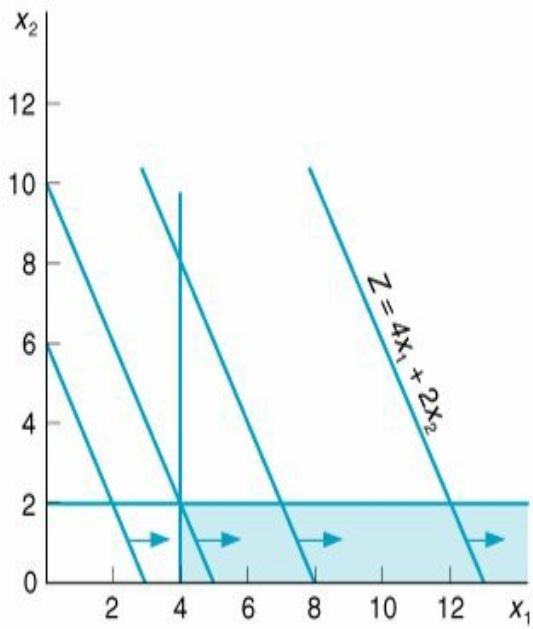
subject to

$$x_1 \geq 4$$

$$x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Figure A-7. An unbounded problem



In [Figure A-7](#) the objective function is shown to increase without bound; thus, a solution is never reached.

A pivot row cannot be selected for an unbounded problem.

The second tableau for this problem is shown in [Table A-28](#). In this simplex tableau, s_1 is chosen as the entering nonbasic variable and pivot

column. However, there is no pivot row or leaving basic variable. One row value is 4 and the other is undefined. This indicates that a "most constrained" point does not exist and that the solution is unbounded. In general, a solution is unbounded if the row value ratios are all negative or undefined.

Table A-28. The Second Simplex Tableau

[\[View full size image\]](#)

			4	2	0	0	
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2	
4	x_1	4	0	0	-1	0	$4 \div -1 = -4$
0	s_2	2	1	1	0	1	$2 \div 0 = \infty$
	z_j	16	4	0	-4	0	
	$c_j - z_j$		0	2	4	0	

Unlimited profits are not possible in the real world; an unbounded solution, like an infeasible solution, typically reflects an error in defining the problem or in formulating the model.

Tie for the Pivot Column

Sometimes when selecting the pivot column, you may notice that the greatest positive $c_j z_j$ (or $z_j c_j$) row values are the same; thus, there is a tie for the pivot column. When this happens, one of the two tied columns should be selected arbitrarily. Even though one choice may require fewer subsequent iterations than the other, there is no way of knowing this beforehand.

A tie for the pivot column is broken arbitrarily.

Tie for the Pivot Row Degeneracy

It is also possible to have a tie for the pivot row (i.e., two rows may have identical lowest nonnegative values). Like a tie for a pivot column, a tie for a

pivot row should be broken arbitrarily. However, after the tie is broken, the basic variable that was the *other* choice for the leaving basic variable will have a quantity value of zero in the next tableau. This condition is commonly referred to as *degeneracy* because theoretically it is possible for subsequent simplex tableau solutions to degenerate so that the objective function value never improves and optimality never results. This occurs infrequently, however.

*A tie for the pivot row is broken arbitrarily and can lead to **degeneracy**.*

In general, tableaus with ties for the pivot row should be treated normally. If the simplex steps are carried out as usual, the solution will evolve normally.

The following is an example of a problem containing a tie for the pivot row.

$$\text{maximize } Z = 4x_1 + 6x_2$$

subject to

$$6x_1 + 4x_2 \leq 24$$

$$x_2 \leq 3$$

$$5x_1 + 10x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

For the sake of brevity we will skip the initial simplex tableau for this problem and go directly to the second simplex tableau in [Table A-29](#), which shows a tie for the pivot row between

the s_1 and s_3 rows.

Table A-29. The Second Simplex Tableau with a Tie for the Pivot Row

			4	2	0	0	
c_j	Basic Variables	Quantity	x_1	x_2	s_1	s_2	
4	x_1	4	0	0	-1	0	$4 \div -1 = -4$
0	s_2	2	1	1	0	1	$2 \div 0 = \infty$
	z_j	16	4	0	-4	0	
	$c_j - z_j$		0	2	4	0	

The s_3 row is selected arbitrarily as the pivot row, resulting in the third simplex tableau, shown in [Table A-30](#).

Table A-30. The Third Simplex Tableau with Degeneracy

		4	6	0	0	0
Basic Variables		Quantity				
C_j		x_1	x_2	s_1	s_2	s_3
0	s_1	0	0	1	8	6/5

6	x_2	3	0	1	0	1	0
4	x_1	2	1	0	0	2	1/5
	z_j	26	4	6	0	2	4/5
	$c_j - z_j$		0	0	0	2	4/5

Note that in [Table A-30](#) a quantity value of zero now appears in the s_1 row, representing the degenerate condition resulting from the tie for the pivot row. However, the

simplex process should be continued as usual: s_2 should be selected as the entering basic variable and the s_1 row should be selected as the pivot row. (Recall that the pivot row value of zero is the minimum nonnegative quotient.) The final optimal tableau is shown in [Table A-31](#).

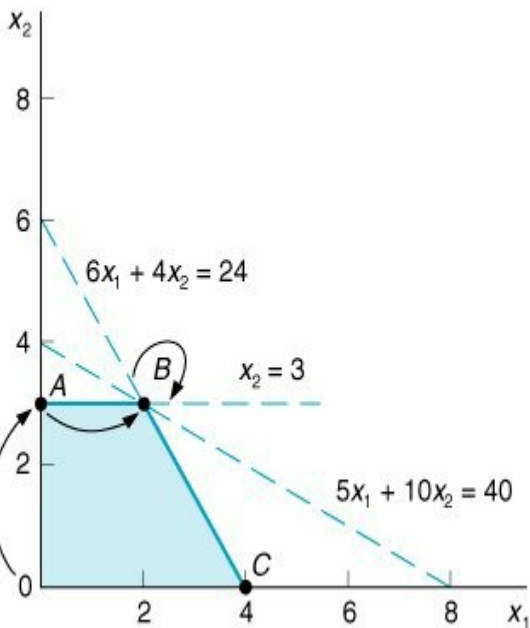
[Page A-29]

Table A-31. The Optimal Simplex Tableau for a Degenerate Problem

Basic Variables		Quantity	4	6	0	0	0
C_j			x_1	x_2	s_1	s_2	s_3
0	s_2	0	0	0	1/8	1	3/20
6	x_2	3	0	1	1/8	0	3/20
4	x_1	2	1	0	1/4	0	1/10
	Z_j	26	4	6	1/4	0	1/2
	$C_j - Z_j$		0	0	1/4	0	1/2

Notice that the optimal solution did not change from the third to the optimal simplex tableau. The graphical analysis of this problem shown in [Figure A-8](#) reveals the reason for this.

Figure A-8. Graph of a degenerate solution



Notice that in the third tableau ([Table A-30](#)) the simplex process went to point B , where all three constraint lines intersect. This is, in fact, what caused the tie for the pivot row and the degeneracy. Subsequently, the simplex process stayed at point B in the optimal tableau ([Table A-31](#)). The two tableaus represent two different basic feasible solutions corresponding to two different sets of model constraint equations.

Degeneracy occurs in

*a simplex problem
when a problem
continually loops back
to the same solution
or tableau.*

Negative Quantity Values

Occasionally a model constraint is formulated with a negative quantity value on the right side of the inequality sign for example,

$$6x_1 + 2x_2 \geq 30$$

*Standard form for
simplex solution
requires positive
right-hand-side
values.*

This is an improper condition for the simplex method, because for the method to work, all quantity values must be positive or zero.

This difficulty can be alleviated by multiplying the inequality by 1, which also changes the direction of the inequality.

$$\begin{aligned} (-1) (-6x_1 + 2x_2 \geq -30) \\ 6x_1 - 2x_2 \leq 30 \end{aligned}$$

A negative right-hand-side value is changed to a positive by multiplying the constraint by -1, which changes the inequality sign.

Now the model constraint is in proper form to be transformed into an equation and solved by the simplex method.

Summary of Simplex Irregularities

Multiple optimal solutions are identified by $c_j z_j$ (or $z_j c_j$) = 0 for a nonbasic variable. To determine the alternate

solution(s), enter the nonbasic variable(s) with a $c_j z_j$ value equal to zero.

An infeasible problem is identified in the simplex procedure when an optimal solution is achieved (i.e., when all $c_j z_j \leq 0$) and one or more of the basic variables are artificial.

An unbounded problem is identified in the simplex procedure when it is not possible to select a pivot row that is, when the values

obtained by dividing the quantity values by the corresponding pivot column values are negative or undefined.

The Dual

Every linear programming model has two forms: the *primal* and the *dual*. The original form of a linear programming model is called the primal. All the examples in this module are primal models. The dual is an alternative model form derived completely from the primal. The dual is useful because it provides the decision maker with an alternative way of looking at a

problem. Whereas the primal gives solution results in terms of the amount of profit gained from producing products, the dual provides information on the *value* of the constrained resources in achieving that profit.

*The original linear programming model is called the **primal**, and the alternative form is the **dual**.*

The following example will demonstrate how the dual form of a model is derived and what it means. The Hickory Furniture Company produces tables and chairs on a daily basis. Each table produced results in \$160 in profit; each chair results in \$200 in profit. The production of tables and chairs is dependent on the availability of limited resources labor, wood, and storage space. The resource requirements for the production of tables and chairs and the total resources available are as follows.

The dual solution variables provide the value of the resources, that is, shadow prices.

Resource Requirements			Total Available per Day
Resource	TABLE	CHAIR	
Labor (hr)	2	4	40

Wood (bd ft)	18	18	216
Storage (ft ²)	24	12	240

The company wants to know the number of tables and chairs to produce per day to maximize profit. The model for this problem is formulated as follows.

maximize $Z = \$160x_1 + 200x_2$

subject to

$$2x_1 + 4x_2 \leq 40 \text{ hr of labor}$$

$$18x_1 + 18x_2 \leq 216 \text{ bd ft of wood}$$

$$24x_1 + 12x_2 \leq 240 \text{ ft}^2 \text{ of storage space}$$

$$x_1, x_2 \geq 0$$

[Page A-31]

where

x_1 = number of tables produced

x_2 = number of chairs produced

This model represents the primal form. For a primal

maximization model, the dual form is a minimization model. The dual form of this example model is

$$\text{minimize } Z = 40y_1 + 216y_2 + 240y_3$$

subject to

$$2y_1 + 18y_2 + 24y_3 \geq 160$$

$$4y_1 + 18y_2 + 12y_3 \geq 200$$

$$y_1, y_2, y_3 \geq 0$$

The dual is formulated entirely from the primal.

The specific relationships between the primal and the dual demonstrated in this example are as follows.

- 1.** The dual variables, y_1 , y_2 , and y_3 , correspond to the model constraints in the primal. For every constraint in the primal there will be a variable in the dual. For example, in this case the primal has *three constraints*; therefore, the dual has *three decision*

variables.

- 2.** The quantity values on the right-hand side of the primal inequality constraints are the objective function coefficients in the dual. The constraint quantity values in the primal, 40, 216, and 240, form the dual objective function: $Z = 40y_1 + 216y_2 + 240y_3$.
- 3.** The model constraint coefficients in the primal are the decision variable coefficients in the dual. For

example, the labor constraint in the primal has the coefficients 2 and 4. These values are the y_1 variable coefficients in the model constraints of the dual: $2y_1$ and $4y_1$.

- 4.** The objective function coefficients in the primal, 160 and 200, represent the model constraint requirements (quantity values on the right-hand side of the constraint) in the dual.

5. Whereas the maximization primal model has $\leq$ constraints, the minimization dual model has $\geq$ constraints.

The primaldual relationships can be observed by comparing the two model forms shown in [Figure A-9](#).

Figure A-9. The primaldual relationships

(This item is displayed on page A-32 in the print version)

[\[View full size image\]](#)

Primal

$$\text{maximize } Z_p = 160 x_1 + 200 x_2$$

subject to:

$$\begin{array}{rcl} 2 x_1 + 4 x_2 & \leq & 40 \\ 18 x_1 + 18 x_2 & \leq & 216 \\ 24 x_1 + 12 x_2 & \leq & 240 \end{array}$$

$$x_1, x_2 \geq 0$$

Dual

$$\text{minimize } Z_d = 40 y_1 + 216 y_2 + 240 y_3$$

subject to:

$$\begin{array}{rcl} 2 y_1 + 18 y_2 + 24 y_3 & \geq & 160 \\ 4 y_1 + 18 y_2 + 12 y_3 & \geq & 200 \end{array}$$

$$y_1, y_2, y_3 \geq 0$$

Now that we have developed the dual form of the model, the next step is determining what

the dual means. In other words, what do the decision variables y_1 , y_2 , and y_3 mean, what do the $\geq$ model constraints mean, and what is being minimized in the dual objective function?

A primal maximization model with $\leq$ constraints converts to a dual minimization model with $\geq$ constraints, and vice versa.

Interpreting the Dual Model

The dual model can be interpreted by observing the simplex solution to the primal form of the model. The simplex solution to the primal model is shown in [Table A-32](#).

Table A-32. The Optimal Simplex Solution for the Primal Model

(This item is displayed on page A-32 in the print version)

		160 200 0 0				
Basic Variables		Quantity				
c_j			x_1	x_2	s_1	s_2
200	x_2	8	0	$1\frac{1}{2}$	$\frac{1}{18}$	
160	x_1	4	1	$0\frac{1}{2}$	$\frac{1}{9}$	
0	s_3	48	0	0	6	2
	z_j	2,240	160	200	20	$\frac{20}{3}$
	$c_j - z_j$		0	0	20	$\frac{20}{3}$

Interpreting this primal solution, we have

$$x_1 = 4 \text{ tables}$$

$$x_2 = 8 \text{ chairs}$$

$$s_3 = 48 \text{ ft}^2 \text{ of storage space}$$

$$Z = \$2,240 \text{ profit}$$

This optimal primal tableau also contains information about the dual. In the $c_j z_j$ row of [Table A-32](#), the negative values of 20 and $20/3$ under the s_1 and s_2 columns indicate that if one unit of either s_1 or s_2 were entered into the solution, profit would *decrease* by \$20 or \$6.67 (i.e., $20/3$), respectively.

Recall that s_1 represents unused labor and s_2 represents unused wood. In the present solution s_1 and s_2 are not basic

variables, so they both equal zero. This means that all of the material and labor are being used to make tables and chairs, and there are no excess (slack) labor hours or board feet of material left over. Thus, if we enter s_1 or s_2 into the solution, then s_1 or s_2 no longer equals zero, we would be decreasing the use of labor or wood. If, for example, one unit of s_1 is entered into the solution, then one unit of labor previously used is not used, and profit is reduced by \$20.

Let us assume that one unit of s_1 has been entered into the solution so that we have one hour of unused labor ($s_1 = 1$). Now let us remove this unused hour of labor from the solution so that all labor is again being used. We previously noted that profit was decreased by \$20 by entering one hour of unused labor; thus, it can be expected that if we take this hour back (and use it again), profit will be increased by \$20. This is analogous to saying that if we could get one more hour of labor, we could increase profit

by \$20. Therefore, if we could purchase one hour of labor, we would be willing to pay up to \$20 for it because that is the amount by which it would increase profit.

[Page A-33]

Time Out: For John Von Neumann

John Von Neumann, the famous Hungarian mathematician, is credited with many contributions in science and mathematics, including crucial work on the development of the atomic bomb during World War II and the development of the computer following the war. In 1947 George Dantzig visited Von Neumann at the Institute for Advanced Study at Princeton and described the linear programming technique to him. Von Neumann grasped the technique immediately, because he had been working on similar concepts himself, and went on to explain duality to Dantzig for the first time.

The negative $c_j z_j$ row values of \$20 and \$6.67 are the *marginal values* of labor (s_1) and wood (s_2), respectively. These dual values are also often referred to as *shadow prices*, since they reflect the maximum "price" one would be willing to pay to obtain one more unit of the resource.

The $c_j z_j$ values for slack variables are the marginal values of the constraint

*resources, i.e.,
shadow prices.*

What happened to the third resource, storage space? The answer can be seen in [Table A-32](#). Notice that the $c_j z_j$ row value for s_3 (which represents unused storage space) is zero. This means that storage space has a marginal value of zero; that is, we would not be willing to pay anything for an extra foot of storage space.

If a resource is not completely used, i.e., there is slack, its marginal value is zero.

The reason more storage space has no marginal value is because storage space was not a limitation in the production of tables and chairs. [Table A-32](#) shows that 48 square feet of storage space were left unused (i.e., $s_3 = 48$) after the 4 tables

and 8 chairs were produced. Since the company already has 48 square feet of storage space left over, an extra square foot would have no additional value; the company cannot even use all of the storage space it has available.

We need to consider one additional aspect of these marginal values. In our discussion of the marginal value of these resources, we have indicated that the marginal value (or *shadow price*) is the *maximum* amount that would be paid for

additional resources. The marginal value of \$60 for one hour of labor is not necessarily what the Hickory Furniture Company would *pay* for an hour of labor. This depends on how the objective function is defined. In this example we are assuming that all of the resources available, 40 hours of labor, 216 board feet of wood, and 240 square feet of storage space, are already paid for. Even if the company does not use all the resources, it still must pay for them. They are *sunk* costs. In other words, the

cost of any additional resources secured are included in the objective function coefficients. As such, the profit values in the objective function for each product are unaffected by how much of a resource is actually used; the profit is independent of the resources used. If the cost of the resources is not included in the profit function, then securing additional resources would reduce the marginal value.

The **shadow price** is the **maximum**

amount that should be paid for one additional unit of a resource.

The **shadow price** is the **maximum** amount that should be paid for one additional unit of a resource.

Continuing our analysis, we note that the profit in the primal model was shown to be \$2,240. For the furniture company, the value of the resources used to produce tables and chairs must be in terms of this profit. In other words, the value of the labor and wood resources is determined by their contribution toward the \$2,240 profit. Thus, if the company wanted to assign a *value* to the resources it used, it could not

assign an amount greater than the profit earned by the resources. Conversely, using the same logic, the total *value* of the resources must also be at least as much as the profit they earn. Thus, the value of all the resources must *exactly equal* the profit earned by the optimal solution.

The total marginal value of the resources equals the optimal profit.

Now let us look again at the dual form of the model.

[Page A-34]

minimize $Z_d = 40y_1 + 16y_2 + 240y_3$

subject to

$$2y_1 + 18y_2 + 24y_3 \geq 160$$

$$4y_1 + 18y_2 + 12y_3 \geq 200$$

$$y_1, y_2, y_3 \geq 0$$

*The dual variables
equal the marginal
value of the
resources, the*

shadow prices.

Given the previous discussion on the value of the model resources, we can now define the decision variables of the dual, y_1 , y_2 , and y_3 , to represent the marginal values of the resources:

y_1 = marginal value of 1 hr of labor = \$20

y_2 = marginal value of 1 bd ft

of wood = \$6.67

y_3 = marginal value of 1 ft² of
storage space = \$0

Use of the Dual

The importance of the dual to the decision maker lies in the information it provides about the model resources. Often the manager is less concerned about profit than about the use of resources because the manager often has more control over the use of

resources than over the accumulation of profits. The dual solution informs the manager of the value of the resources, which is important in deciding whether or not to secure more resources and how much to pay for these additional resources.

The dual provides the decision maker with a basis for deciding how much to pay for more resources.

If the manager secures more resources, the next question is "How does this affect the original solution?" The feasible solution area is determined by the values forming the model constraints, and if those values are changed, it is possible for the feasible solution area to change. The effect on the solution of changes to the model is the subject of sensitivity analysis, the next topic to be presented here.

Sensitivity Analysis

In this section we will show how sensitivity ranges are mathematically determined using the simplex method. While this is not as efficient or quick as using the computer, close examination of the simplex method for performing sensitivity analysis can provide a more thorough understanding of the topic.

Changes in Objective Function Coefficients

To demonstrate sensitivity analysis for the coefficients in the objective function, we will use the Hickory Furniture Company example developed in the previous section. The model for this example was formulated as

maximize $Z = \$160x_1 + 200x_2$

subject to

$$2x_1 + 4x_2 \leq 40 \text{ hr of labor}$$

$$18x_1 + 18x_2 \leq 216 \text{ bd ft of wood}$$

$$24x_1 + 12x_2 \leq 240 \text{ ft}^2 \text{ of storage space}$$

$$x_1, x_2 \geq 0$$

where

x_1 = number of tables produced

x_2 = number of chairs produced

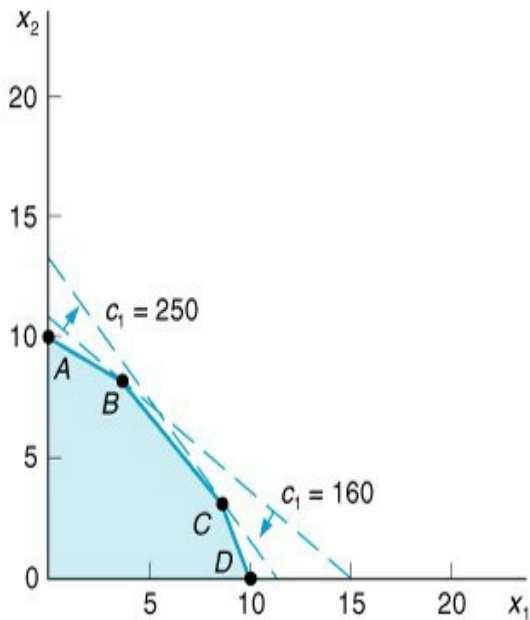
[Page A-35]

The coefficients in the objective function will be represented

symbolically as c_j (the same notation used in the simplex tableau). Thus, $c_1 = 160$ and $c_2 = 200$. Now, let us consider a change in one of the c_j values by an amount Δ . For example, let us change $c_1 = 160$ by $\Delta = 90$. In other words, we are changing c_1 from \$160 to \$250. The effect of this change on the solution of this model is shown graphically in [Figure A-10](#).

Figure A-10. A change

in c_1



Originally, the solution to this problem was located at point B in [Figure A-10](#), where $x_1 = 4$ and $x_2 = 8$. However, increasing c_1 from \$160 to \$250 shifts the slope of the objective function so that point C ($x_1 = 8, x_2 = 4$) becomes the optimal solution. This demonstrates that a change in one of the coefficients of the objective function can change the optimal solution. Therefore, sensitivity analysis is performed to determine the range over which c_j can be changed without altering the

optimal solution.

The sensitivity range for a c_j value is the range of values over which the current optimal solution will remain optimal.

The range of c_j that will maintain the optimal solution can be determined directly from the optimal simplex tableau. The optimal simplex

tableau for our furniture company example is shown in [Table A-33](#).

Table A-33. The Optimal Simplex Tableau

		160	200	0	0
C_j	Basic Variables	Quantity			
		x_1	x_2	s_1	s_2
200	x_2	8	0	$1\frac{1}{2}$	$1/18$
160	x_1	4	1	$0\frac{1}{2}$	$1/9$
0	s_3	48	0	0	2

z_j	2,240	160	200	20	20/3
-------	-------	-----	-----	----	------

$c_j \ z_j$	0	0	20	20/3	
-------------	---	---	----	------	--

Δ is added to c_j in the optimal simplex tableau.

First, consider a Δ change for c_1 . This will change the c_1

value from $c_1 = 160$ to $c_1 = 160 + \Delta$, as shown in [Table A-34](#). Notice that when c_1 is changed to $160 + \Delta$, the new value is included not only in the top c_j row but also in the left-hand c_j column. This is because x_1 is a basic solution variable. Since $160 + \Delta$ is in the left-hand column, it becomes a multiple of the column values when the new z_j row values and the subsequent $c_j z_j$ row values, also shown in [Table A-34](#), are computed.

Table A-34. The Optimal Simplex Tableau with $c_1 = 160 + \Delta$

		$\begin{matrix} 160 \\ + \Delta \end{matrix} \quad 200 \quad 0$			
Basic Variables		Quantity			
c_j			x_1	x_2	s_1
200	x_2	8	0	1	$1/2$
$160 + \Delta$	x_1	4	1	0	$1/2$
0	s_3	48	0	0	6

z_j	$2,240 + 160 \Delta$	200Δ	$20 \Delta / 2$	20
	4Δ	$+$	Δ	
$c_j - z_j$	0	0	$20 + \Delta / 2$	20

For the solution to remain optimal all values in the $c_j - z_j$ row must be ≤ 0 .

The solution shown in [Table A-34](#) will remain optimal as long as the $c_j z_j$ row values remain negative. (If $c_j z_j$ becomes positive, the product mix will change, and if it becomes zero, there will be an alternative solution.) Thus, for the solution to remain optimal,

$$20 + \Delta/2 \leq 0$$

and

$$20/3 - \Delta/9 \leq 0$$

Both of these inequalities must be solved for Δ .

$$\begin{aligned}
 -20 + \Delta/2 &\leq 0 \\
 \Delta/2 &\leq 20 \\
 \Delta &\leq 40
 \end{aligned}$$

and

$$\begin{aligned}
 -20/3 - \Delta/9 &\leq 0 \\
 -\Delta/9 &\leq 20/3 \\
 -\Delta &\leq 60 \\
 \Delta &\geq -60
 \end{aligned}$$

Thus, $\Delta \leq 40$ and $\Delta \geq -60$. Now recall that $c_1 = 160 + \Delta$; therefore, $\Delta = c_1 - 160$.

Substituting the amount $c_1 - 160$ for Δ in these inequalities,

$$\begin{aligned}
 \Delta &\leq 40 \\
 c_1 - 160 &\leq 40 \\
 c_1 &\leq 200
 \end{aligned}$$

and

$$\Delta \geq -60$$

$$c_1 - 160 \geq -60$$

$$c_1 \geq 100$$

Therefore, the range of values for c_1 over which the solution basis will remain optimal (although the value of the objective function may change) is

$$100 \leq c_1 \leq 200$$

Next, consider a Δ change in c_2 so that $c_2 = 200 + \Delta$. The effect of this change in the final

simplex tableau is shown in [Table A-35](#).

Table A-35. The Optimal Simplex Tableau with $c_2 = 200 + \Delta$

(This item is displayed on page A-37 in the print version)

		160	$200 + \Delta$	0	0	
Basic Variables		Quantity				
c_j		x_1	x_2	s_1	s_2	
$200 + \Delta$	x_2	8	0	$1\frac{1}{2}$	$1/18$	
160	x_1	4	1	$0\frac{1}{2}$	$1/9$	

$$0 \quad s_3 \quad 48 \quad 0 \quad 0 \quad 6 \quad 2$$

$$z_j \quad 2,240 + \frac{160}{8\Delta} \quad 200 + \frac{20}{\Delta/2} \quad \frac{20}{\Delta/18}$$

$$c_j - z_j \quad 0 \quad 0 \quad \frac{20}{\Delta/2} + \frac{20}{\Delta/18}$$

[Page A-37]

As before, the solution shown in [Table A-35](#) will remain

optimal as long as the $c_j z_j$ row values remain negative or zero. Thus, for the solution to remain optimal, we must have

$$20 - \Delta/2 \leq 0$$

and

$$20/3 + \Delta/18 \leq 0$$

Solving these inequalities for Δ gives

$$-20 - \Delta/2 \leq 0$$

$$-\Delta/2 \leq 20$$

$$\Delta \geq -40$$

and

$$-20/3 + \Delta/18 \leq 0$$

$$\Delta/18 \leq 20/3$$

$$\Delta \leq 120$$

Thus, $\Delta \geq 40$ and $\Delta \leq 120$.

Since $c_2 = 200 + \Delta$, we have $\Delta = c_2 - 200$. Substituting this value for Δ in the inequalities yields

$$\Delta \geq -40$$

$$c_2 - 200 \geq -40$$

$$c_2 \geq 160$$

and

$$\Delta \leq 120$$

$$c_2 - 200 \leq 120$$

$$c_2 \leq 320$$

Therefore, the range of values for c_2 over which the solution will remain optimal is

$$160 \leq c_2 \leq 320$$

The ranges for both objective function coefficients are as follows.

$$100 \leq c_1 \leq 200$$

$$160 \leq c_2 \leq 320$$

However, these ranges reflect a possible change in either c_1 or c_2 , not simultaneous changes in

both c_1 and c_2 . Both of the objective function coefficients in this example were for basic solution variables. Determining the c_j sensitivity range for a decision variable that is *not basic* is much simpler. Because it is not in the basic variable column, the Δ change does not become a multiple of the z_j row. Thus, the Δ change will show up in only one column in the c_j z_j row.

Changes in Constraint Quantity Values

To demonstrate the effect of a change in the quantity values of the model constraints, we will again use the Hickory Furniture Company example.

$$\text{maximize } Z = 160x_1 + 200x_2$$

subject to

$$2x_1 + 4x_2 \leq 40 \text{ hr of labor}$$

$$18x_1 + 18x_2 \leq 216 \text{ bd ft of wood}$$

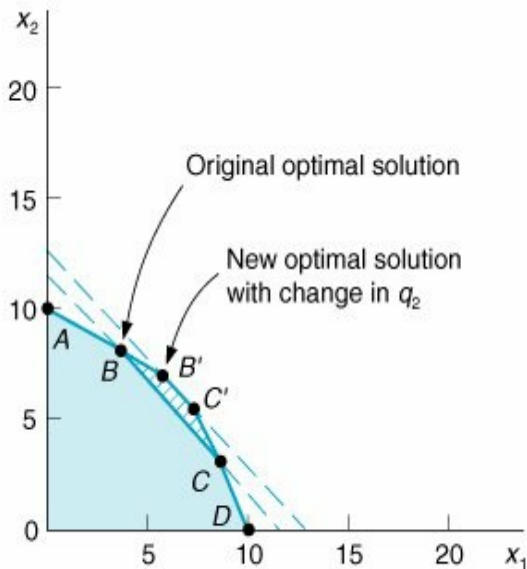
$$24x_1 + 12x_2 \leq 240 \text{ ft}^2 \text{ of storage space}$$

$$x_1, x_2 \geq 0$$

The quantity values 40, 216,

and 240 will be represented symbolically as q_i . Thus, $q_1 = 40$, $q_2 = 216$, and $q_3 = 240$. Now consider a Δ change in q_2 . For example, let us change $q_2 = 216$ by $\Delta = 18$. In other words, q_2 is changed from 216 board feet to 234 board feet. The effect of this change is shown graphically in [Figure A-11](#).

Figure A-11. A Δ change in q_2



In [Figure A-11](#) a change in q_2 is shown to have the effect of changing the feasible solution area from $0ABCD$ to $0AB'C'D$. Originally, the optimal solution point was B ; however, the change in q_2 causes B to be the new optimal solution point. At point B the optimal solution is

$$x_1 = 4$$

$$x_2 = 8$$

$$s_3 = 48$$

$$s_1 \text{ and } s_2 = 0$$

$$Z = \$2,240$$

At point B' the new optimal solution is

[Page A-39]

$$x_1 = 6$$

$$x_2 = 7$$

$$s_3 = 12$$

$$s_1 \text{ and } s_2 = 0$$

$$Z = \$2,360$$

Thus, a change in a q_i value can change the values of the optimal solution. At some point an increase or decrease in q_i will change the variables in the

optimal solution mix, including the slack variables. For example, if q_2 increases to 240 board feet, then the optimal solution point will be at

$$x_1 = 6.67$$

$$x_2 = 6.67$$

$$s_3 = 0$$

$$s_1 \text{ and } s_2 = 0$$

$$Z = \$2,400$$

Notice that s_3 has left the solution; thus, the optimal solution mix has changed. At this point, where $q_2 = 240$, which is the upper limit of the sensitivity range for q_2 , the shadow price will also change. Therefore, the purpose of sensitivity analysis is to determine the range for q_i over which the optimal variable mix will remain the same and the shadow price will remain the

same.

The sensitivity range for a q_i value is the range of values over which the right-hand-side values can vary without changing the solution variable mix, including slack variables and the shadow prices.

As in the case of the c_j values,

the range for q_i can be determined directly from the optimal simplex tableau. As an example, consider a Δ increase in the number of labor hours. The model constraints become

$$\begin{aligned}2x_1 + 4x_2 &\leq 40 + 1\Delta \\18x_1 + 18x_2 &\leq 216 + 0\Delta \\24x_1 + 12x_2 &\leq 240 + 0\Delta\end{aligned}$$

Notice in the initial simplex tableau for our example ([Table A-36](#)) that the changes in the quantity column are the same as the coefficients in the s_1 column.

Table A-36. The Initial Simplex Tableau

			160	200	0	0	0	
Basic Variables			Quantity					
C_j			x_1	x_2	s_1	s_2	s_3	
0	1	40	$+\frac{1}{1}\Delta$	2	4	1	0	0
0	s_2	216	$+\frac{0}{0}\Delta$	18	18	0	1	0
0	s_3	240	$+\frac{0}{0}\Delta$	24	12	0	0	1
	Z_j		0	0	0	0	0	0

$c_j \ z_j$

160 200 0 0 0

A Δ in a q_i value is duplicated in the s_i column in the final tableau.

This duplication will carry through each subsequent tableau, so the s_1 column values will duplicate the Δ

changes in the quantity column in the final tableau ([Table A-37](#)).

Table A-37. The Final Simple Tableau

(This item is displayed on page A-40 in the print version)

			160	200	0	0
Basic Variables		Quantity				
C_j			x_1	x_2	s_1	s_2
200	x_2	$8 + \frac{\Delta}{2}$	0	$1\frac{1}{2}$	$\frac{1}{18}$	
160	x_1	$4\frac{\Delta}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{9}$

0	s_3	48	+	0	0	6	2
		6Δ					
	z_j	2,240	+	160	200	20	20/3
		20Δ					
	$c_j - z_j$			0	0	20	20/3

In effect, the Δ changes form a separate column identical to the s_1 column. Therefore, to determine the Δ change, we need only observe the slack (s_i) column corresponding to the

model constraint quantity (q_i) being changed.

Recall that a requirement of the simplex method is that the quantity values not be negative. If any q_i value becomes negative, the current solution will no longer be *feasible* and a new variable will enter the solution. Thus, the inequalities

[Page A-40]

$$8 + \Delta/2 \geq 0$$

$$4 - \Delta/2 \geq 0$$

$$48 + 6\Delta \geq 0$$

are solved for Δ :

$$8 + \Delta/2 \geq 0$$

$$\Delta/2 \geq -8$$

$$\Delta \geq -16$$

and

$$4 - \Delta/2 \geq 0$$

$$-\Delta/2 \geq -4$$

$$\Delta \leq 8$$

and

$$48 + 6\Delta \geq 0$$

$$6\Delta \geq -48$$

$$\Delta \geq -8$$

Since

$$q_1 = 40 + \Delta$$

then

$$\Delta = q_1 - 40$$

These values are substituted into the inequalities $\Delta \geq -16$, $\Delta \leq 8$, and $\Delta \geq -8$ as follows.

$$\Delta \geq -16$$

$$q_1 - 40 \geq -16$$

$$q_1 \geq 24$$

$$\Delta \leq 8$$

$$q_1 - 40 \leq 8$$

$$q_1 \leq 48$$

$$\Delta \geq -8$$

$$q_1 - 40 \geq -8$$

$$q_1 \geq 32$$

Summarizing these inequalities, we have

$$24 \leq 32 \leq q_1 \leq 48$$

[Page A-41]

The value of 24 can be eliminated, since q_1 must be greater than 32; thus,

$$32 \leq q_1 \leq 48$$

As long as q_1 remains in this range, the present basic solution variables will remain the same and feasible.

However, the quantity values of those basic variables may change. In other words, although the variables in the basis remain the same, their values can change.

To determine the range for q_2 (where $q_2 = 216 + \Delta$), the s_2 column values are used to develop the Δ inequalities.

$$8 - \Delta/18 \geq 0$$

$$4 + \Delta/9 \geq 0$$

$$48 - 2\Delta \geq 0$$

The inequalities are solved as follows.

$$\begin{aligned}
 8 - \Delta/18 &\geq 0 \\
 -\Delta/18 &\geq -8 \\
 \Delta &\leq 144
 \end{aligned}$$

$$\begin{aligned}
 4 + \Delta/9 &\geq 0 \\
 \Delta/9 &\geq -4 \\
 \Delta &\geq -36
 \end{aligned}$$

$$\begin{aligned}
 48 - 2\Delta &\geq 0 \\
 -2\Delta &\geq -48 \\
 \Delta &\leq 24
 \end{aligned}$$

Since

$$q_2 = 216 + \Delta$$

we have

$$\Delta = q_2 - 216$$

Substituting this value into the inequalities $\Delta \leq 144$, $\Delta \geq 36$, and $\Delta \leq 24$ gives a range of possible values for q_2 :

$$\Delta \leq 144$$

$$q_2 - 216 \leq 144$$

$$q_2 \leq 360$$

$$\Delta \geq -36$$

$$q_2 - 216 \geq -36$$

$$q_2 \geq 180$$

$$\Delta \leq 24$$

$$q_2 - 216 \leq 24$$

$$q_2 \leq 240$$

That is,

$$180 \leq q_2 \leq 240 \leq 360$$

The value 360 can be eliminated, because q_2 cannot exceed 240. Thus, the range over which the basic solution variables will remain the same is

$$180 \leq q_2 \leq 240$$

The range for q_3 is

$$192 \leq q_3 \leq \infty$$

The upper limit of ∞ means that q_3 can increase indefinitely

(without limit) without changing the optimal variable solution mix in the shadow price.

Sensitivity analysis of constraint quantity values can be used in conjunction with the dual solution to make decisions regarding model resources. Recall from our analysis of the dual solution of the Hickory Furniture Company example that

$y_1 = \$20$, marginal value of labor

$y_2 = \$6.67$, marginal value of wood

$y_3 = \$0$, marginal value of storage space

The shadow prices are only valid with the sensitivity range for the right-hand-side values.

Because the resource with the greatest marginal value is

labor, the manager might desire to secure some additional hours of labor. How many hours should the manager get? Given that the range for q_1 is $32 \leq q_1 \leq 48$, the manager could secure up to an additional 8 hours of labor (i.e., 48 total hours) before the solution basis changes and the shadow price also changes. If the manager did purchase 8 more hours, the solution values could be found by observing the quantity values in [Table A-37](#).

$$x_2 = 8 + \Delta/2$$

$$x_1 = 4\Delta/2$$

$$s_3 = 48 + 6\Delta$$

Since $\Delta = 8$,

$$x_2 = 8 + (8)/2$$

$$= 12$$

$$x_1 = 4 (8)/2$$

$$= 0$$

$$s_3 = 48 + 6(8)$$

$$= 96$$

Total profit will be increased by \$20 for each extra hour of labor.

$$Z = \$2,240 + 20\Delta$$

$$= \frac{2,240}{20(8)} +$$

$$= 2,240 + 160$$

$$= \$2,400$$

In this example for the Hickory Furniture Company, we considered only $\leq$ constraints in determining the sensitivity ranges for q_i values. To compute the q_i sensitivity range, we observed the slack

column, s_j , since a Δ change in q_j was reflected in the s_j column. However, recall that with a $\geq$ constraint we subtract a surplus variable rather than adding a slack variable to form an equality (in addition to adding an artificial variable). Thus, for a $\geq$ constraint we must consider a Δ change in q_j in order to use the s_j (surplus) column to perform sensitivity analysis. In that case sensitivity analysis would be performed exactly as shown in this example, except that the value of $q_j \Delta$ would be used

instead of $q_i + \Delta$ when
computing the sensitivity range
for q_i .

Problems

Following is a simplex tableau for a linear programming model.

c_j	Basic Variables	Quantity	2	10	6
			x_1	x_2	x_3
2	x_1	10	1	0	2
10	x_2	40	0	1	2

1.

0	s_3	30	0	0	8
	z_j	420	2	10	16
	$c_j - z_j$		0	0	10

1. What is the solution given in this tableau?
2. Is the solution in this tableau optimal?
3. What does x_3 equal in this tableau?
4. Write out the original objective function and constraints for the linear programming model, using x_1 and x_2 as variables.
5. How many constraints are in the linear programming model?

6. Explain briefly why it would have to solve this problem graphically.

The following is a simplex tableau for a programming model.

c_j	Basic Variables	Quantity	6	20
			x_1	x_2
6	x_1	20	1	1
12	x_3	10	0	1/3
0	s_1	10	0	1/3

z_j	240	6	10
-------	-----	---	----

z_j c_j		0	10
-------------	--	---	----

2.

1. Is this a maximization or a minimization problem? Why?
2. What is the solution given in this tableau?
3. Is the solution given in this tableau optimal? Why?

[Page A-44]

4. Write out the original objective function and constraints for the linear programming model using x_1 and x_2 variables.
5. How many constraints are in the model?

programming model?

6. Were any of the constraints origin Why?
7. What is the value of x_2 in this tabl

The following is a simplex tableau for a programming problem.

			60	50	45	50
c_j	Basic Variables	Quantity	x_1	x_2	x_3	x_4
0	s_1	20	0	1	0	0
50	x_4	15	0	0	0	1

60	x_1	12	1	1/2	0	0
----	-------	----	---	-----	---	---

0	s_4	45	0	0	8	6
---	-------	----	---	---	---	---

3.

z_j	1,470	60	30	0	50
-------	-------	----	----	---	----

$c_j - z_j$	0	20	45	0
-------------	---	----	----	---

1. Is this a maximization or a minimization problem?
2. What are the values of the decision variables in this tableau?
3. What are the values of the slack variables in this tableau?

4. What does the $c_j z_j$ value of "20" column mean?
5. Is this solution optimal? Why? If the solution is not optimal, determine the optimal solution.

The following is a simplex tableau for a linear programming problem.

			8	10	4	0
c_j	Basic Variables	Quantity				
			x_1	x_2	x_3	s
M	A_1	30	2/3	0	1	1
10	x_2	10	1/3	1	0	0

M	A_3	100	0	0	1	0
-----	-------	-----	---	---	---	---

$$z_j \quad \frac{130M + 2M/3}{100} + 10 \quad \frac{2M}{10/3}$$

4.

$$z_j \quad c_j \quad \frac{2M/3}{14/3} \quad 0 \quad \frac{2M}{4} \quad M$$

1. Is this a maximization or a minimization problem?
2. What is the value of x_3 in this table?
3. What does the value " $M/6 \ 5/3$ " in the $z_j \ c_j$ column of the $z_j \ c_j$ row mean?
4. What is the minimum number of artificial variables?

simplex iterations that this problem
through to determine a feasible opti

- 5.** Is this solution optimal? Why? If th
not optimal, compute the optimal

[Page A-45]

Given is the following simplex tableau for
programming problem.

c_j	Basic Variables	Quantity	4		6		0	
			x_1		x_2		s_1	
M	A_1	2	0		1/2		1	
4	x_1	6	1		1/2		0	

$$z_j \quad 24 + 2M \quad 4 \quad M/2 + M/2 \quad M$$

5.

$$z_j c_j \quad 0 \quad M/2 \quad 4 \quad M$$

1. Is this a maximization or a minimization problem? Why?
2. What are the values of the decision variables in this tableau?
3. Were any of the constraints in this tableau originally equations? Why?
4. What is the value of s_2 in this tableau?
5. Is this solution optimal? Why? If not optimal, complete the next iteration.

(tableau) and indicate if it is optim

Following is a simplex tableau for a linear programming problem.

c_j	Basic Variables	Quantity	10	5	0
			x_1	x_2	s_1
10	x_1	5	1	$1/2$	$1/2$
M	A_2	4	0	1	0
0	s_2	15	0	$7/2$	$1/2$
	z_j	$4M + 50$	10	$M + 5$	5

6.

$c_j z_j$

0

M

5

1. Is this a maximization or a minimization problem? Why?
2. What is the value of x_2 in this tableau?
3. Does the fact that x_1 has a $c_j z_j$ value of "0" in this tableau mean that multiple solutions exist? Why?
4. What does the $c_j z_j$ value for the slack variable mean?
5. Is this solution optimal? Why? If not, find the optimal solution and indicate if multiple optima exist.

- The Munchies Cereal Company makes several ingredients. Two of the ingredients, rice and oats, provide vitamins A and B. The company knows how many ounces of oats and rice to include in each box of cereal to meet the requirements of 48 milligrams of vitamin A and 12 milligrams of vitamin B while minimizing cost. An ounce of oats contributes 8 milligrams of vitamin A and 1 milligram of vitamin B, whereas an ounce of rice contributes 6 milligrams of vitamin A and 2 milligrams of vitamin B. An ounce of oats costs \$0.05 and an ounce of rice costs \$0.03. Formulate a linear programming model for this problem and solve it using the simplex method.
- 7.

[Page A-46]

A company makes product 1 and product 2 using two resources. The linear programming model for determining the amounts of product 1 and product 2 to produce (x_1 and x_2) is

maximize $Z = 8x_1 + 2x_2$ (profit, \$)

8. subject to

$$4x_1 + 5x_2 \leq 20 \text{ (resource 1, lb)}$$

$$2x_1 + 6x_2 \leq 18 \text{ (resource 2, lb)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

A company produces two products that are assembled on two assembly lines. Assembly line 1 has 100 hours available, and assembly line 2 has 80 hours. Each product requires 10 hours

9. time on line 1, while on line 2 product 1 requires 4 hours and product 2 requires 3 hours. The profit per unit of product 1 is \$6 per unit, and the profit per unit of product 2 is \$4 per unit. Formulate a linear programming model for this problem and solve using the simplex method.

The Pinewood Furniture Company produces chairs and tables from two resources: labor and wood. The company has 80 hours of labor and 36 board feet of wood available each week. Each chair requires 8 hours of labor and 6 board feet of wood, and each table requires 10 hours of labor and 8 board feet of wood. The profit is \$12 per chair and \$20 per table. Formulate a linear programming model for this problem and solve using the simplex method.

10.

company has 80 hours of labor and 36 pounds of wood available each day. Demand for chairs is at least 6 per day. Each chair requires 8 hours of labor and 2 pounds of wood to produce, while a table requires 10 hours of labor and 6 pounds of wood. The profit derived from each chair is \$400 and from each table is \$100. The company wants to determine the number of chairs and tables to produce each day to maximize profit. Formulate a linear programming problem and solve using the simplex method.

11.

The Crumb and Custard Bakery makes coffee cakes and Danish in large pans. The main ingredients are flour and sugar. There are 25 pounds of flour and 16 pounds of sugar available and the demand for coffee cakes is 8. Five pounds of flour and 3 pounds of sugar are required to make one pan of coffee cake, and 5 pounds of flour and 4 pounds of sugar are required to make one pan of Danish. One coffee cake has a profit of \$1, and one Danish has a profit of \$5. Determine the number of coffee cakes and Danish that the bakery must produce to maximize profit.

day so that profit will be maximized. Formulate a linear programming model for this problem and solve using the simplex method.

- 12.** The Kalo Fertilizer Company makes a fertilizer from two chemicals that provide nitrogen, phosphate, and potassium. A pound of ingredient 1 contains 4 ounces of nitrogen and 6 ounces of phosphate, whereas a pound of ingredient 2 contains 2 ounces of nitrogen, 6 ounces of phosphate, and 4 ounces of potassium. Ingredient 1 costs \$3 per pound and ingredient 2 costs \$5 per pound. The company wants to know how many pounds of each chemical to use to produce a bag of fertilizer that meets the minimum requirements of 20 ounces of nitrogen, 24 ounces of phosphate, and 16 ounces of potassium while minimizing cost. Formulate a linear programming model for this problem and solve using the simplex method.

Solve the following model using the simplex method.

$$\text{minimize } Z = 0.06x_1 + 0.10x_2$$

13. subject to

$$4x_1 + 3x_2 \geq 12$$

$$3x_1 + 6x_2 \geq 12$$

$$5x_1 + 2x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

The Copperfield Mining Company owns both of which produce three grades of medium, and low. The company has a supply a smelting company with at least high-grade ore, 8 tons of medium-grade ore, and 4 tons of low-grade ore. Each mine produces a certain amount of each type of ore each hour of operation. Mine 1 produces 6 tons of high-grade ore, 4 tons of medium-grade, and 4 tons of low-grade ore per hour. Mine 2 produces 2 tons of high-grade ore, 12 tons of medium-grade, and 12 tons of low-grade ore per hour.

14. per hour. Mine 2 produces 2 tons of high-grade ore, 12 tons of medium-grade, and 12 tons of low-grade ore per hour.

per hour. It costs \$200 per hour to mine ore from mine 1, and it costs \$160 per hour to mine a ton of ore from mine 2. The company must determine the number of hours it needs to operate each mine so that contractual obligations are met at the lowest cost. Formulate a linear programming model for this problem and solve using the simplex method.

A marketing firm has contracted to do a survey on a political issue for a Spokane television station. The firm conducts interviews during the day and at night by telephone and in person. Each hour of work at each type of interview results in a certain number of interviews. In order to have a representative survey, the firm has determined that there must be at least 400 day interviews, 200 personal interviews, and 1,200 interviews in total. The company has developed the following linear programming model to determine the number of hours of telephone interviews during the day (x_1), telephone interviews at night (x_2), pers

during the day (x_3), and personal interview (x_4) that should be conducted to minimize

$$\text{minimize } Z = 2x_1 + 3x_2 + 5x_3 + 7x_4 \text{ (cost, \$)}$$

subject to

$$10x_1 + 4x_3 \geq 400 \text{ (day interviews)}$$

$$4x_3 + 5x_4 \geq 100 \text{ (personal interviews)}$$

$$x_1 + x_2 + x_3 + x_4 \geq 1,200 \text{ (total interviews)}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solve this model using the simplex method

A jewelry store makes both necklaces from gold and platinum. The store has the following linear programming model for the number of necklaces and bracelets that it needs to make to maximize profit.

16. maximize $Z = 300x_1 + 400x_2$ (profit, \$)

subject to

$$3x_1 + 2x_2 \leq 18 \text{ (gold, oz)}$$

$$2x_1 + 4x_2 \leq 20 \text{ (platinum, oz)}$$

$$x_2 \leq 4 \text{ (demand, bracelets)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

A sporting goods company makes baseballs and softballs on a daily basis from leather and other materials. The company has developed the following linear programming model for determining the number of baseballs and softballs to produce (x_1 and x_2) to maximize profits:

[Page A-48]

17.

maximize $Z = 5x_1 + 4x_2$ (profit, \$)

subject to

$$0.3x_1 + 0.5x_2 \leq 150 \text{ (leather, ft}^2\text{)}$$

$$10x_1 + 4x_2 \leq 2,000 \text{ (yarn, yd)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

A clothing shop makes suits and blazers. The resources used are: material, rack space. The shop has developed this linear programming model for determining the number of suits (x_1) and blazers (x_2) to make (x_1 and x_2) to maximize profit.

18.

maximize $Z = 100x_1 + 150x_2$ (profit, \$)

subject to

$$10x_1 + 4x_2 \leq 160 \text{ (material, yd}^2\text{)}$$

$$x_1 + x_2 \leq 20 \text{ (rack space)}$$

$$10x_1 + 20x_2 \leq \text{(labor, hr)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex met

Solve the following linear programming
the simplex method.

maximize $Z = 100x_1 + 20x_2 + 60x_3$

subject to

$$3x_1 + 5x_2 \leq 60$$

$$2x_1 + 2x_2 + 2x_3 \leq 100$$

$$x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0$$

19.

The following is a simplex tableau for a programming model.

c_j	Basic Variables	Quantity	1	2	1
			x_1	x_2	x_3
2	x_2	10	0	1	$1/4$
0	s_2	20	0	0	$3/4$
1	x_1	10	1	0	1
	z_j	30	1	2	$3/2$
	$c_j - z_j$		0	0	$5/2$

20.

- 1.** Is this a maximization or a minimization problem? Why?
- 2.** What is the solution given in this tableau?
- 3.** Write out the original objective function for the linear programming model, using the original variables.
- 4.** How many constraints are in the original linear programming model?
- 5.** Were any of the constraints originally binding? Why?
- 6.** What does s_1 equal in this tableau?

[Page A-49]

- 7.** This solution is optimal. Are there any other optimal solutions?

optimal solutions? Why?

- 8.** If there are multiple optimal solutions, find the alternate solutions.

A wood products firm in Oregon plants trees white pines, spruce, and ponderosa pines. The white pines produce pulp for paper products and wood for lumber. The spruce and ponderosa pines produce pulp for paper products and wood for lumber. The company wants to plant enough acres of each type of tree to produce at least 27 tons of pulp and 30 tons of lumber. The company has developed the following linear programming model to determine the number of acres of white pines (x_1), spruce (x_2), and ponderosa pines (x_3) to plant to minimize cost.

21.

minimize $Z = 120x_1 + 40x_2 + 240x_3$ (cost, \$)

subject to

$$4x_1 + x_2 + 3x_3 \geq 27 \text{ (pulp, tons)}$$

$$2x_1 + 6x_2 + 3x_3 \geq 30 \text{ (lumber, tons)}$$

$$x_1, x_2, x_3 \geq 0$$

Solve this model using the simplex met

A baby products firm produces a strain containing liver and milk, each of which protein and iron to the baby food. Each food must have 36 milligrams of protein and 50 milligrams of iron. The company has developed the following linear programming model to determine the number of ounces of liver (x_1) and milk (x_2)

22. in each jar of baby food to meet the requirements for protein and iron at the minimum cost.

$$\text{minimize } Z = 0.05x_1 + 0.10x_2 \text{ (cost, \$)}$$

subject to

$$6x_1 + 2x_2 \geq 36 \text{ (protein, mg)}$$

$$5x_1 + 5x_2 \geq 50 \text{ (iron, mg)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex met

- 23.** Solve the linear programming model in graphically, and identify the points on the feasible region that correspond to each simplex tableau.

The Cookie Monster Store at South Ac has three types of cookies: chocolate chip, peanut butter, and pecan sandies. Three primary ingredients are used: chocolate chips, pecans, and sugar. The store has 40 pounds of chocolate chips, 40 pounds of pecans, and 300 pounds of sugar. The following linear programming model has been developed to determine the number of batches of each type of cookie to make to maximize profit.

Let x_1 = number of batches of peanut butter cookies, x_2 = number of batches of chocolate chip cookies, and x_3 = number of batches of pecan sandies.

24.

maximize $Z = 10x_1 + 12x_2 + 7x_3$ (profit, \$)

subject to

$$20x_1 + 15x_2 + 10x_3 \leq 300 \text{ (sugar, lb)}$$

$$10x_1 + 5x_2 \leq 120 \text{ (chocolate chips, lb)}$$

$$x_1 + 2x_3 \leq 40 \text{ (pecans, lb)}$$

$$x_1, x_2, x_3 \geq 0$$

Solve this model using the simplex method.

[Page A-50]

The Eastern Iron and Steel Company makes nails, bolts, and washers from leftover steel and zinc. The company has 24 tons of steel and 12 tons of zinc. The following linear program has been developed for determining the number of batches of nails (x_1), bolts (x_2), and washers (x_3) to produce to maximize profit.

25.

maximize $Z = 6x_1 + 2x_2 + 12x_3$ (profit, \$1,000s)

subject to

$$4x_1 + x_2 + 3x_3 \leq 24 \text{ (steel, tons)}$$

$$2x_1 + 6x_2 + 3x_3 \leq 30 \text{ (zinc, tons)}$$

$$x_1, x_2, x_3 \geq 0$$

Solve this model using the simplex method.

Solve the following linear programming problem using the simplex method.

$$\text{maximize } Z = 100x_1 + 75x_2 + 90x_3 + 95x_4$$

26.

subject to

$$3x_1 + 2x_2 \leq 40$$

$$4x_3 + x_4 \leq 25$$

$$200x_1 + 250x_3 \leq 2,000$$

$$100x_1 + 200x_4 \leq 2,200$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solve the following linear programming the simplex method.

$$\text{minimize } Z = 20x_1 + 16x_2$$

27. subject to

$$3x_1 + x_2 \geq 6$$

$$x_1 + x_2 \geq 4$$

$$2x_1 + 6x_2 \geq 12$$

$$x_1, x_2 \geq 0$$

28. Solve the linear programming model in graphically, and identify the points on the feasible region that correspond to each simplex tableau.

Transform the following linear programming model into proper form for solution by the simplex method.

$$\text{minimize } Z = 8x_1 + 2x_2 + 7x_3$$

29.

subject to

$$2x_1 + 6x_2 + x_3 = 30$$

$$3x_2 + 4x_3 \geq 60$$

$$4x_1 + x_2 + 2x_3 \leq 50$$

$$x_1 + 2x_2 \geq 20$$

$$x_1, x_2, x_3 \geq 0$$

Transform the following linear program into proper form for solution by the simplex method.

[Page A-51]

30.

$$\text{minimize } Z = 40x_1 + 55x_2 + 30x_3$$

subject to

$$x_1 + 2x_2 + 3x_3 \leq 60$$

$$2x_1 + x_2 + x_3 = 40$$

$$x_1 + 3x_2 + x_3 \geq 50$$

$$5x_2 + 3x_3 \geq 100$$

$$x_1, x_2, x_3 \geq 0$$

A manufacturing firm produces two products, each requiring labor and material. The company has a production capacity of 60 units of labor and 40 units of material. The company has developed the following linear programming model to determine the optimal production levels of product 1 (x_1) and product 2 (x_2) to maximize profit.

maximize $Z = 40x_1 + 60x_2$ (profit, \$)

subject to

$$x_1 + 2x_2 \leq 30 \text{ (material, lb)}$$

$$4x_1 + 4x_2 \leq 72 \text{ (labor, hr)}$$

$$x_1 \geq 5 \text{ (contract, product 1)}$$

$$x_2 \geq 12 \text{ (contract, product 2)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

A custom tailor makes pants and jackets from imported Irish wool cloth. To get any cloth, the tailor must purchase at least 25 square feet of material each week. Each pair of pants and each jacket requires 1 square foot of material. The tailor has 100 square feet of material available each week to make pants and jackets. The demand for pants is never more than 50 pairs per week. The tailor has developed the following linear programming model to determine the number of pairs of pants (x_1) and jackets (x_2) to make each week.

32. maximize profit.

$$\text{maximize } Z = x_1 + 5x_2 \text{ (profit, \$100s)}$$

subject to

$$5x_1 + 5x_2 \geq 25 \text{ (wool, ft}^2\text{)}$$

$$2x_1 + 4x_2 \leq 16 \text{ (labor, hr)}$$

$$x_1 \leq 5 \text{ (demand, pants)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

A sawmill in Tennessee produces cherry boards for a large furniture manufacturer. The sawmill must deliver at least 5 tons of cherry to the manufacturer. It takes the sawmill 1 day to produce a ton of cherry and 2 days to produce a ton of oak, and the sawmill can allocate 18 days a month for this contract. The sawmill can produce 4 tons of wood and 1 ton of cherry to make 4 tons of wood and 1 ton of cherry. The sawmill owns 7 tons of wood. The sawmill owns 7 tons of wood.

developed the following linear program
determine the number of tons of cherry
(x_2) to produce to minimize cost.

33.

[Page A-52]

$$\text{minimize } Z = 3x_1 + 6x_2 \text{ (cost, \$)}$$

subject to

$$3x_1 + 2x_2 \leq 18 \text{ (production time, days)}$$

$$x_1 + x_2 \geq 5 \text{ (contract, tons)}$$

$$x_1 \leq 4 \text{ (cherry, tons)}$$

$$x_2 \leq 7 \text{ (oak, tons)}$$

$$x_1, x_2 \geq 0$$

Solve this model using the simplex method.

Solve the following linear programming problem using the simplex method.

$$\text{maximize } Z = 10x_1 + 5x_2$$

34.

subject to

$$2x_1 + x_2 \geq 10$$

$$x_2 = 4$$

$$x_1 + 4x_2 \leq 20$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming the simplex method.

$$\text{maximize } Z = x_1 + 2x_2 - x_3$$

35. subject to

$$4x_2 + x_3 \leq 40$$

$$x_1 - x_2 \leq 20$$

$$2x_1 + 4x_2 + 3x_3 \leq 60$$

$$x_1, x_2, x_3 \geq 0$$

Solve the following linear programming the simplex method.

$$\text{maximize } Z = x_1 + 2x_2 + 2x_3$$

36. subject to

$$\begin{aligned}x_1 + x_2 + 2x_3 &\leq 12 \\2x_1 + x_2 + 5x_3 &= 20 \\x_1 + x_2 - x_3 &\geq 8 \\x_1, x_2, x_3 &\geq 0\end{aligned}$$

A farmer has a 40-acre farm in Georgia trying to determine how many acres of corn and cotton to plant. Each crop requires fertilizer and insecticide. The farmer has developed the following linear programming model to determine the number of acres of corn (x_1), peanuts (x_2), and cotton (x_3) to plant to maximize profit.

37. maximize $Z = 400x_1 + 350x_2 + 450x_3$ (profit, \$)

subject to

$$2x_1 + 3x_2 + 2x_3 \leq 120 \text{ (labor, hr)}$$

$$4x_1 + 3x_2 + x_3 \leq 160 \text{ (fertilizer, tons)}$$

$$3x_1 + 2x_2 + 4x_3 \leq 100 \text{ (insecticide, tons)}$$

$$x_1 + x_2 + x_3 \leq 40 \text{ (acres)}$$

$$x_1, x_2, x_3 \geq 0$$

Solve this model using the simplex method

Solve the following linear programming graphically and (b) using the simplex method

38.

$$\text{maximize } Z = 3x_1 + 2x_2$$

subject to

$$x_1 + x_2 \leq 1$$

$$x_1 + x_2 \geq 2$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming graphically and (b) using the simplex m

$$\text{maximize } Z = x_1 + x_2$$

39. subject to

$$x_1 - x_2 \geq -1$$

$$-x_1 + 2x_2 \leq 4$$

$$x_1, x_2 \geq 0$$

Solve the following linear programming

the simplex method.

$$\text{maximize } Z = 7x_1 + 5x_2 + 5x_3$$

40. subject to

$$x_1 + x_2 + x_3 \leq 25$$

$$2x_1 + x_2 + x_3 \leq 40$$

$$x_1 + x_2 \leq 25$$

$$x_3 \leq 6$$

$$x_1, x_2, x_3 \geq 0$$

Solve the following linear programming
the simplex method.

41.

$$\text{minimize } Z = 15x_1 + 25x_2$$

subject to

$$3x_1 + 4x_2 \geq 12$$

$$2x_1 + x_2 \geq 6$$

$$3x_1 + 2x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

The Old English Metal Crafters Company produces trays and buckets. The number of trays (x_1) and buckets (x_2) that can be produced daily is limited by the availability of brass and labor, as shown in the following linear programming model.

maximize $Z = 6x_1 + 10x_2$ (profit, \$)

subject to

$$x_1 + 4x_2 \leq 90 \text{ (brass, lb)}$$

$$2x_1 + 2x_2 \leq 60 \text{ (labor, hr)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem follows.

42.	c_j	Basic Variables	Quantity	6	10
				x_1	x_2
	10	x_2	20	0	1
	6	x_1	10	1	0

z_j	260	6	10
-------	-----	---	----

$c_j z_j$		0	0
-----------	--	---	---

1. Formulate the dual of this model.
2. Define the dual variables and indicate their units.
3. Determine the optimal ranges for the right-hand side values (b_1 (ounces of brass) and q_2 (labor hours)).
4. Determine the feasible ranges for the objective function coefficients (c_1 (dollars per ounce of brass) and c_2 (dollars per labor hour)).
5. What is the maximum price the company would be willing to pay for additional labor hours? How many hours could be purchased at this price?

The Southwest Foods Company produces chiliRazorback and Longhorn from selected ingredients, including chili beans and ground beef. The number of 100-gallon batches of Razorback and Longhorn chili (x_2) that can be produced is constrained by the availability of chili beans and ground beef, as shown in the following programming model.

$$\text{maximize } Z = 200x_1 + 300x_2 \text{ (profit, \$)}$$

subject to

$$10x_1 + 50x_2 \leq 500 \text{ (chili beans, lb)}$$

$$34x_1 + 20x_2 \leq 800 \text{ (ground beef, lb)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem follows.

		200	300
c_j	Basic	Quantity	

Variables

x_1

x_2

300 x_2

6 0

1

200 x_1

20 1

0

z_j

5,800 200 300

43.

$c_j z_j$

0

0

[Page A-55]

- 1.** Formulate the dual of this model and what the dual variables equal.

- 2.** What profit for Razorback chili will Longhorn chili being produced? What optimal solution values be?
- 3.** Determine what the effect will be amount of beans in Razorback chili pounds per batch to 15 pounds per batch?
- 4.** Determine the optimal ranges for
- 5.** Determine the feasible ranges for beans) and q_2 (pounds of ground
- 6.** What is the maximum price the company be willing to pay for additional pounds of beans, and how many pounds could be purchased at that price?
- 7.** If the company could secure an additional pounds of only one of the ingredients: ground beef, which should it be?
- 8.** If the company changed the selling price of Longhorn chili so that the profit w

instead of \$300, would the optimum be affected?

The Agrimaster Company produces two types of fertilizer spreaders: regular and cyclone. Each spreader must undergo two production processes. Let x_1 be the number of regular spreaders produced and x_2 be the number of cyclone spreaders produced. The linear programming problem can be formulated as follows.

$$\text{maximize } Z = 9x_1 + 7x_2 \text{ (profit, \$)}$$

subject to

$$12x_1 + 4x_2 \leq 60 \text{ (process 1, production hr)}$$

$$4x_1 + 8x_2 \leq 40 \text{ (process 2, production hr)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem is as follows.

			9	7	
Basic Variables			Quantity		
C_j			x_1	x_2	
44.	9	x_1	4	1	0
	7	x_2	3	0	1
		Z_j	57	9	7
		$C_j - Z_j$		0	0

1. Formulate the dual for this problem
2. Define the dual variables and indicate

values.

- 3.** Determine the optimal ranges for
- 4.** Determine the feasible ranges for (production hours for processes₁ respectively).
- 5.** What is the maximum price the A Company would be willing to pay hours of process₁ production time many hours could be purchased a

[Page A-56]

The Stratford House Furniture Company makes two kinds of tables: end tables (x_1) and coffee tables (x_2). The manufacturer is restricted by material and labor constraints, as shown in the following linear programming formulation.

maximize $Z = 200x_1 + 300x_2$ (profit, \$)

subject to

$$2x_1 + 5x_2 \leq 180 \text{ (labor, hr)}$$

$$3x_1 + 3x_2 \leq 135 \text{ (wood, bd ft)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem follows.

			200	300
Basic Variables			Quantity	
C_j			x_1	x_2
300	x_2	30	0	
200	x_1	15	1	

45. z_j 12,000 200 30

c_j z_j 0

- 1.** Formulate the dual for this problem.
- 2.** Define the dual variables and indicate their units.
- 3.** What profit for coffee tables will result if 100 coffee tables are being produced, and what will be the optimal solution values?
- 4.** What will be the effect on the optimal solution if the available wood is increased from 12,000 to 12,500 board feet?
- 5.** Determine the optimal ranges for the available wood.

- 6.** Determine the feasible ranges for hours) and q_2 (board feet of wood)
- 7.** What is the maximum price the S Furniture Company would be willing to pay for additional wood, and how many board feet of wood could be purchased at that price?
- 8.** If the furniture company wanted to purchase additional units of only one of the resources, labor or wood, which should it be purchased?

A manufacturing firm produces electric washing machines and vacuum cleaner motors. The linear programming model for the number of washing machine motors (x_1) and vacuum cleaner motors (x_2) to produce is formulated as follows.

maximize $Z = 70x_1 + 80x_2$ (profit, \$)

subject to

$2x_1 + x_2 \leq 19$ (production, hr)

$x_1 + x_2 \leq 14$ (steel, lb)

$x_1 + 2x_2 \leq 20$ (wire, ft)

$x_1, x_2 \geq 0$

The final optimal simplex tableau for this problem follows.

[Page A-57]

C_j	Basic Variables	Quantity	70	80
			x_1	x_2
46. 70	x_1	6	1	0

0	s_2	1	0	0
80	x_2	7	0	1
	z_j	980	70	80
	$c_j - z_j$		0	0

- 1.** Formulate the dual for this problem
- 2.** What do the dual variables equal, they mean?
- 3.** Determine the optimal ranges for
- 4.** Determine the feasible ranges for hours), q_2 (pounds of steel), and

wire).

- 5.** Managers at the firm have determined that the firm can purchase a new production machine that will increase available production time from 19 to 25 hours. Would this change the optimal solution?

A manufacturer produces products 1 and 2. The unit profits are \$9 and \$12, respectively. Each product must undergo two production processes. The production is subject to labor constraints. There are also material and storage limitations. The linear programming model for determining the number of product 1 (x_1) and the number of product 2 to produce is given as follows.

maximize $Z = 9x_1 + 12x_2$ (profit, \$)

subject to

$$4x_1 + 8x_2 \leq 64 \text{ (process 1, labor hr)}$$

$$5x_1 + 5x_2 \leq 50 \text{ (process 2, labor hr)}$$

$$15x_1 + 8x_2 \leq 120 \text{ (material, lb)}$$

$$x_1 \leq 7 \text{ (storage space, ft}^2\text{)}$$

$$x_2 \leq 7 \text{ (storage space, ft}^2\text{)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem follows.

C_j	Basic Variables	Quantity	9	12	0
			x_1	x_2	s_1
9	x_1	4	1	0	1/4

0	s_5	1	0	0	$1/4$
0	s_3	12	0	0	$7/4$
0	s_4	3	0	0	$1/4$
12	x_2	6	0	1	$1/4$
	z_j	108	9	12	$3/4$
	c_j		0	0	$3/4$

- 1.** Formulate the dual for this problem
- 2.** What do the dual variables equal, and what does this dual solution mean?
- 3.** Determine the optimal ranges for the right-hand side of the constraints.
- 4.** Determine the range for q_1 (processing time for product 1) so that the current optimal basis remains optimal.
- 5.** Due to a problem with a supplier, only 100 pounds of material will be available for production instead of 120 pounds. What is the optimal solution mix?

A manufacturer produces products 1, 2, and 3. The three products are processed through two production operations that have time costs. The finished products are then stored. The following linear programming model has been formulated to determine the number of product 1 (x_1), product 2 (x_2), and product 3 (x_3) to produce.

maximize $Z = 40x_1 + 35x_2 + 45x_3$ (profit, \$)

subject to

$$2x_1 + 3x_2 + 2x_3 \leq 120 \text{ (operation 1, hr)}$$

$$4x_1 + 3x_2 + x_3 \leq 160 \text{ (operation 2, hr)}$$

$$3x_1 + 2x_2 + 4x_3 \leq 100 \text{ (operation 3, hr)}$$

$$x_1 + x_2 + x_3 \leq 40 \text{ (storage, ft}^2\text{)}$$

$$x_1, x_2, x_3 \geq 0$$

The final optimal simplex tableau for this follows.

		40	35	45
c_j	Basic Variables	Quantity		
		x_1	x_2	x_3

48.

0	s_1	10	$1/2$	0	0
0	s_2	60	2	0	0
45	x_3	10	$1/2$	0	1
35	x_2	30	$1/2$	1	0
	z_j	1,500	40	35	45
	$c_j - z_j$			0	0

1. Formulate the dual for this problem
2. What do the dual variables equal,

they mean?

- 3.** How does the fact that this is a maximum optimum solution affect the interpretation of the dual solution values?
- 4.** Determine the optimal range for c_1 (cost of storage space).
- 5.** Determine the feasible range for c_2 (cost of storage space).
- 6.** What is the maximum price the manufacturer would be willing to pay to lease additional storage space, and how many additional feet could be leased at that price?

A school dietitian is attempting to plan a menu that will minimize cost and meet certain dietary requirements. The two staples are meat and potatoes, which provide protein and carbohydrates. The following linear programming model has been formulated to determine

ounces of meat (x_1) and ounces of potatoes (x_2) put in a lunch.

subject to

$$4x_1 + 5x_2 \geq 20 \text{ (protein, mg)}$$

$$12x_1 + 3x_2 \geq 30 \text{ (iron, mg)}$$

$$3x_1 + 2x_2 \geq 12 \text{ (carbohydrates, mg)}$$

$$x_1, x_2 \geq 0$$

The final optimal simplex tableau for this problem follows.

c_j	Basic Variables	Quantity	0.03 0.02 0		
			x_1	x_2	s
49. 0.02	x_2	3.6	0	1	0

0.03	x_1	1.6	1	0	0
0	s_1	4.4	0	0	1
	z_j	0.12	0.03	0.02	0
	z_j	c_j	0	0	0

- 1.** Formulate the dual for this problem. What are the dual variables equal to, and what do they mean?
- 2.** Determine the optimal ranges for the right-hand side values.
- 3.** Determine the ranges for q_1 , q_2 , and q_3 (milligrams of protein, iron, and calcium respectively).

4. What would it be worth for the sc
be able to reduce the requirement
carbohydrates, and what is the sr
of milligrams of carbohydrates tha
required at that value?

The Overnight Food Processing Compa
sandwiches (among other processed fo
vending machines, markets, and busine
around the city. The sandwiches are ma
delivered early the following morning. A
not purchased during the previous day
away. Three kinds of sandwiches are m
a basic cheese sandwich (x_1), a ham s
(x_2), and a pimento cheese sandwich (
are \$1.25, \$2.00, and \$1.75, respectiv
0.5 minutes to make a cheese sandwich
to make a ham salad sandwich, and 0.3
make a pimento cheese sandwich. The
20 hours of labor available to produce
each night. The demand for ham salad
at least as great as the demand for the

cheese sandwiches combined. However, the company has only enough ham salad to produce 1,200 sandwiches per night. The company has the following linear programming model to determine how many of each type of sandwich to make to maximize profit.

$$\text{maximize } Z = \$1.25x_1 + 2.00x_2 + 1.75x_3$$

subject to

$$0.5x_1 + 1.2x_2 + 0.8x_3 \leq 1,200 \text{ (production time, min)}$$

$$x_1 + x_3 \leq x_2 \text{ (demand for ham salad sandwiches)}$$

$$x_2 \leq 500 \text{ (ham salad sandwich limit)}$$

$$x_1, x_2, x_3 \geq 0$$

[Page A-60]

The optimal simplex tableau follows.

50.

c_j	Basic	Quantity	1.25	2.00	1.75
-------	-------	----------	------	------	------

Variables			x_1	x_2	x_3
0	s_1	200	0.3	0	0
1.75	x_3	500	1	0	1
2.00	x_2	500	0	1	0
	z_j	1,875	1.75	2.00	1.75
	$c_j - z_j$		.5	0	0

1. Formulate the dual for this problem and find the optimal values of the dual variables.

- 2.** Determine the optimal ranges for
- 3.** Determine the range for q_3 (ham sandwiches).
- 4.** Overnight Foods is considering adding cheese sandwiches to increase demand. The company estimates that spending on some leaflets that would be packed with other sandwiches would increase demand for both kinds of cheese sandwiches. Should it make this expenditure?

Given the linear programming model,

$$\text{minimize } Z = 3x_1 + 5x_2 + 2x_3$$

subject to

$$x_1 + x_2 - 3x_3 \geq 35$$

$$x_1 + 2x_2 \geq 50$$

$$-x_1 + x_2 \geq 25$$

$$x_1, x_2, x_3 \geq 0$$

and its optimal simplex tableau,

51.

c_j	Basic Variables	Quantity	3	5	2
			x_1	x_2	x_3
0	s_2	15	0	0	4

3	x_1	5	1	0	2
5	x_2	30	0	1	1
	z_j	165	3	5	15
	$z_j - c_j$		0	0	15

[Page A-61]

1. Find the optimal ranges for all c_j v
2. Find the feasible ranges for all q_i v

The Sunshine Food Processing Compar

three canned fruit products: mixed fruit (cocktail (x_2), and fruit delight (x_3). The ingredients in each product are pears and peaches. Each product is produced in lots and must pass through three processes: mixing, canning, and packaging. The resource requirements for each product and process are shown in the following linear programming formulation.

$$\text{maximize } Z = 10x_1 + 6x_2 + 8x_3 \text{ (profit, \$)}$$

subject to

$$20x_1 + 10x_2 + 16x_3 \leq 320 \text{ (pears, lb)}$$

$$10x_1 + 20x_2 + 16x_3 \leq 400 \text{ (peaches, lb)}$$

$$x_1 + 2x_2 + 2x_3 \leq 43 \text{ (mixing, hr)}$$

$$x_1 + x_2 + x_3 \leq 60 \text{ (canning, hr)}$$

$$2x_1 + x_2 + x_3 \leq 40 \text{ (packaging, hr)}$$

$$x_1, x_2, x_3 \geq 0$$

The optimal simplex tableau is as follows

52.	c_j	Basic Variables	Quantity	10 6		8
				x_1	x_2	x_3
	10	x_1	8	1	0	$8/15$
	6	x_2	16	0	1	$8/15$
	0	s_3	3	0	0	$2/5$
	0	s_4	36	0	0	$1/15$
	0	s_5	8	0	0	$3/5$
	z_j		176	10	6	$128/15$

- 1.** What is the maximum price the company would be willing to pay for additional packaging time if it could be purchased at that price?
- 2.** What is the marginal value of packaging time, and what range is this price valid?
- 3.** The company can purchase a new packaging machine that would increase the available packaging time from 43 to 60 hours. Would this be an optimal solution?
- 4.** The company can also purchase a new packaging machine that would increase the available packaging time from 40 to 60 hours. Would this affect the optimal solution?
- 5.** If the manager were to attempt to purchase a new packaging machine that would increase the available packaging time from 40 to 60 hours, would this affect the optimal solution?

additional units of only one of the
which should it be?

The Evergreen Products Firm produces pressed paneling from pine and spruce. types of paneling are Western (x_1), Old and Colonial (x_3). Each sheet must be pressed. The resource requirements are following linear programming formulation

[Page A-62]

maximize $Z = 4x_1 + 10x_2 + 8x_3$ (profit, \$)

subject to

$$5x_1 + 4x_2 + 4x_3 \leq 200 \text{ (pine, lb)}$$

$$2x_1 + 5x_2 + 2x_3 \leq 160 \text{ (spruce, lb)}$$

$$x_1 + x_2 + 2x_3 \leq 50 \text{ (cutting, hr)}$$

$$2x_1 + 4x_2 + 2x_3 \leq 80 \text{ (pressing, hr)}$$

$$x_1, x_2, x_3 \geq 0$$

The optimal simplex tableau is as follow

c_j	Basic Variables	Quantity	4 10 8			
			x_1	x_2	x_3	s_1
0	s_1	80	$7/3$	0	0	
0	s_2	70	$1/3$	0	0	
8	x_3	20	$1/3$	0	1	
10	x_2	10	$1/3$	1	0	
	z_j	260	6	10	8	

53.

c_j z_j

2 0 0

- 1.** What is the marginal value of an additional unit of spruce? Over what range is this valid?
- 2.** What is the marginal value of an additional unit of cutting? Over what range is this valid?
- 3.** Given a choice between securing 100 hours or more pressing hours, what would management select? Why?
- 4.** If the amount of spruce available were decreased from 160 to 100, would this reduction affect the solution?
- 5.** What unit profit would have to be obtained from Western paneling before manager would consider producing it?

- 6.** Management is considering changing Colonial paneling from \$8 to \$13. How will this change affect the solution?

A manufacturing firm produces four products. Each product requires material and machine time. The linear programming model formulated to determine the number of product 1 (x_1), product 2 (x_2), product 3 (x_3), and product 4 (x_4) to produce is

$$\text{maximize } Z = 2x_1 + 8x_2 + 10x_3 + 6x_4 \text{ (profit, \$)}$$

subject to

$$2x_1 + x_2 + 4x_3 + 2x_4 \leq 200 \text{ (material, lb)}$$

$$x_1 + 2x_2 + 2x_3 + x_4 \leq 160 \text{ (machine processing, hr)}$$

$$x_1, x_2, x_3, x_4 \geq 0$$

- 54.** The optimal simplex tableau is as follows

C_j	Basic Variables	Quantity	2	8	10
			x_1	x_2	x_3
6	x_4	80	1	0	2
8	x_2	40	0	1	0
	Z_j	800	6	8	12
	$C_j - Z_j$		4	0	2

What is the marginal value of an additic

material? Over what range is this value

Module B.

Transportation and Assignment Solution Methods

[Solution of the
Transportation Model](#)

[Solution of the Assignment
Model](#)

[Problems](#)

Solution of the Transportation Model

The following example was used in [chapter 6](#) of the text to demonstrate the formulation of the transportation model.

Wheat is harvested in the Midwest and stored in grain elevators in three different cities: Kansas City, Omaha, and Des Moines. These grain elevators supply three flour mills, located in Chicago, St. Louis, and Cincinnati. Grain is

shipped to the mills in railroad cars, each car capable of holding one ton of wheat. Each grain elevator is able to supply the following number of tons (i.e., railroad cars) of wheat to the mills on a monthly basis.

<i>Grain Elevator</i>	<i>Supply</i>
------------------------------	----------------------

1. Kansas City	150
----------------	-----

2. Omaha	175
----------	-----

3. Des Moines	275
---------------	-----

Total	600 tons
-------	----------

Each mill demands the following number of tons of wheat per month.

<i>Mill</i>	<i>Demand</i>
A. Chicago	200
B. St. Louis	100
C. Cincinnati	300

Total

600 tons

The cost of transporting one ton of wheat from each grain elevator (source) to each mill (destination) differs according to the distance and rail system. These costs are shown in the following table. For example, the cost of shipping one ton of wheat from the grain elevator at Omaha to the mill at Chicago is \$7.

Grain Elevator	Mill		
	<i>A. Chicago</i>	<i>B. St. Louis</i>	<i>C. Cincinnati</i>
1. Kansas City	\$6	\$ 8	\$10
2. Omaha	7	11	11
3. Des Moines	4	5	12

The problem is to determine how many tons of wheat to

transport from each grain elevator to each mill on a monthly basis in order to minimize the total cost of transportation.

The linear programming model for this problem is formulated in the equations that follow.

[Page B-3]

$$\begin{aligned} \text{minimize } Z = & \$6x_{1A} + 8x_{1B} + \\ & 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} \\ & + 4x_{3A} + 5x_{3B} + 12x_{3C} \end{aligned}$$

subject to

$$x_{1A} + x_{1B} + x_{1C} = 150$$

$$x_{2A} + x_{2B} + x_{2C} = 175$$

$$x_{3A} + x_{3B} + x_{3C} = 275$$

$$x_{1A} + x_{2A} + x_{3A} = 200$$

$$x_{1B} + x_{2B} + x_{3B} = 100$$

$$x_{1C} + x_{2C} + x_{3C} = 300$$

$$x_{ij} \geq 0$$

In this model the decision variables, x_{ij} , represent the number of tons of wheat

transported from each grain elevator, i (where $i = 1, 2, 3$), to each mill, j (where $j = A, B, C$). The objective function represents the total transportation cost for each route. Each term in the objective function reflects the cost of the tonnage transported for one route. For example, if 20 tons are transported from elevator 1 to mill A, the cost of \$6 is multiplied by $x_{1A}(= 20)$, which equals \$120.

The first three constraints in the linear programming model

represent the supply at each elevator; the last three constraints represent the demand at each mill. As an example, consider the first supply constraint, $x_{1A} + x_{1B} + x_{1C} = 150$. This constraint represents the tons of wheat transported from Kansas City to all three mills: Chicago (x_{1A}), St. Louis (x_{1B}), and Cincinnati (x_{1C}). The amount transported from Kansas City is limited to the 150 tons available. Note that this constraint (as well as all others) is an equation ($=$) rather than a $\leq$ inequality

because all of the tons of wheat available will be needed to meet the *total demand* of 600 tons. In other words, the three mills demand 600 total tons, which is the exact amount that can be supplied by the three grain elevators. Thus, all that *can* be supplied *will be*, in order to meet demand. This type of model, in which supply exactly equals demand, is referred to as a *balanced transportation* model. The balanced model will be used to demonstrate the solution of a transportation problem.

Transportation problems are solved manually within a tableau format.

Transportation models are solved manually within the context of a *tableau*, as in the simplex method. The tableau for our wheat transportation model is shown in [Table B-1](#).

Table B-1. The

Transportation Tableau

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Each cell in the tableau represents the amount transported from one source to one destination. Thus, the amount placed in each cell is the value of a decision variable for that cell. For example, the cell at the intersection of row 1 and column A represents the decision variable x_{1A} . The smaller box within each cell contains the unit transportation cost for that route. For example, in cell 1A the value, \$6, is the cost of transporting one ton of wheat from Kansas City to Chicago. Along the

outer rim of the tableau are the supply and demand constraint quantity values, which are referred to as *rim requirements*.

Each cell in a transportation tableau is analogous to a decision variable that indicates the amount allocated from a source to a destination.

*The supply and demand values along the outside rim of a tableau are called **rim requirements**.*

[Page B-4]

The two methods for solving a transportation model are the *stepping-stone method* and the *modified distribution method* (also known as MODI). In applying the simplex method,

an initial solution had to be established in the initial simplex tableau. This same condition must be met in solving a transportation model. In a transportation model, an initial feasible solution can be found by several alternative methods, including the *northwest corner method*, the *minimum cell cost method*, and *Vogel's approximation model*.

Transportation models do not start at the origin where all decision variables

equal zero; they must be given an initial feasible solution.

The Northwest Corner Method

With the northwest corner method, an initial allocation is made to the cell in the upper lefthand corner of the tableau

(i.e., the "northwest corner"). The amount allocated is the most possible, *subject* to the supply and demand constraints for that cell. In our example, we first allocate as much as possible to cell 1A (the northwest corner). This amount is 150 tons, since that is the maximum that can be supplied by grain elevator 1 at Kansas City, even though 200 tons are demanded by mill A at Chicago. This initial allocation is shown in [Table B-2](#).

Table B-2. The Initial NW Corner Solution

From \ To	A	B	C	Supply
1	150	0	0	150
2	50	100	25	175
3	0	0	275	275
Demand	200	100	300	600

*In the **northwest corner method** the largest possible allocation is made to the cell in the upper left-hand corner of the tableau, followed by allocations to adjacent **feasible** cells.*

We next allocate to a cell

adjacent to cell 1A, in this case either cell 2A or cell 1B.

However, cell 1B no longer represents a feasible allocation, because the total tonnage of wheat available at source 1 (i.e., 150 tons) has already been allocated. Thus, cell 2A represents the only feasible alternative, and as much as possible is allocated to this cell. The amount allocated at 2A can be either 175 tons, the supply available from source 2 (Omaha), or 50 tons, the amount *now* demanded at destination A. (Recall that 150

of the 200 tons demanded at A have already been supplied.) Because 50 tons is the most constrained amount, it is allocated to cell 2A, as shown in [Table B-2](#).

The third allocation is made in the same way as the second allocation. The only feasible cell adjacent to cell 2A is cell 2B. The most that can be allocated is either 100 tons (the amount demanded at mill B) or 125 tons (175 tons minus the 50 tons allocated to cell 2A). The smaller (most constrained) amount, 100 tons, is allocated

to cell 2B, as shown in [Table B-2](#).

The fourth allocation is 25 tons to cell 2C, and the fifth allocation is 275 tons to cell 3C, both of which are shown in [Table B-2](#). Notice that all of the row and column allocations add up to the appropriate rim requirements.

The initial solution is complete when all rim requirements are satisfied.

The transportation cost of this solution is computed by substituting the cell allocations (i.e., the amounts transported),

$$x_{1A} = 150$$

$$x_{2A} = 50$$

$$x_{2B} = 100$$

$$x_{2C} = 25$$

$$x_{3C} = 275$$

into the objective function.

$$\begin{aligned}
 Z &= \$6x_{1A} + 8x_{1B} + 10x_{1C} + 7x_{2A} + 11x_{2B} + 11x_{2C} + 4x_{3A} + 5x_{3B} + 12x_{3C} \\
 &= 6(150) + 8(0) + 10(0) + 7(50) + 11(100) + 11(25) + 4(0) + 5(0) + 12(275) \\
 &= \$5,925
 \end{aligned}$$

The steps of the northwest corner method are summarized here.

- 1.** Allocate as much as possible to the cell in the upper left-hand corner, subject to the supply and demand constraints.
- 2.** Allocate as much as possible to the next adjacent feasible cell.
- 3.** Repeat step 2 until all rim

requirements have been met.

The Minimum Cell Cost Method

With the minimum cell cost method, the basic logic is to allocate to the cells with the lowest costs. The initial allocation is made to the cell in the tableau having the lowest cost. In the transportation tableau for our example problem, cell 3A has the minimum cost of \$4. As much

as possible is allocated to this cell; the choice is either 200 tons or 275 tons. Even though 275 tons could be supplied to cell 3A, the most we can allocate is 200 tons, since only 200 tons are demanded. This allocation is shown in [Table B-3](#).

Table B-3. The Initial Minimum Cell Cost Allocation

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

*In the **minimum cell cost method** as much as possible is allocated to the cell with the minimum cost.*

Notice that all of the remaining cells in column A have now been eliminated, because all of the wheat demanded at destination A, Chicago, has now been supplied by source 3, Des Moines.

The next allocation is made to the cell that has the minimum cost and also is feasible. This is cell 3B which has a cost of \$5. The most that can be allocated is 75 tons (275 tons minus the 200 tons already supplied). This allocation is shown in [Table B-4](#).

Table B-4. The Second Minimum Cell Cost Allocation

(This item is displayed on page B-6 in the print version)

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

The third allocation is made to cell 1B, which has the minimum cost of \$8. (Notice that cells with lower costs, such

as 1A and 2A, are not considered because they were previously ruled out as infeasible.) The amount allocated is 25 tons. The fourth allocation of 125 tons is made to cell 1C, and the last allocation of 175 tons is made to cell 2C. These allocations, which complete the initial minimum cell cost solution, are shown in [Table B-5](#).

[Page B-6]

Table B-5. The Initial

Solution

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

The total cost of this initial

solution is \$4,550, as compared to a total cost of \$5,925 for the initial northwest corner solution. It is not a coincidence that a lower total cost is derived using the minimum cell cost method; it is a logical occurrence. The northwest corner method does not consider cost at all in making allocations the minimum cell cost method does. It is therefore quite natural that a lower initial cost will be attained using the latter method. Thus, the initial solution achieved by using the

minimum cell cost method is usually better in that, because it has a lower cost, it is closer to the optimal solution; fewer subsequent iterations will be required to achieve the optimal solution.

*The **minimum cell cost method** will provide a solution with a lower cost than the northwest corner solution because it considers cost in the allocation process.*

The specific steps of the minimum cell cost method are summarized next.

- 1.** Allocate as much as possible to the feasible cell with the minimum transportation cost, and adjust the rim requirements.
- 2.** Repeat step 1 until all rim requirements have been met.

Vogel's Approximation Model

The third method for determining an initial solution, Vogel's approximation model (also called VAM), is based on the concept of penalty cost or *regret*. If a decision maker incorrectly chooses from several alternative courses of action, a penalty may be suffered (and the decision maker may regret the decision that was made). In a transportation problem, the

courses of action are the alternative routes, and a wrong decision is allocating to a cell that does not contain the lowest cost.

A penalty cost is the difference between the largest and next largest cell cost in a row (or column).

In the VAM method, the first

step is to develop a penalty cost for each source and destination. For example, consider column A in [Table B-6](#). Destination A, Chicago, can be supplied by Kansas City, Omaha, and Des Moines. The best decision would be to supply Chicago from source 3 because cell 3A has the minimum cost of \$4. If a wrong decision were made and the next higher cost of \$6 were selected at cell 1A, a "penalty" of \$2 per ton would result (i.e., $\$6 - \$4 = \$2$). This demonstrates how the penalty

cost is determined for each row and column of the tableau. The general rule for computing a penalty cost is to subtract the minimum cell cost from the next higher cell cost in each row and column. The penalty costs for our example are shown at the right and at the bottom of [Table B-6](#).

[Page B-7]

Table B-6. The VAM Penalty Costs

From \ To	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	4
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	1		

VAM allocates as much as possible to the minimum cost cell

*in the row or column
with the largest
penalty cost.*

The initial allocation in the VAM method is made in the row or column that has the highest penalty cost. In [Table B-6](#), row 2 has the highest penalty cost of \$4. We allocate as much as possible to the feasible cell in this row with the minimum cost. In row 2, cell 2A has the lowest cost of \$7, and the most

that can be allocated to cell 2A is 175 tons. With this allocation the greatest penalty cost of \$4 has been avoided because the best course of action has been selected. The allocation is shown in [Table B-7](#).

Table B-7. The Initial VAM Allocation

From \ To	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	
3	4	5	12	275	1
Demand	200	100	300	600	
	2	3	1		

After the initial allocation is made, *all* of the penalty costs must be recomputed. In some

allocacases the penalty costs will change or be eliminated; in other cases they will not change. For example, the penalty cost in row 2 was eliminated altogether (because no more allocations are possible for that row).

After each VAM cell allocation, all row and column penalty costs are recomputed

Next, we repeat the previous step and allocate to the row or column with the highest penalty cost, which is now column B with a penalty cost of \$3 (see [Table B-7](#)). The cell in column B with the lowest cost is 3B, and we allocate as much as possible to this cell, 100 tons. This allocation is shown in [Table B-8](#).

Table B-8. The Second VAM

Allocation

From \ To	A	B	C	Supply	
1	6	8	10	150	2
2	7	11	11	175	
3	4	5	12	275	8
Demand	200	100	300	600	
	2		1		

Note that all penalty costs have been recomputed in [Table B-8](#). Since the highest penalty cost is now \$8 for row 3 and since cell 3A has the minimum cost of \$4, we allocate 25 tons to this cell, as shown in [Table B-9](#).

Table B-9. The Third VAM Allocation

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

1

Table B-9 also shows the recomputed penalty costs after the third allocation. Notice that

by now only column C has a penalty cost. Rows 1 and 3 have only one feasible cell, so a penalty does not exist for these rows. Thus, the last two allocations are made to column C. First, 150 tons are allocated to cell 1C because it has the lowest cell cost. This leaves only cell 3C as a feasible possibility, so 150 tons are allocated to this cell. Both of these allocations are shown in [Table B-10](#).

Table B-10. The

Initial VAM Solution

(This item is displayed on page B-9 in the print version)

From \ To				Supply
	A	B	C	
1	6 150	8 150	10 150	150
2	7 175	11 175	11 175	175
3	4 25	5 100	12 150	275
Demand	200	100	300	600

The total cost of this initial Vogel's approximation model solution is \$5,125, which is not as high as the northwest corner initial solution of \$5,925. It is also not as low as the minimum cell cost solution of \$4,550. Like the minimum cell cost method, VAM typically results in a lower cost for the initial solution than does the northwest corner method.

VAM and minimum cell cost both provide better initial solutions than the northwest

The steps of Vogel's approximation model can be summarized in the following list.

- 1.** Determine the penalty cost for each row and column by subtracting the lowest cell cost in the row or column from the next lowest cell

cost in the same row or column.

- 2.** Select the row or column with the highest penalty cost (breaking ties arbitrarily or choosing the lowest-cost cell).
- 3.** Allocate as much as possible to the feasible cell with the lowest transportation cost in the row or column with the highest penalty cost.
- 4.** Repeat steps 1, 2, and 3 until all rim requirements

have been met.

The Stepping-Stone Solution Method

Once an initial *basic feasible solution* has been determined by any of the previous three methods, the next step is to solve the model for the optimal (i.e., minimum total cost) solution. There are two basic solution methods: the *stepping-stone solution method* and the modified distribution method (MODI). The stepping-stone

solution method will be demonstrated first. Because the initial solution obtained by the minimum cell cost method had the lowest total cost of the three initial solutions, we will use it as the starting solution. [Table B-11](#) repeats the initial solution that was developed from the minimum cell cost method.

Table B-11. The Minimum Cell Cost Solution

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

Once an initial solution is derived, the problem must be solved using either

the stepping-stone method or MODI.

The basic solution principle in a transportation problem is to determine whether a transportation route not at present being used (i.e., an empty cell) would result in a lower total cost if it were used. For example, [Table B-11](#) shows four empty cells (1A, 2A, 2B, 3C) representing unused routes. Our first step in the

stepping-stone method is to evaluate these empty cells to see whether the use of any of them would reduce total cost. If we find such a route, then we will allocate as much as possible to it.

[Page B-10]

*The **stepping-stone method** determines if there is a cell with no allocation that would reduce cost if used.*

First, let us consider allocating one ton of wheat to cell 1A. If one ton is allocated to cell 1A, cost will be increased by \$6 the transportation cost for cell 1A. However, by allocating one ton to cell 1A, we increase the supply in row 1 to 151 tons, as shown in [Table B-12](#).

Table B-12. The Allocation of One Ton to Cell 1A

From \ To	A	B	C	Supply	
1	+1 6	8	10	150	151
2	7	11	11	175	
3	4	5	12	275	
Demand	200	100	300	600	

The constraints of the problem cannot be violated, and feasibility must be maintained. If we add one ton to cell 1A, we

must subtract one ton from another allocation along that row. Cell 1B is a logical candidate because it contains 25 tons. By subtracting one ton from cell 1B, we now have 150 tons in row 1, and we have satisfied the supply constraint again. At the same time, subtracting one ton from cell 1B has reduced total cost by \$8.

However, by subtracting one ton from cell 1B, we now have only 99 tons allocated to column B, where 100 tons are demanded, as shown in [Table](#)

B-13. To compensate for this constraint violation, one ton must be added to *a cell that already has an allocation*. Since cell 3B has 75 tons, we will add one ton to this cell, which again satisfies the demand constraint of 100 tons.

Table B-13. The Subtraction of One Ton from Cell 1B

From \ To	A		B		C		Supply
	+1	6	-1	8		10	
1			25		125		150
		7		11		11	
2					175		175
		4		5		12	
3	200		75				275
Demand	200		100		300		600

99

A requirement of this solution method is that units can only be added to and subtracted

from cells that already have allocations. That is why one ton was added to cell 3B and not to cell 2B. It is from this requirement that the method derives its name. The process of adding and subtracting units from allocated cells is analogous to crossing a pond by stepping on stones (i.e., only allocated-to cells).

[Page B-11]

By allocating one extra ton to cell 3B we have increased cost by \$5, the transportation cost

for that cell. However, we have also increased the supply in row 3 to 276 tons, a violation of the supply constraint for this source. As before, this violation can be remedied by subtracting one ton from cell 3A, which contains an allocation of 200 tons. This satisfies the supply constraint again for row 3, and it also reduces the total cost by \$4, the transportation cost for cell 3A. These allocations and deletions are shown in [Table B-14](#).

**Table B-14. The
Addition of One Ton
to Cell 3B and the
Subtraction of One
Ton from Cell 3A**

From \ To	A	B	C	Supply
1	+1 ↑ 6 → -1	8	10	150
2	7	11	11	175
3	-1 ← 4	+1 ↓ 5	12	275
Demand	200	100	300	600

Notice in [Table B-14](#) that by subtracting one ton from cell 3A, we did not violate the demand constraint for column

A, since we previously added one ton to cell 1A.

Now let us review the increases and reductions in costs resulting from this process. We initially increased cost by \$6 at cell 1A, then reduced cost by \$8 at cell 1B, then increased cost by \$5 at cell 3B, and, finally, reduced cost by \$4 at cell 3A.

$$1A \rightarrow 1B \rightarrow 3B \rightarrow 3A$$
$$+\$6 - 8 + 5 - 4 = -\$1$$

In other words, for each ton allocated to cell 1A (a route not

at present used), total cost will be reduced by \$1. This indicates that the initial solution is not optimal because a lower cost can be achieved by allocating additional tons of wheat to cell 1A (i.e., cell 1A is analogous to a pivot column in the simplex method). Our goal is to determine the cell or entering "variable" that will reduce cost the most. Another variable (empty cell) may result in an even greater decrease in cost than cell 1A. If such a cell exists, it will be selected as the entering

variable; if not, cell 1A will be selected. To identify the appropriate entering variable, the remaining empty cells must be tested as cell 1A was.

An empty cell that will reduce cost is a potential entering variable.

Before testing the remaining empty cells, let us identify a few of the general

characteristics of the stepping-stone process. First, we always start with an empty cell and form a *closed path* of cells that now have allocations. In developing the path, it is possible to skip over both unused and used cells. In any row or column there can be only *one* addition and *one* subtraction. (For example, in row 1, wheat is added at cell 1A and is subtracted at cell 1B.)

To evaluate the cost reduction potential of

an empty cell, a closed path connecting used cells to the empty cell is identified.

Let us test cell 2A to see if it results in a cost reduction. The stepping-stone closed path for cell 2A is shown in [Table B-15](#). Notice that the path for cell 2A is slightly more complex than the path for cell 1A. Notice also that the path crosses itself at

one point, which is perfectly acceptable. An allocation to cell 2A will reduce cost by \$1, as shown in the computation in [Table B-15](#). Thus, we have located another possible entering variable, although it is no better than cell 1A.

Table B-15. The Stepping-Stone Path for Cell 2A

(This item is displayed on page B-12 in the print version)

From \ To	A	B	C	Supply
1	6 -	8 +	10	150
2	+ 7	11 -	11	175
3	- 4	+ 5	12	275
Demand	200	100	300	600

$2A \rightarrow 2C \rightarrow 1C \rightarrow 1B \rightarrow 3B \rightarrow 3A$
 $+\$7 - 11 + 10 - 8 + 5 - 4 = -\1

The remaining stepping-stone paths and the resulting

computations for cells 2B and 3C are shown in [Tables B-16](#) and [B-17](#), respectively.

[Page B-12]

Table B-16. The Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	6 +	8 -	10	150
2	7 -	11 +	11	175
3	4	5	12	275
Demand	200	100	300	600

$2B \rightarrow 2C \rightarrow 1C \rightarrow 1B$

$+\$11 - 11 + 10 - 8 = +\2

Table B-17. The

Stepping-Stone Path for Cell 3C

From \ To	A	B	C	Supply
1	6 +	8 -	10	150
2	7	11	11	175
3	4 -	5 +	12	275
Demand	200	100	300	600

3C → 1C → 1B → 3B

$$+\$12 - 10 + 8 - 5 = +\$5$$

Notice that after all four unused routes are evaluated, there is a tie for the entering variable between cells 1A and 2A. Both show a reduction in cost of \$1 per ton allocated to that route. The tie can be broken arbitrarily. We will select cell 1A (i.e., x_{1A}) to enter the solution.

After all empty cells are evaluated, the one with the greatest cost reduction

*potential is the
entering variable.*

[Page B-13]

Because the total cost of the model will be reduced by \$1 for each ton we can reallocate to cell 1A, we naturally want to reallocate as much as possible. To determine how much to allocate, we need to look at the path for cell 1A again, as shown in [Table B-18](#).

Table B-18. The Stepping-Stone Path for Cell 1A

From \ To	A	B	C	Supply
1	+ 6	- 8	10	150
2	7	11	11	175
3	- 4	+ 5	12	275
Demand	200	100	300	600

The stepping-stone path in [Table B-18](#) shows that tons of wheat must be subtracted at cells 1B and 3A to meet the rim requirements and thus satisfy the model constraints. Because we cannot subtract more than is available in a cell, we are limited by the 25 tons in cell 1B. In other words, if we allocate more than 25 tons to cell 1A, then we must subtract more than 25 tons from 1B, which is impossible because only 25 tons are available.

Therefore, 25 tons is the amount we reallocate to cell 1A according to our path. That is, 25 tons are added to 1A, subtracted from 1B, added to 3B, and subtracted from 3A. This reallocation is shown in [Table B-19](#).

Table B-19. The Second Iteration of the Stepping-Stone Method

From \ To	A	B	C	Supply
1	25		125	150
2			175	175
3	175	100		275
Demand	200	100	300	600

When reallocating units to the entering variable (cell), the amount is the

*minimum amount
subtracted on the
stepping-stone path.*

The process culminating in [Table B-19](#) represents one *iteration* of the stepping-stone method. We selected x_{1A} as the entering variable, and it turned out that x_{1B} was the leaving variable (because it now has a value of zero in [Table B-19](#)). Thus, at each iteration one variable enters and one leaves

(just as in the simplex method).

Now we must check to see whether the solution shown in [Table B-19](#) is, in fact, optimal. We do this by plotting the paths for the unused routes (i.e., empty cells 2A, 1B, 2B, and 3C) that are shown in [Table B-19](#). These paths are shown in [Tables B-20](#) through [B-23](#).

[Page B-14]

Table B-20. The

Stepping-Stone Path for Cell 2A

From \ To	A	B	C	Supply
1	- 6 25	8	+ 10 125	150
2	+ 7	11	- 11 175	175
3	4 175	5 100	12	275
Demand	200	100	300	600

2A → 2C → 1C → 1A

$$+\$7 - 11 + 10 - 6 = \$0$$

**Table B-21. The
Stepping-Stone Path
for Cell 1B**

From \ To	A		B		C	Supply
	-	6	+	8	10	
1	25				125	150
		7		11	11	
2					175	175
	+	4	-	5	12	
3	175		100			275
Demand	200		100		300	600

$1B \rightarrow 3B \rightarrow 3A \rightarrow 1A$

$+\$8 - 5 + 4 - 6 = +\1

Table B-22. The

Stepping-Stone Path for Cell 2B

From \ To	A	B	C	Supply
1	- 6 25	8	+ 10 125	150
2	7	+ 11	- 11 175	175
3	+ 4 175	- 5 100	12	275
Demand	200	100	300	600

2B → 3B → 3A → 1A → 1C → 2C

$$+\$11 - 5 + 4 - 6 + 10 - 11 = +\$3$$

**Table B-23. The
Stepping-Stone Path
for Cell 3C**

From \ To	A	B	C	Supply
1	+ 6 25	8	- 10 125	150
2	7	11	11 175	175
3	- 4 175	5 100	+ 12	275
Demand	200	100	300	600

$3C \rightarrow 3A \rightarrow 1A \rightarrow 1C$

$$+\$12 - 4 + 6 - 10 = +\$4$$

Our evaluation of the four paths indicates no cost

reductions; therefore, the solution shown in [Table B-19](#) is optimal. The solution and total minimum cost are

$$x_{1A} = 25 \text{ tons}$$

$$x_{2C} = 175 \text{ tons}$$

$$x_{3A} = 175 \text{ tons}$$

$$x_{1C} = 125 \text{ tons}$$

$$x_{3B} = 100 \text{ tons}$$

$$\begin{aligned} Z &= \$6(25) + 8(0) + 10(125) + 7(0) + 11(0) + 11(175) + 4(175) + 5(100) + 12(0) \\ &= \$4,525 \end{aligned}$$

The stepping-stone process is repeated until none of the empty cells will reduce cost (i.e., an optimal solution).

However, notice in [Table B-20](#) that the path for cell 2A resulted in a cost change of \$0. In other words, allocating to this cell would neither increase nor decrease total cost. This situation indicates that the problem has *multiple optimal solutions* of the text. Thus, x_{2A} could be entered into the solution and there would not be a change in the total minimum cost of \$4,525. To identify the alternative solution, we would allocate as much as possible to

cell 2A, which in this case is 25 tons of wheat. The alternative solution is shown in [Table B-24](#).

Table B-24. The Alternative Optimal Solution

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Demand	200	100	300	600

A multiple optimal solution occurs when an empty cell has a cost change of

*zero and all other
empty cells are
positive.*

*An alternate optimal
solution is
determined by
allocating to the
empty cell with a
zero cost change.*

The solution in [Table B-24](#) also results in a total minimum cost of \$4,525. The steps of the stepping-stone method are summarized here.

- 1.** Determine the stepping-stone paths and cost changes for each empty cell in the tableau.
- 2.** Allocate as much as possible to the empty cell with the greatest net decrease in cost.
- 3.** Repeat steps 1 and 2 until all empty cells have

positive cost changes that indicate an optimal solution.

The Modified Distribution Method

The *modified distribution method (MODI)* is basically a modified version of the steppingstone method. However, in the MODI method the individual cell cost changes are determined mathematically, without identifying all of the stepping-stone paths for the

empty cells.

[Page B-16]

To demonstrate *MODI*, we will again use the initial solution obtained by the minimum cell cost method. The tableau for the initial solution with the modifications required by *MODI* is shown in [Table B-25](#).

Table B-25. The Minimum Cell Cost Initial Solution

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	6	8	10	150
$u_2 =$	2	7	11	11	175
$u_3 =$	3	4	5	12	275
	Demand	200	100	300	600

MODI is a modified
version of the
stepping-stone

*method in which
math equations
replace the stepping-
stone paths.*

The extra left-hand column with the u_i symbols and the extra top row with the v_j symbols represent column and row values that must be computed in *MODI*. These values are computed for all *cells with allocations* by using the following formula.

$$u_i + v_j = c_{ij}$$

The value c_{ij} is the unit transportation cost for cell ij . For example, the formula for cell 1B is

$$u_1 + v_B = c_{1B}$$

and, since $c_{1B} = 8$,

$$u_1 + v_B = 8$$

The formulas for the remaining cells that presently contain allocations are

$$x_{1C}: u_1 + v_C = 10$$

$$x_{2C}: u_2 + v_C = 11$$

$$x_{3A}: u_3 + v_A = 4$$

$$x_{3B}: u_3 + v_B = 5$$

Now there are five equations with six unknowns. To solve these equations, it is necessary to assign only one of the unknowns a value of zero.

Thus, if we let $u_1 = 0$, we can solve for all remaining u_i and v_j values.

$$x_{1B}: u_1 + v_B = 8$$

$$0 + v_B = 8$$

$$v_B = 8$$

$$x_{1C}: u_1 + v_C = 10$$

$$0 + v_C = 10$$

$$v_C = 10$$

$$x_{2C}: u_2 + v_C = 11$$

$$u_2 + 10 = 11$$

$$u_2 = 1$$

$$x_{3B}: u_3 + v_B = 5$$

$$u_3 + 8 = 5$$

$$u_3 = -3$$

$$x_{3A}: u_3 + v_A = 4$$

$$-3 + v_A = 4$$

$$v_A = 7$$

Notice that the equation for cell 3B had to be solved before the cell 3A equation could be solved. Now all the u_i and v_j values can be substituted into the tableau, as shown in [Table B-26](#).

Table B-26. The Initial Solution with All u_i and v_j Values

	v_j	$v_A = 7$	$v_B = 8$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	6	8	10	150
$u_2 = 1$	2	7	11	11	175
$u_3 = -3$	3	4	5	12	275
	Demand	200	100	300	600

Next, we use the following formula to evaluate all *empty cells*:

$$c_{ij} u_i v_j = k_{ij}$$

where k_{ij} equals the cost increase or decrease that would occur by allocating to a cell.

For the *empty cells* in [Table B-26](#), the formula yields the following values:

$$x_{1A}: k_{1A} = c_{1A} u_1 v_A = 6 \ 0 \ 7 = 1$$

$$x_{2A}: k_{2A} = c_{2A} u_2 v_A = 7 \ 1 \ 7 = 1$$

$$x_{2B}: k_{2B} = c_{2B} u_2 v_B = 11 \ 1 \ 8 = +2$$

$$x_{3C}: k_{3C} = c_{3C} u_3 v_C = 12 (3) 10 = + 5$$

These calculations indicate that either cell 1A or cell 2A will decrease cost by \$1 per allocated ton. Notice that those are exactly the same cost changes for all four empty cells as were computed in the stepping-stone method. That is, the same information is obtained by evaluating the paths in the stepping-stone method and by using the mathematical formulas of the

MODI.

Each MODI allocation replicates the stepping-stone allocation.

We can select either cell 1A or 2A to allocate to because they are tied at 1. If cell 1A is selected as the entering nonbasic variable, then the stepping-stone path for that cell must be determined so that

we know how much to reallocate. This is the same path previously identified in [Table B-18](#). Reallocating along this path results in the tableau shown in [Table B-27](#) (and previously shown in [Table B-19](#)).

Table B-27. The Second Iteration of the MODI Solution Method

(This item is displayed on page B-18 in the print version)

	v_j	$v_A =$	$v_B =$	$v_C =$	
u_i	From \ To	A	B	C	Supply
$u_1 =$	1	25	6	8	150
$u_2 =$	2	7	11	11	175
$u_3 =$	3	4	5	12	275
	Demand	200	100	300	600

After each allocation to an empty cell, the u_i and v_j values must

be recomputed.

The u_i and v_j values for [Table B-27](#) must now be recomputed using our formula for the allocated-to cells.

$$\begin{array}{lcl} x_{1A}: & u_1 + v_A = 6 \\ & 0 + v_A = 6 \\ & v_A = 6 \\ x_{1C}: & u_1 + v_C = 10 \\ & 0 + v_C = 10 \\ & v_C = 10 \\ x_{2C}: & u_2 + v_C = 11 \\ & u_2 + 10 = 11 \\ & u_2 = 1 \end{array}$$

$$\begin{aligned}x_{3A}: \quad u_3 + v_A &= 4 \\u_3 + 6 &= 4 \\u_3 &= -2 \\x_{3B}: \quad u_3 + v_B &= 5 \\-2 + v_B &= 5 \\v_B &= 7\end{aligned}$$

These new u_i and v_j values are shown in [Table B-28](#).

Table B-28. The New u_i and v_j Values for the Second Iteration

	v_j	$v_A = 6$	$v_B = 7$	$v_C = 10$	
u_i	From \ To	A	B	C	Supply
$u_1 = 0$	1	25	6	8	150
$u_2 = 1$	2	7	11	11	175
$u_3 = -2$	3	4	5	12	275
	Demand	200	100	300	600

The cost changes for the empty cells are now computed using the formula $c_{ij} u_i v_j = k_{ij}$.

$$x_{1A}: k_{1B} = c_{1B} u_1 v_B = 8 \ 0 \ 7 = +1$$

$$x_{2A}: k_{2A} = c_{2A} u_2 v_A = 7 \ 1 \ 6 = 0$$

$$x_{2B}: k_{2B} = c_{2B} u_2 v_B = 11 \ 1 \ 7 = +3$$

$$x_{3C}: k_{3C} = c_{3C} u_3 v_C = 12 \ (2) \ 10 = +4$$

Because none of these values is negative, the solution shown in [Table B-28](#) is optimal. However, as in the stepping-stone method, cell 2A with a zero

cost change indicates a multiple optimal solution.

[Page B-19]

The steps of the modified distribution method can be summarized as follows.

- 1.** Develop an initial solution using one of the three methods available.
- 2.** Compute u_i and v_j values for each row and column by applying the formula $u_i + v_j = c_{ij}$ to each cell that has

an allocation.

- 3.** Compute the cost change, k_{ij} , for each empty cell using $c_{ij} - u_i - v_j = k_{ij}$.
- 4.** Allocate as much as possible to the empty cell that will result in the greatest net decrease in cost (most negative k_{ij}).
Allocate according to the stepping-stone path for the selected cell.
- 5.** Repeat steps 2 through 4 until all k_{ij} values are

positive or zero.

The Unbalanced Transportation Model

Thus far, the methods for determining an initial solution and an optimal solution have been demonstrated within the context of a balanced transportation model.

Realistically, however, an *unbalanced problem* is a more likely occurrence. Consider our example of transporting wheat. By changing the demand at

Cincinnati to 350 tons, we create a situation in which total demand is 650 tons and total supply is 600 tons.

To compensate for this difference in the transportation tableau, a "dummy" row is added to the tableau, as shown in [Table B-29](#). The dummy row is assigned a supply of 50 tons to balance the model. The additional 50 tons demanded, which cannot be supplied, will be allocated to a cell in the dummy row. The transportation costs for the cells in the dummy row are zero because

the tons allocated to these cells are not amounts really transported but the amounts by which demand was not met. These dummy cells are, in effect, *slack variables*.

**Table B-29. An
Unbalanced Model
(Demand > Supply)**

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	175
3	4	5	12	275
Dummy	0	0	0	50
Demand	200	100	350	650

When demand exceeds supply, a

*dummy row is added
to the tableau.*

Now consider our example with the supply at Des Moines increased to 375 tons. This increases total supply to 700 tons, while total demand remains at 600 tons. To compensate for this imbalance, we add a dummy column instead of a dummy row, as shown in [Table B-30](#).

Table B-30. An Unbalanced Model (Supply > Demand)

(This item is displayed on page B-20 in the print version)

From \ To	A	B	C	Dummy	Supply
1	6	8	10	0	150
2	7	11	11	0	175
3	4	5	12	0	375
Demand	200	100	300	100	700

When a supply exceeds demand, a dummy column is added to the tableau.

The addition of a dummy row or a dummy column has no effect on the initial solution methods or on the methods for determining an optimal solution. The dummy row or column cells are treated the

same as any other tableau cell. For example, in the minimum cell cost method, three cells would be tied for the minimum cost cell, each with a cost of zero. In this case (or any time there is a tie between cells) the tie would be broken arbitrarily.

[Page B-20]

Degeneracy

In all the tableaus showing a solution to the wheat transportation problem, the

following condition was met.

m rows + n columns - 1 = the number of cells with allocations

For example, in any of the balanced tableaus for wheat transportation, the number of rows was three (i.e., $m = 3$) and the number of columns was three (i.e., $n = 3$); thus, $3 + 3 - 1 = 5$ cells with allocations.

These tableaus always had five cells with allocations; thus, our condition for normal solution was met. When this condition is not met and fewer than $m + n$

- 1 cells have allocations, the tableau is said to be *degenerate*.

*In a transportation tableau with m rows and n columns, there must be $m + n - 1$ cells with allocations; if not, it is **degenerate**.*

Consider the wheat transportation example with

the supply values changed to the amounts shown in [Table B-31](#). The initial solution shown in this tableau was developed using the minimum cell cost method.

Table B-31. The Minimum Cell Cost Initial Solution

From \ To	A	B	C	Supply
1	6	8	10	150
2	7	11	11	250
3	4	5	12	200
Demand	200	100	300	600

The tableau shown in [Table B-31](#) does not meet the condition

$m + n - 1 =$ the number of cells with allocations

$$3 + 3 - 1 = 5 \text{ cells}$$

because there are only four cells with allocations. The difficulty resulting from a degenerate solution is that neither the stepping-stone method nor MODI will work unless the preceding condition is met (there is an appropriate number of cells with allocations). When the tableau is degenerate, a closed path cannot be completed for all cells in the stepping-stone method, and not all the $u_i + v_j = c_{ij}$ computations can be

completed in MODI. For example, a closed path cannot be determined for cell 1A in [Table B-31](#).

[Page B-21]

In a degenerate tableau, not all of the stepping-stone paths or MODI equations can be developed.

To create a closed path, one of

the empty cells must be artificially designated as a cell with an allocation. Cell 1A in [Table B-32](#) is designated arbitrarily as a cell with artificial allocation of zero. (However, any symbol, such as $\emptyset$, could be used to signify the artificial allocation.) This indicates that this cell will be treated as a cell with an allocation in determining stepping-stone paths or MODI formulas, although there is no real allocation in this cell. Notice that the location of 0 was *arbitrary* because there is

no general rule for allocating the artificial cell. Allocating zero to a cell does not guarantee that all of the steppingstone paths can be determined.

Table B-32. The Initial Solution

From \ To	A	B	C	Supply
1	0	100	50	150
2			250	250
3	200			200
Demand	200	100	300	600

To rectify a degenerate tableau, an empty cell must artificially be treated

as an occupied cell.

For example, if zero had been allocated to cell 2B instead of to cell 1A, none of the stepping-stone paths could have been determined, even though technically the tableau would no longer be degenerate. In such a case, the zero must be reallocated to another cell and all paths determined again. This process must be repeated until an artificial allocation has

been made that will enable the determination of all paths. In most cases, however, there is more than one possible cell to which such an allocation can be made.

The stepping-stone paths and cost changes for this tableau follow.

$$\begin{array}{lcl}
 & 2A & 2C \quad 1C \quad 1A \\
 x_{2A}: & 7 - 11 + 10 - 6 = 0 \\
 & 2B & 2C \quad 1C \quad 1B \\
 x_{2B}: & 11 - 11 + 10 - 8 = +2 \\
 & 3B & 1B \quad 1A \quad 3A \\
 x_{3B}: & 5 - 8 + 6 - 4 = -1 \\
 & 3C & 1C \quad 1A \quad 3A \\
 x_{3C}: & 12 - 10 + 6 - 4 = +4
 \end{array}$$

Because cell 3B shows a \$1 decrease in cost for every ton of wheat allocated to it, we will allocate 100 tons to cell 3B. This results in the tableau shown in [Table B-33](#).

Table B-33. The Second Stepping-Stone Iteration

(This item is displayed on page B-22 in the print version)

From \ To	A	B	C	Supply
1	100		50	150
2			250	250
3	100	100		200
Demand	200	100	300	600

Notice that the solution in [Table B-33](#) now meets the condition $m + n - 1 = 5$. Thus, in applying the stepping-stone method (or MODI) to this

tableau, it is not necessary to make an artificial allocation to an empty cell. It is quite possible to begin the solution process with a normal tableau and have it become degenerate or begin with a degenerate tableau and have it become normal. If it had been indicated that the cell with the zero should have units subtracted from it, no actual units could have been subtracted. In that case the zero would have been moved to the cell that represents the entering variable. (The solution shown

in [Table B-33](#) is optimal;
however, a multiple optimal
solution exists at cell 2A.)

*A normal problem
can become
degenerate at any
iteration and vice
versa.*

[Page B-22]

Prohibited Routes

Sometimes one or more of the routes in the transportation model are *prohibited*. That is, units cannot be transported from a particular source to a particular destination. When this situation occurs, we must make sure that no units in the optimal solution are allocated to the cell representing this route. In our study of the simplex tableau, we learned that assigning a large relative cost or a coefficient of m to a variable would keep it out of the final solution. This same principle can be used in a

transportation model for a prohibited route. A value of m is assigned as the transportation cost for a cell that represents a prohibited route. Thus, when the prohibited cell is evaluated, it will always contain a large positive cost change of m , which will keep it from being selected as an entering variable.

A **prohibited route** is assigned a large cost such as m so that it will never

receive an allocation.

Solution of the Assignment Model

The assignment model is a special form of a linear programming model that is similar to the transportation model. There are differences, however. In the assignment model, the supply at each source and the demand at each destination are each limited to one unit.

*An **assignment***

problem is a special form of transportation problem where all supply and demand values equal one.

The following example from the text will be used to demonstrate the assignment model and its special solution method. The Atlantic Coast Conference has four basketball games on a particular night.

The conference office wants to assign four teams of officials to the four games in a way that will minimize the total distance traveled by the officials. The distances in miles for each team of officials to each game location are shown in [Table B-34](#).

Table B-34. The Travel Distances to Each Game for Each Team of Officials

Game Sites

Officials RALEIGH ATLANTA DURHAM CLEMSON

A	210	90	180	160
B	100	70	130	200
C	175	105	140	170
D	80	65	105	120

The supply is always one team of officials, and the demand is for only one team of officials at each game. [Table B-34](#) is already in the proper form for the assignment.

*An **opportunity cost table** is developed by first subtracting the minimum value in each row from all other row values and then repeating this process for each column.*

The first step in the assignment method of solution is to develop an [opportunity cost](#)

table. We accomplish this by first subtracting the minimum value in each row from every value in the row. These computations are referred to as *row reductions*. We applied a similar principle in the VAM method when we determined penalty costs. In other words, the best course of action is determined for each row, and the penalty or "lost opportunity" is developed for all other row values. The row reductions for this example are shown in [Table B-35](#).

**Table B-35. The Assignment
Tableau with Row
Reductions**

Game Sites				
Officials	RALEIGH	ATLANTA	DURHAM	CLEMSON
A	120	0	90	70
B	30	0	60	130
C	70	0	35	65
D	15	0	40	55

Next, the minimum value in each column is subtracted from all column values. These computations are called *column reductions* and are shown in [Table B-36](#). It represents the completed opportunity cost table for our example.

Assignments can be made in this table wherever a zero is present. For example, team A can be assigned to Atlanta. An *optimal solution* results when each of the four teams can be

uniquely assigned to a different game.

Table B-36. The Tableau with Column Reductions

Game Sites

Officials RALEIGH ATLANTA DURHAM CLEMSON

A	105	0	55	15
B	15	0	25	75
C	55	0	0	10
D	0	0	5	0

Notice in [Table B-36](#) that the assignment of team A to Atlanta means that no other team can be assigned to that game. Once this assignment is made, the zero in row B is infeasible, which indicates that there is not a unique optimal assignment for team B. Therefore, [Table B-36](#) does not contain an optimal solution.

*Assignments are
made to locations*

with zeros in the opportunity cost table.

An optimal solution occurs when the number of independent unique assignments equals the number of rows or columns.

A test to determine if four unique assignments exist in [Table B-36](#) is to draw the minimum number of horizontal or vertical lines necessary to cross out all zeros through the rows and columns of the table. For example, [Table B-37](#) shows that three lines are required to cross out all zeros.

Table B-37. The Opportunity Cost Table with the Line Test

Game Sites

Officials RALEIGH ATLANTA DURHAM CLEMSON

A	105	0	55	15
B	15	0	25	75
C	55	0	0	10
D	0	0	5	0

The three lines indicate that there are only three unique assignments, whereas four are required for an optimal solution. (Note that even if the three lines could have been drawn differently, the subsequent solution method would not be affected.) Next, subtract the minimum value that is not crossed out from all values not crossed out. Then add this minimum value to those cells where two lines intersect. The minimum value not crossed out in [Table B-37](#) is 15. The second iteration for

this model with the appropriate changes is shown in [Table B-38](#).

Table B-38. The Second Iteration

Game Sites

Officials RALEIGH ATLANTA DURHAM CLEMSON

A	90	0	40	0
B	0	0	10	60
C	55	15	0	10
D	0	15	5	0

If the number of unique assignments is less than the number of rows (or columns), a line test must be used.

No matter how the lines are drawn in [Table B-38](#), at least four are required to cross out all the zeros. This indicates

that four unique assignments can be made and that an optimal solution has been reached. Now let us make the assignments from [Table B-38](#).

In a line test, all zeros are crossed out by horizontal and vertical lines; the minimum uncrossed value is subtracted from all uncrossed values and added to cell values where two lines cross.

First, team A can be assigned to either the Atlanta game or the Clemson game. We will assign team A to Atlanta first. This means that team A cannot be assigned to any other game, and no other team can be assigned to Atlanta. Therefore, row A and the Atlanta column can be eliminated. Next, team B is assigned to Raleigh. (Team B cannot be assigned to Atlanta, which has already been eliminated.) The third assignment is of team C to the

Durham game. This leaves team D for the Clemson game. These assignments and their respective distances (from [Table B-34](#)) are summarized as follows.

Assignment	Distance
Team A → Atlanta	90
Team B → Raleigh	100
Team C → Durham	140
Team D → Clemson	120

450 miles

Now let us go back and make the initial assignment of team A to Clemson (the alternative assignment we did not initially make). This will result in the following set of assignments.

Assignment	Distance
Team A → Clemson	160

Team B → Atlanta 70

Team C → Durham 140

Team D → Raleigh 80

450 miles

These two assignments represent *multiple optimal solutions* for our example problem. Both assignments will result in the officials traveling a minimum total distance of 450

miles.

Like a transportation problem, an assignment model can be unbalanced when supply exceeds demand or demand exceeds supply. For example, assume that, instead of four teams of officials, there are five teams to be assigned to the four games. In this case a *dummy column* is added to the assignment tableau to balance the model, as shown in [Table B-39](#).

Table B-39. An Unbalanced

- . - -

Assignment Tableau with a Dummy Column

(This item is displayed on page B-25 in the previous version)

Game Sites					
Officials	RALEIGH	ATLANTA	DURHAM	CLEMSON	DURHAM
A	210	90	180	160	0
B	100	70	130	200	0
C	175	105	140	170	0
D	80	65	105	120	0
E	95	115	120	100	0

When supply exceeds demands, a dummy column is added to the assignment tableau.

[Page B-25]

In solving this model, one team of officials would be assigned to

the dummy column. If there were five games and only four teams of officials, a *dummy row* would be added instead of a dummy column. The addition of a dummy row or column does not affect the solution method.

When demand exceeds supply, a dummy row is added to the assignment tableau.

Prohibited assignments are also possible in an assignment problem, just as prohibited routes can occur in a transportation model. In the transportation model, an M value was assigned as a large cost for the cell representing the prohibited route. This same method is used for a prohibited assignment. A value of M is placed in the cell that represents the prohibited assignment.

A prohibited assignment is given

*a large relative cost
of M so that it will
never be selected.*

The steps of the assignment solution method are summarized here.

- 1.** Perform row reductions by subtracting the minimum value in each row from all row values.
- 2.** Perform column reductions

by subtracting the minimum value in each column from all column values.

- 3.** In the completed opportunity cost table, cross out all zeros, using the minimum number of horizontal or vertical lines.
- 4.** If fewer than m lines are required (where m = the number of rows or columns), subtract the minimum uncrossed value from all uncrossed values, and add this same

minimum value to all cells where two lines intersect. Leave all other values unchanged, and repeat step 3.

- 5.** If m lines are required, the tableau contains the optimal solution and m unique assignments can be made. If fewer than m lines are required, repeat step 4.

Problems

Green Valley Mills produces carpet at pl carpet is then shipped to two outlets lo cost per ton of shipping carpet from ea warehouses is as follows.

From

Chicago

St. Louis

\$40

1.

Richmond

70

The plant at St. Louis can supply 250 tons per week, the plant at Richmond can supply 400 tons per week, and the outlet at Kansas City can handle a total of 300 tons per week, and the outlet at St. Louis can handle a total of 300 tons per week. The company wants to know the cost of shipping from each plant to each outlet in order to solve this transportation problem.

A transportation problem involves the following data:

		To
From	1	2
1	\$500	\$750

	2	650	800
2.	3	400	700
Demand		10	10

1. Find the initial solution using the n minimum cell cost method, and V Compute total cost for each.
2. Using the VAM initial solution, find modified distribution method (MO

Consider the following transportation ta

From \ To	A	B	C	Supply
1	12	10	6	600
2	4	15	3	400
3	9	7	M	300
4	11	8	6	800
Dummy	0	0	0	200
Demand	900	500	900	2,300

3.

1. Is this a balanced or an unbalanced problem?
2. Is this solution degenerate? Explain. If not, it would be put into proper form.

3. Is there a prohibited route in this |
4. Compute the total cost of this so |
5. What is the value of x_{2B} in this so

[Page 1]

Solve the following transportation prob

		To	
From		1	2
4.	1	\$ 40	\$ 10
	2	15	20

	3	20	25
Demand	1,050	500	

Given a transportation problem with the above supply and demand, find the initial solution using the Vogel's approximation model. Is the VA solution optimal?

		To	
	From	1	2
5.	A	\$ 6	\$ 5
	B	\$ 5	\$ 4

C

8

Demand

135

17

Consider the following transportation p

To**From****1****2**

A

\$ 6

9

B

12

3

6.

C	4	8
Demand	80	110

1. Find the initial solution by using V_A stepping-stone method.
2. Formulate this problem as a gene

Solve the following linear programming

[Page 1

7.

minimize $Z = 3x_{11} + 12x_{12} + 8x_{13} + 10x_{21} + 5x_{22} + 6x_{23} + 6x_{31} + 7x_{32} + 10x_{33}$

subject to

$$x_{11} + x_{12} + x_{13} = 90$$

$$x_{11} + x_{21} + x_{31} \leq 70$$

$$x_{21} + x_{22} + x_{23} = 30$$

$$x_{12} + x_{22} + x_{32} \leq 110$$

$$x_{31} + x_{32} + x_{33} = 100$$

$$x_{13} + x_{23} + x_{33} \leq 80$$

$$x_{ij} \geq 0$$

Consider the following transportation p

		To	
From		1	2
8.	A	\$ 6	\$ 9
	B	12	3
	C	4	11

Demand

80

110

1. Find the initial solution using the n
2. Solve using the stepping-stone m

Steel mills in three cities produce the fo

Location

***Weekly
Production
(tons)***

A. Bethlehem

150

B. Birmingham

210

C. Gary	320
---------	-----

	680
--	-----

These mills supply steel to four cities with the following demand.

<i>Location</i>	<i>Weekly Demand (tons)</i>
1. Detroit	130
2. St. Louis	70

3. Chicago

180

9. 4. Norfolk

240

620

Shipping costs per ton of steel are as follows

From

1

2

A

\$14

9

B	11	8
C	16	12

Because of a truckers' strike, shipment from Birmingham to Chicago.

1. Set up a transportation tableau for an initial solution. Identify the method

[Page

2. Solve this problem using MODI.
3. Are there multiple optimal solutions?
4. Formulate this problem as a general

- 10.** In [problem 9](#), what would be the effect reduction in production capacity at the per week?

Coal is mined and processed at the folk Virginia, and Virginia.

<i>Location</i>	<i>Capacity (tons)</i>
------------------------	-----------------------------------

A. Cabin Creek	90
----------------	----

B. Surry	50
----------	----

C. Old Fort	80
-------------	----

D. McCoy	60
----------	----

280

These mines supply the following amount to three cities.

Plant

Demand (tons)

1. Richmond

120

2. Winston-Salem

100

3. Durham

110

11.

The railroad shipping costs (\$1,000s) per ton are given in the following table. Because of railroad construction, shipping is prohibited from Cabin Creek to Richmond.

	To	
	1	2
From		
A	\$ 7	\$10
B	12	9
C	7	3

- 1.** Set up the transportation tableau solution using VAM, and compute
- 2.** Solve using MODI.
- 3.** Are there multiple optimal solution solutions, identify them.
- 4.** Formulate this problem as a linear

Oranges are grown, picked, and then shipped to Miami, and Fresno. These warehouses have shipping costs per truckload (\$100s), s York, Philadelphia, Chicago, and Boston agreement between distributors, shipm Chicago.

To**From*****New
York******Philadelphia C***

Tampa

\$ 9

\$ 14

12.

Miami

11

10

Fresno

12

8

Demand

130

170

1. Set up the transportation tableau initial solution using the minimum
2. Solve using MODI.
3. Are there multiple optimal solution
4. Formulate this problem as a linear

A manufacturing firm produces diesel engines in St. Louis, and Detroit. The company is numbers of engines per month.

Plant

Production

1. Phoenix

5

2. Seattle

25

3. St. Louis	20
--------------	----

4. Detroit	25
------------	----

Three trucking firms purchase the following plants in three cities.

<i>Firm</i>	<i>Demand</i>
--------------------	----------------------

A. Greensboro	10
---------------	----

B. Charlotte	20
--------------	----

C. Louisville	15
---------------	----

The transportation costs per engine (\$) are shown in the following table. However, engines made in Seattle, and the Louisville-Detroit; therefore, these routes are prohibited.

From	To	A
1	2	\$ 7
2	3	6
3	4	10
4	1	3

-
1. Set up the transportation tableau solution using VAM.
 2. Solve for the optimal solution using the stepping stone method. Compute the total minimum cost.
 3. Formulate this problem as a linear programming model.

The Interstate Truck Rental firm has acquired several new truck leasing outlets, as shown in the following table.

[Page 1 of 1]

Leasing Outlet	Extra Trucks
----------------	--------------

1. Atlanta	70
2. St. Louis	115
3. Greensboro	60
Total	245

The firm also has four outlets with short

Leasing Outlet	Trucks Shortage
A. New Orleans	80

B. Cincinnati	50
---------------	----

14.

C. Louisville	90
---------------	----

D. Pittsburgh	25
---------------	----

Total	245
-------	-----

The firm wants to transfer trucks from
with shortages at the minimum total cost
transporting these trucks from city to city

From

A

B

1	\$ 70	80
---	-------	----

2	120	40
---	-----	----

3	110	60
---	-----	----

1. Find the initial solution using the n
2. Solve using the stepping-stone m

The Sholtz Beer Company has breweries that supply the following numbers of barrels to distributors each month.

Brewery

Monthly S

A. Tampa	3,
B. St. Louis	5,
Total	8,

[Page 1]

The distributors, which are spread throughout the total monthly demand.

Distributor	Monthly D
1. Tennessee	1,

2. Georgia	1,
------------	----

3. North Carolina	1,
-------------------	----

15. 4. South Carolina	9,
------------------------------	----

5. Kentucky	1,
-------------	----

6. Virginia	1,
-------------	----

Total	8,
-------	----

The company must pay the following s

To

From	<i>1</i>	<i>2</i>	<i>3</i>
A	\$0.50	0.35	0.60
B	0.25	0.65	0.40

- 1.** Find the initial solution using VAM.
- 2.** Solve using the stepping-stone m

In [problem 15](#), the Shotz Beer Compar

- 16.** new shipping contract with a trucking firm as its distributor in Kentucky that reduces the cost from \$0.80 per barrel to \$0.65 per barrel. How is this an optimal solution?

Computers Unlimited sells microcomputers on the East Coast and ships them from three warehouses. It is able to supply the following numbers of microcomputers to various universities by the beginning of the academic year.

Distribution Warehouse	Supply (microcomputers)
1. Richmond	420
2. Atlanta	610
3. Washington, D.C.	340

Total	1,370
-------	-------

Four universities have ordered microcom installed by the beginning of the academic

[Page 1]

University	Demand (microcom)
------------	-------------------

A. Tech	520
---------	-----

17. B. A and M 250

C. State	400
----------	-----

D. Central

380

Total

1,550

The shipping and installation costs per r
to each university are as follows.

		To	
From	A	B	
1	\$22	17	
2	15	35	

1. Find the initial solution using VAM.
2. Solve using MODI.

In [problem 17](#), Computers Unlimited w four universities it supplies. It is conside warehouse at Richmond to a capacity (additional \$6 in handling and shipping p

- 18.** warehouse in Charlotte that can supply to Tech, \$26 to A and M, \$22 to State, alternative should management select (i.e., no capital costs)?

Computers Unlimited in [problem 17](#) has to meet the demand for microcompute

universities tend to purchase microcom
the firm has estimated a shortage cost
but not supplied that reflects the loss o
costs for each university are as follows

University	Cost/Micro
A. Tech	\$4
B. A and M	65
C. State	25
D. Central	50

19.

Solve [problem 17](#) with these shortage

transportation cost and the total shorta

A severe winter ice storm has swept ac
followed by over a foot of snow and fri
weather conditions have resulted in nur
power outages in the region causing da
population. Local utility companies have
requested assistance from unaffected u
Southeast. The following table shows t
crews available from five different com
and Florida; the demand for crews in so
companies cannot get to; and the wee
to a specific area (based on the visiting
distance the crew has to come, and livi

[Page 1

Area (Cost = \$:

Crew ***NC-E*** ***NC-SW*** ***NC-P*** ***NC-W***

20.

GA-1 15.2 14.3 13.9 13.5

GA-2 12.8 11.3 10.6 12.0

SC-1 12.4 10.8 9.4 11.3

FL-1 18.2 19.4 18.2 17.9

FL-2 19.3 20.2 19.5 20.2

Crews
Needed 9 7 6 8

Determine the number of crews that sh
each affected area that will minimize to

A large manufacturing company is closi
intends to transfer some of its more sk
will remain open. The number of emplo
closing plant is as follows.

Closing Plant	Transferable Employees
1	60
2	105
3	70
Total	235

The following number of employees can
plants remaining open.

Open Plants	Employees Demanded
--------------------	-------------------------------

A	45
---	----

B	90
---	----

C	35
---	----

Total	170
-------	-----

21.

Each transferred employee will increase plant as shown in the following table. The employees so as to ensure the maximum

[Page 1]

		To	
From	A	B	
1	5	8	
2	10	9	
3	7	6	

- 1.** Find the initial solution using VAM.
- 2.** Solve using MODI.

The Sav-Us Rental Car Agency has six lots, each with a certain number of cars available at each lot for local rental. The agency would like a computer program at the end of each day that would tell it how to rent cars from the lots in the minimum total time. The times for each lot are as follows.

	To (minutes)			
From	1	2	3	4
1		12	17	18

2	14		10	19
3	14	10		12
4	8	16	14	
5	11	21	16	18
6	24	12	9	17

22.

The agency would like the following nur the day. Also shown is the number of a a particular day. Determine the optimal initial solution approach and any solutio

Cars	<i>1</i>	<i>2</i>	<i>3</i>
Available	37	20	14
Desired	30	25	20

Bayville has built a new elementary school of four schools Addison, Beeks, Canfield 400 students. The school wants to assign travel time by bus is as short as possible town into five districts conforming to p west, and central. The average bus travel school is shown as follows.

Travel Time (mins)

23.

District	Addison	Beeks	Canfield
North	12	23	35
South	26	15	21
East	18	20	22
West	20	24	25

	West	23	24	33
Central		15	10	23

Determine the number of children that to each school to minimize total student

24. In [problem 23](#), the school board has decided one school to be more crowded than a assign students from each district to each evenly balanced between the four schools concerned that this might significantly increase the number of students to be assigned from school enrollments are evenly balanced result in a significant increase in travel time

The Easy Time Grocery chain operates eastern seaboard. The stores have a "1

overhead and high volume. They generate low prices. However, in some cases they acquire goods in other areas and ship it in. They can do this because they can operate in areas with lower costs compared with costs in other areas. For example, Easy Time purchases baby food and toys in Clarendon, Dover, and Edison, and then ships them to New York City. The stores in the outlying areas are too far to, so they limit the number of cases of each item. The following table shows the profit Easy Time makes on baby food based on where the chain purchases the food, plus the available baby food per week and the space available at each Easy Time store.

Purchase Location	Easy Time Store			
	1	2	3	4
25. Albany	9	8	11	12

Binghamton	10	10	8	6
Claremont	8	6	6	5
Dover	4	6	9	5
Edison	12	10	8	9
Demand	25	15	30	18

Determine where Easy Time should purchase and how much should be distributed in order to maximize profit. Use the branch and bound approach and any solution method.

Suppose that in [problem 25](#) Easy Time needs from a New York City distributor

26. \$9 per case at stores 1, 3, and 4, \$8 per case at store 5. Should Easy Time buy baby food from the distributor rather than trucking it in?

27. In [problem 25](#), if Easy Time could arrange to buy from one of the outlying locations, which location would be best, how many cases could be purchased, and how much would it cost?

[Page 1]

The Roadnet Transport Company has been purchasing 90 trailer trucks from a company subsequently located 30 of the shipping warehouses in Charlotte, Memphis, Atlanta, and New York. Each truck is used to make shipments from each of these warehouses. The terminal managers have indicated that the manager at St. Louis can handle 30 trucks per week, the manager at Atlanta can handle 30 trucks, and the manager at New York can

trucks. The company makes the follow
 from each warehouse to each terminal
 differences in products shipped, shipping

28.

Terminal

Warehouse *St. Louis* *Atlanta* *Memphis*

Charlotte \$1,800 \$2,100

Memphis 1,000 700

Louisville 1,400 800

Determine how many trucks to assign

terminal) in order to maximize profit.

During the Gulf War, Operation Desert Storm, the United States shipped military matériel and supplies to be shipped to bases in the Middle East. The movement of these supplies was speeded by the number of plane loads of supplies available at the depots and the number of daily loads delivered. (Each plane load is approximately equal to the transport hours per plane, including loading and unloading and refueling.

	Military Base		
	<i>A</i>	<i>B</i>	<i>C</i>
Supply Depot			
1	36	40	32

29.	2	28	27	29	4
	3	34	35	41	5
	4	41	42	35	6
	5	25	28	40	7
	6	31	30	43	8
Demand	9	6	12		

Determine the optimal daily flight schedule time.

PM Computer Services produces personal computers. The company buys component parts it buys on the open market. The company has a production schedule for the first six months of 300 personal computers per month. The company knows the cost of component parts will be 5% of the overall cost to produce a PC will be 5% holding a computer in inventory is \$15 per month. The demand for the company's computers

[Page 1]

30. Month	Demand	Month
January	180	April
February	260	May
March	340	June

Determine a production schedule for PM

In [problem 30](#), suppose the demand for each month is as follows.

Month	Demand
January	410
February	320
March	500
31. April	620

May

430

June

380

In addition to the regular production capacity, Computer Services can also produce an additional 100 units a month using overtime. Overtime production cost is \$100 per personal computer.

Determine a production schedule for PM

National Foods Company has five plants that produce fruits and vegetables. It has suppliers in California, Alabama, and Florida. The company has its own trucking system in the past for transporting products to its plants. However, it is now considering shipping to outside trucking firms and g

currently spends \$245,000 per month. It has determined monthly shipping costs from each of its suppliers to each of the following processing plants. The following table.

	Processing Plants (\$1000)			
	Denver	St. Paul	Louisville	Portland
Sacramento	3.7	4.6	4.9	3.8
Bakersfield	3.4	5.1	4.4	3.5
San Antonio	3.3	4.1	3.7	3.6
Montgomery	1.9	4.2	2.7	3.9

32.

Jacksonville	6.1	5.1	3.8
Ocala	6.6	4.8	3.5
Demand (tons)	20	15	15

Should National Foods continue to operate its trucks and outsource its shipping to

[Page 1]

33. In [problem 32](#), National Foods would like to see if it can improve on the optimal solution and the company it negotiates with its suppliers in Sacramento. Suppose the company increases their capacity to 25 tons per region and the company is negotiating instead with its suppliers at

increase their capacity to 25 tons each

Orient Express is a global distribution company that ships consumer products to customers in Hong Kong, Singapore, and the U.S. The products Orient Express ships are stored in three U.S. centers: one in Los Angeles, one in Savannah, and one in Galveston. Each month the company has 450 containers available at the Los Angeles center, 600 containers available in Galveston. The containers are shipped from Hong Kong, 500 containers from Hong Kong, 500 containers from Taipei. The shipping costs to each of the overseas ports are shown below.

Shipping Costs (per container)			
		Overseas	
U.S. Center Distribution		Hong Kong	Singapore
34.	Los Angeles	\$300	\$210
	Savannah	\$250	\$180
	Galveston	\$200	\$150

Savannah	490	520
Galveston	360	320

The Orient Express as the overseas branch responsible for unfulfilled orders, and it charges overseas customers if it does not meet their requirements. Singapore customers charge a penalty cost of \$800 per container. Taipei customers charge \$1,100 per container. Use the transportation model to determine the optimal distribution center to each overseas port. Indicate what portion of the total cost is allocated to each port.

Binford Tools manufactures garden tools and is considering subcontracting to absorb demand fluctuations and overtime production capacity, and

in the following table for the next four quarters of demand for garden tools.

Quarter	Demand	Regular Capacity
1	9,000	9,000
2	12,000	10,000
3	16,000	12,000
4	19,000	12,000

35.

The regular production cost per unit is \$25, the cost to subcontract a unit is \$

is \$2 per unit. The company has 300 u
the year.

Determine the optimal production sche
minimize total costs.

[Page I

Solve the following linear programming

$$\begin{aligned} \text{minimize } Z = & 18x_{11} + 30x_{12} + 20x_{13} + 18x_{14} + 25x_{21} + 27x_{22} + 22x_{23} \\ & + 16x_{24} + 30x_{31} + 26x_{32} + 19x_{33} + 32x_{34} + 40x_{41} + 36x_{42} \\ & + 27x_{43} + 29x_{44} + 30x_{51} + 26x_{52} + 18x_{53} + 24x_{54} \end{aligned}$$

subject to

36.

$$\begin{aligned}
x_{11} + x_{12} + x_{13} + x_{14} &\leq 1 \\
x_{21} + x_{22} + x_{23} + x_{24} &\leq 1 \\
x_{31} + x_{32} + x_{33} + x_{34} &\leq 1 \\
x_{41} + x_{42} + x_{43} + x_{44} &\leq 1 \\
x_{51} + x_{52} + x_{53} + x_{54} &\leq 1 \\
x_{11} + x_{21} + x_{31} + x_{41} + x_{51} &= 1 \\
x_{12} + x_{22} + x_{32} + x_{42} + x_{52} &= 1 \\
x_{13} + x_{23} + x_{33} + x_{43} + x_{53} &= 1 \\
x_{14} + x_{24} + x_{34} + x_{44} + x_{54} &= 1 \\
x_{ij} &\geq 0
\end{aligned}$$

A plant has four operators to be assign (minutes) required by each worker to p is shown in the following table. Determin compute total minimum time.

Machir

Operator

A

B

37.

1

10

12

2

5

10

3

12

14

4

8

15

A shop has four machinists to be assigned to four machines. The cost of having each machine operated

Machine**Machinist*****A******B******C***

38.

1	\$12	11	8
2	10	9	10
3	14	8	7
4	6	8	10

However, because he does not have er
operate machine B.

- 1.** Determine the optimal assignmen
- 2.** Formulate this problem as a gene

The Omega pharmaceutical firm has five salespersons and wants to assign them to five sales regions. Given that the salespersons are able to cover the regions, the amount of time (days) required by each salesperson to cover each region is shown in the following table. Which salesperson should be assigned to each region to minimize total time? Identify the assignment and compute total minimum time.

[Page 1]

		Region		
		<i>A</i>	<i>B</i>	<i>C</i>
39.	Salesperson			
	1	17	10	15
	2	12	9	11

3	11	16	1
4	14	10	1
5	13	12	9

The Bunker Manufacturing firm has five wants to assign the employees to the table showing the cost incurred by each. Because of union rules regarding department cannot be assigned to machine E and employee machine B. Solve this problem, indicate compute total minimum cost.

Machi

Employee	<i>A</i>	<i>B</i>	<i>C</i>
40.			
1	\$12	\$ 7	\$20
2	10	14	13
3	5	3	6
4	9	11	7
5	10	6	14

Given the following cost table for an assignment problem, find an optimal assignment and compute total cost. If there are multiple optimal solutions, give any one of them.

		Machin	
Operator		<i>A</i>	<i>B</i>
41.	1	\$10	\$2
	2	9	5
	3	12	7
	4	3	1

An electronics firm produces electronic various electrical manufacturers. Quality

different employees produce different r
average number of defects produced b
components is given in the following ta
assignment that will minimize the total
by the firm per month.

[Page 1

Compo

Employee

A

B

C

42.

1

30

24

16

2

22

28

14

3

18

16

25

	4	14	22	18
	5	25	18	14
	6	32	14	10

A dispatcher for the Citywide Taxi Company has three taxi locations and five customers who have requested rides. The mileage from each taxi's present location to each customer's location is shown in the following table. Determine the optimal assignment of taxis to customers to minimize the total mileage traveled.

	Customers			
Cab	1	2	3	4

43.

A	7	2	4
B	5	1	5
C	8	7	6
D	2	5	2
E	3	3	5
F	6	2	4

The Southeastern Conference has nine assigned to three conference games, the office wants to assign the officials so th

be minimized. The distance (in miles) e game is given in the following table.

	Game		
	Official	<i>Athens</i>	<i>Columbia</i>
44.	1	165	90
	2	75	210
	3	180	170
	4	220	80
	5	410	140

6	150	170
7	170	110
8	105	125
9	240	200

[Page]

- 1.** Should this problem be solved by assignment method? Explain.
- 2.** Determine the optimal assignment distance traveled by the officials.

45.

In [problem 44](#), officials 2 and 8 have had the coaches in the game in Athens. The after several technical fouls. The conference not be a good idea to have these two (soon after this confrontation, so they should not be assigned to the Athens game. How solution to this problem?

State University has planned six special Saturday of its homecoming football game brunch, a parent's brunch, a booster club season ticket holders, a lettermen's dinner major contributors. The university wants as the university catering service to cater caterers to bid on each event. The bids guidelines for the events prepared by the following table.

Caterer	Alumni Brunch	Parent's Brunch	Booster Club Lunch
---------	---------------	-----------------	--------------------

Al's	\$12.6	\$10.3	\$14.0
------	--------	--------	--------

46. Bon Apétit	14.5	13.0	16.5
----------------	------	------	------

Custom	13.0	14.0	17.6
--------	------	------	------

Divine	11.5	12.6	13.0
--------	------	------	------

Epicurean	10.8	11.9	12.9
-----------	------	------	------

Fouchés	13.5	13.5	15.5
---------	------	------	------

University	12.5	14.3	16.0
------------	------	------	------

The Bon Apétit, Custom, and University whereas the other four caterers can be confident all the caterers will do a high-quality caterers for the events that will result in

Determine the optimal selection of caterers

A university department head has five instructors for five different courses. All of the instructors have been evaluated by the students and have been evaluated by the students. The average grade for each course is given in the following table. The department head wants to know the optimal assignment of courses that will maximize the overall average grade. Who is not assigned to teach a course? Solve this problem using the assignment method.

Instructor

A

B

47.

1

80

75

2

95

90

3

85

95

4

93

91

5

91

92

The coach of the women's swim team at the conference swim meet and must choose a swimmer to assign to the 800-meter medley relay team. The coach has computed the average times each swimmer has achieved in each of the four strokes: the backstroke, breaststroke, butterfly, and freestyle, during the previous swim meets during the season.

Stroke

Swimmer	<i>Backstroke</i>	<i>Breaststroke</i>
----------------	--------------------------	----------------------------

Annie	2.56	3.07
-------	------	------

Beth	2.63	3.01
------	------	------

Carla	2.71	2.95
-------	------	------

Debbie	2.60	2.87
Erin	2.68	2.97
Fay	2.75	3.10

Determine the medley relay team and i coach.

Biggio's Department Store has six emp departments in the store home furnishi Most of the six employees have worke several occasions in the past, and have better in some departments than in oth

of the six employees in each of the four following table.

		Department Sales		
	Employee	Department Sales		
		<i>Home Furnishings</i>	<i>China</i>	<i>Appliances</i>
49.	1	340	160	
	2	560	370	
	3	270		
	4	360	220	
	5	450	190	

Employee 3 has not worked in the chin manager does not want to assign this (

Determine which employee to assign to total expected daily sales.

The Vanguard Publishing Company has salespeople to sell encyclopedias during to allocate them to three sales territories salespeople, and territories 2 and 3 req estimated that each salesperson will be dollar sales per day in each of the three table. The company desires to allocate territories so that sales will be maximiz method to determine the initial solution the maximum total sales per day.

Salesperson	Territory	
	<i>1</i>	<i>2</i>

50.

A	\$110	\$150
B	90	120
C	205	160
D	125	100
E	140	105

F	100	140
G	180	210
H	110	120

Carolina Airlines, a small commuter airline, has three flight attendants whom it wants to assign to three different flights that will minimize the number of nights each attendant will be away from home. The numbers of nights each attendant will be away from home on each schedule are given in the following table. The airline wants a schedule that will minimize the total number of nights the attendants will be away from home.

	Attendant	<i>A</i>	<i>B</i>	<i>C</i>
--	------------------	-----------------	-----------------	-----------------

51.

1	7	4	6
2	4	5	5
3	9	9	11
4	11	6	8
5	5	8	6
6	10	12	11

The football coaching staff at Tech focuses on three states, including Georgia, Florida, and Virginia. The staff includes seven assistants, with one assistant responsible for Florida, a high school teacher assigned to each of the other five states, and a coach with a long time and at one time or another a coach in all of the states. The head coach has accumulated a 52% success rate (i.e., percentage of target yards per play) in each state as shown in the following table.

	State		
	GA	FL	VA
Allen	62	56	65
Bush	65	70	63

Crumb	46	53	62
Doyle	58	66	70
Evans	77	73	69
Fouch	68	73	72
Goins	72	60	74

Determine the optimal assignment of c
maximize the overall-success rate and
success rate for the staff with this assign

Module C. Integer Programming: The Branch and Bound Method

[The Branch and Bound Method](#)

[Problems](#)

The Branch and Bound Method

The branch and bound method is not a solution technique specifically limited to integer programming problems. It is a *solution approach* that can be applied to a number of different types of problems. The branch and bound approach is based on the principle that the total set of feasible solutions can be partitioned into smaller subsets of solutions. These smaller

subsets can then be evaluated systematically until the best solution is found. When the branch and bound approach is applied to an integer programming problem, it is used in conjunction with the normal noninteger solution approach. We will demonstrate the branch and bound method using the following example.

The branch and bound method is a solution approach that partitions the feasible solution

*space into smaller
subsets of solutions.*

The owner of a machine shop is planning to expand by purchasing some new machines presses and lathes. The owner has estimated that each press purchased will increase profit by \$100 per day and each lathe will increase profit by \$150 daily. The number of machines the owner can purchase is limited by the

cost of the machines and the available floor space in the shop. The machine purchase prices and space requirements are as follows.

Machine	Required Floor Space (ft²)	Purchase Price
Press	15	\$8,000
Lathe	30	4,000

The owner has a budget of \$40,000 for purchasing machines and 200 square feet of available floor space. The owner wants to know how many of each type of machine to purchase to maximize the daily increase in profit.

The linear programming model for an integer programming problem is formulated in exactly the same way as the linear programming examples in [chapters 2](#) and [4](#) of the text. The only difference is that in this problem, the decision variables are restricted to

integer values because the owner cannot purchase a fraction, or portion, of a machine. The linear programming model follows.

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq \$40,000$$

$$15x_1 + 30x_2 \leq 200 \text{ ft}^2$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

[Page C-3]

where

x_1 = number of presses

x_2 = number of lathes

The decision variables in this model are restricted to whole machines. The fact that *both* decision variables, x_1 and x_2 , can assume any integer value greater than or equal to zero is what gives this model its designation as a total integer model.

We begin the branch and bound method by first solving the problem as a regular linear programming model without integer restrictions (i.e., the integer restrictions are *relaxed*). The linear programming model for the problem and the optimal relaxed solution is

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq \$40,000$$

$$15x_1 + 30x_2 \leq 200 \text{ ft}^2$$

$$x_1, x_2 \geq 0$$

and

$x_1 = 2.22$, $x_2 = 5.56$, and $Z = 1,055.56$

*A linear programming model solution with no integer restrictions is called a **relaxed** solution.*

The branch and bound method employs a diagram consisting of *nodes* and *branches* as a framework for the solution process. The first node of the

branch and bound diagram, shown in [Figure C-1](#) contains the relaxed linear programming solution shown earlier *and* the rounded-down solution.

Figure C-1. The initial node in the branch and bound diagram

UB = 1,055.56 ($x_1 = 2.22$, $x_2 = 5.56$)

LB = 950 ($x_1 = 2$, $x_2 = 5$)



*The branch and bound method uses a tree diagram of **nodes** and **branches** to organize the solution partitioning.*

Notice that this node has two designated bounds: an upper bound (UB) of \$1,055.56 and a lower bound (LB) of \$950. The

lower bound is the Z value for the rounded-down solution, $x_1 = 2$ and $x_2 = 5$; the upper bound is the Z value for the relaxed solution, $x_1 = 2.22$ and $x_2 = 5.56$. The *optimal integer solution* will be between these two bounds.

*The **optimal integer solution** will always be between the upper bound of the relaxed solution and a lower bound of the rounded-down integer solution.*

Rounding down might result in a suboptimal solution. In other words, we are hoping that a Z value greater than \$950 might be possible. We are not concerned that a value *lower than* \$950 might be available. Thus, \$950 represents a *lower bound* for our solution.

Alternatively, since $Z = \$1,055.56$ reflects an optimal solution point *on the solution space boundary*, a greater Z value cannot possibly be attained. Hence, $Z = \$1,055.56$

is the *upper bound* of our solution.

Branch on the variable with the solution value with the greatest fractional part.

Now that the possible feasible solutions have been narrowed to values between the upper and lower bounds, we must test the solutions within these

bounds to determine the best one. The first step in the branch and bound method is to create *two* solution subsets from the present relaxed solution. This is accomplished by observing the relaxed solution value for each variable,

$$x_1 = 2.22$$

$$x_2 = 5.56$$

[Page C-4]

and seeing which one is the

farthest from the rounded-down integer value (i.e., which variable has the greatest fractional part). The .56 portion of 5.56 is the greatest fractional part; thus, x_2 will be the variable that we will "branch" on.

Because x_2 must be an integer value in the optimal solution, the following constraints can be developed.

$$x_2 \leq 5$$

$$x_2 \geq 6$$

Create two constraints (or subsets) to eliminate the fractional part of the solution value

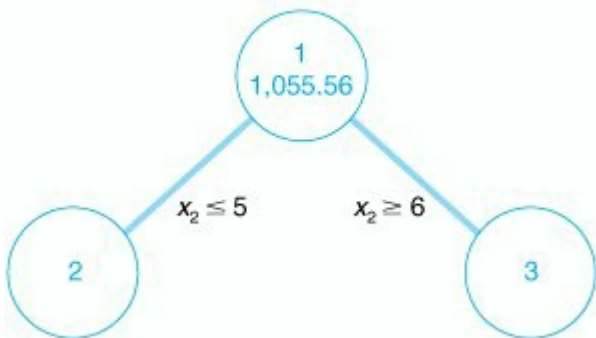
In other words, x_2 can be 0, 1, 2, 3, 4, 5, or 6, 7, 8, etc., but it cannot be a value between 5 and 6, such as 5.56. These two new constraints represent the two solution subsets for our solution approach. Each of these constraints will be added

to our linear programming model, which will then be solved normally to determine a relaxed solution. This sequence of events is shown on the branch and bound diagram in [Figure C-2](#). The solutions at nodes 2 and 3 will be the relaxed solutions obtained by solving our example model with the appropriate constraints added.

Figure C-2. Solution subsets x_2

UB = 1,055.56 ($x_1 = 2.22$, $x_2 = 5.56$)

LB = 950 ($x_1 = 2$, $x_2 = 5$)



First, the solution at node 2 is found by solving the following model with the constraint $x_2 \leq 5$ added.

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \leq 5$$

$$x_1, x_2 \geq 0$$

The optimal solution for this model with integer restrictions relaxed (solved using the computer) is $x_1 = 2.5$, $x_2 = 5$, and $Z = 1,000$.

Next, the solution at node 3 is found by solving the model with $x_2 \geq 6$ added.

$$\text{maximize } Z = \$100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1, x_2 \geq 0$$

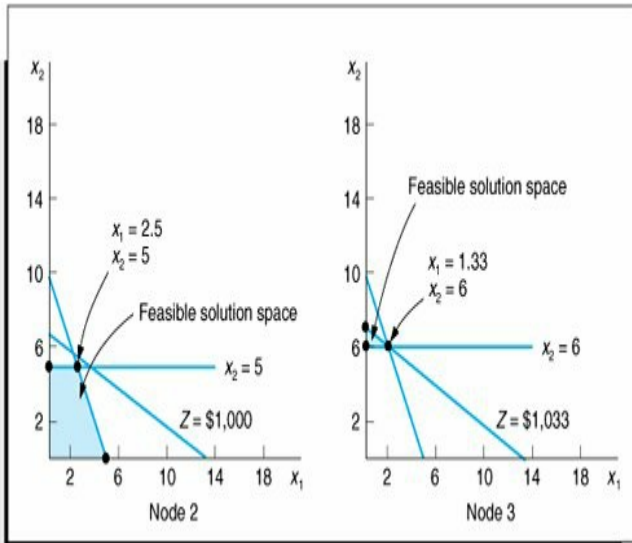
The optimal solution for this model with integer restrictions relaxed is $x_1 = 1.33$, $x_2 = 6$, and $Z = 1,033.33$.

[Page C-5]

These solutions with $x_2 \leq 5$ and $x_2 \geq 6$ reflect the partitioning of the original relaxed model

into two subsets formed by the addition of the two constraints. The resulting solution sets are shown in the graphs in [Figure C-3](#).

Figure C-3. Feasible solution spaces for nodes 2 and 3



Notice that in the node 2 graph in [Figure C-3](#), the solution point $x_1 = 2.5$, $x_2 = 5$ results in

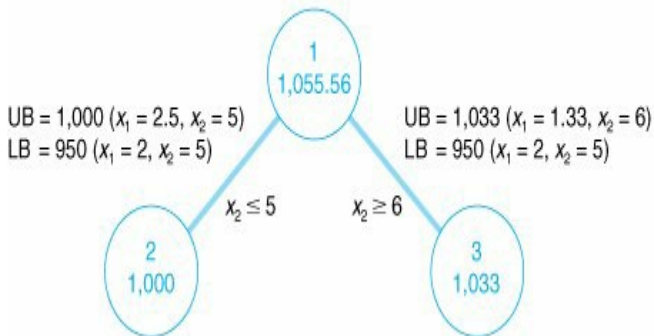
a maximum Z value of \$1,000, which is the upper bound for this node. Next, notice that in the node 3 graph, the solution point $x_1 = 1.33$, $x_2 = 6$ results in a maximum Z value of \$1,033. Thus, \$1,033 is the upper bound for node 3. The lower bound at each of these nodes is the maximum *integer* solution. Since neither of these relaxed solutions is totally integer, the lower bound remains \$950, the integer solution value already obtained at node 1 for the rounded-down integer solution. The diagram

in [Figure C-4](#) reflects the addition of the upper and lower bounds at each node.

Figure C-4. Branch and bound diagram with upper and lower bounds at nodes 2 and 3

$$UB = 1,055.56 \ (x_1 = 2.22, x_2 = 5.56)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$



Since we do not have an optimal and feasible integer solution yet, we must continue to branch (i.e., partition) the model, from either node 2 or node 3. A look at [Figure C-4](#)

reveals that if we branch from node 2, the maximum value that can possibly be achieved is \$1,000 (the upper bound). However, if we branch from node 3, a higher maximum value of \$1,033 is possible. Thus, we will branch from node 3. In general, *always branch from the node with the maximum upper bound.*

Now the steps for branching previously followed at node 1 are repeated at node 3. First, the variable that has the value with the greatest fractional part

is selected. Because x_2 has an integer value, x_1 , with a fractional part of .33, is the only variable we can select. Thus, two new constraints are developed for x_1 ,

[Page C-6]

$$x_1 \leq 1$$

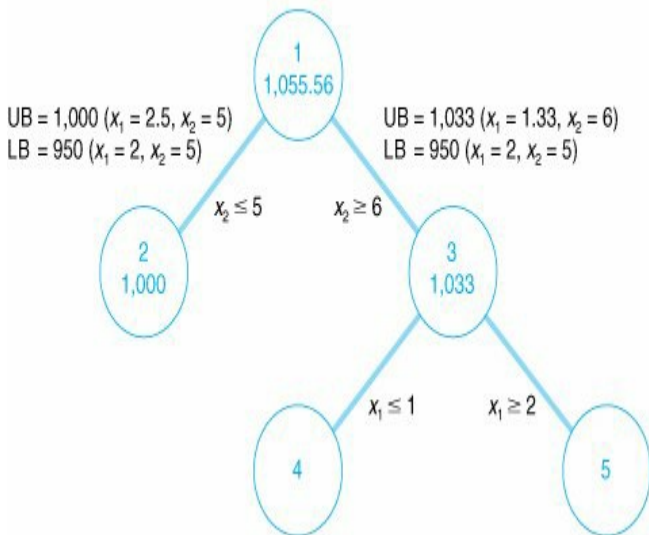
$$x_1 \geq 2$$

This process creates the new branch and bound diagram shown in [Figure C-5](#).

Figure C-5. Solution subsets for x_1

UB = 1,055.56 ($x_1 = 2.22$, $x_2 = 5.56$)

LB = 950 ($x_1 = 2$, $x_2 = 5$)



Next, the relaxed linear programming model with the new constraints added must be solved at nodes 4 and 5. (However, do not forget that the model is not the original, but the original with the constraint previously added, $x_2 \geq 6$.) Consider the node 4 model first.

maximum $Z = 100x_1 + 150x_2$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1 \leq 1$$

$$x_1, x_2 \geq 0$$

The optimal solution for this model with integer restrictions relaxed is $x_1 = 1$, $x_2 = 6.17$, and $Z = 1,025$.

Next, consider the node 5 model.

maximize $Z = 100x_1 + 150x_2$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1 \geq 2$$

$$x_1, x_2 \geq 0$$

However, there is no feasible solution for this model.

Therefore, no solution exists at node 5, and we have only to evaluate the solution at node 4. The branch and bound diagram reflecting these results is shown in [Figure C-6](#).

**Figure C-6. Branch
and bound diagram
with upper and lower
bounds at nodes 4
and 5**

$$UB = 1,055.56 \ (x_1 = 2.22, x_2 = 5.56)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$

$$UB = 1,000 \ (x_1 = 2.5, x_2 = 5)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$

$$UB = 1,033 \ (x_1 = 1.33, x_2 = 6)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$

$$x_2 \leq 5$$

$$x_2 \geq 6$$



$$UB = 1,025.50 \ (x_1 = 1, x_2 = 6.17)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$

$$x_1 \leq 1$$

$$x_1 \geq 2$$

Infeasible



The branch and bound diagram in [Figure C-6](#) indicates that we

still have not reached an optimal integer solution; thus, we must repeat the branching steps followed earlier. Since a solution does not exist at node 5, there is no comparison between the upper bounds at nodes 4 and 5. Comparing nodes 2 and 4, we must branch from node 4 because it has the greater upper bound. Next, since x_1 has an integer value, x_2 , with a fractional part of .17, is selected by default. The two new constraints developed for x_2 are

$$x_2 \leq 6$$

$$x_2 \geq 7$$

This creates the new branch and bound diagram in [Figure C-7](#).

Figure C-7. Solution subsets for x_2

UB = 1,055.56 ($x_1 = 2.22, x_2 = 5.56$)

LB = 950 ($x_1 = 2, x_2 = 5$)

UB = 1,000 ($x_1 = 2.5, x_2 = 5$)

LB = 950 ($x_1 = 2, x_2 = 5$)

UB = 1,033 ($x_1 = 1.33, x_2 = 6$)

LB = 950 ($x_1 = 2, x_2 = 5$)

$x_2 \leq 5$

$x_2 \geq 6$



UB = 1,025.50 ($x_1 = 1, x_2 = 6.17$)

LB = 950 ($x_1 = 2, x_2 = 5$)

$x_1 \leq 1$

$x_1 \geq 2$

Infeasible



$x_2 \leq 6$

$x_2 \geq 7$



The relaxed linear programming model with the new constraints added must be solved at nodes 6 and 7. Consider the node 6 model first.

maximize $Z = 100x_1 + 150x_2$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1 \leq 1$$

$$x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

The optimal solution for this relaxed linear programming model is $x_1 = 1$, $x_2 = 6$, and $Z = 1,000$.

Next, consider the node 7 model.

$$\text{maximize } Z = 100x_1 + 150x_2$$

subject to

$$8,000x_1 + 4,000x_2 \leq 40,000$$

$$15x_1 + 30x_2 \leq 200$$

$$x_2 \geq 6$$

$$x_1 \leq 1$$

$$x_2 \geq 7$$

$$x_1, x_2 \geq 0$$

However, the solution to this model is infeasible and no solution exists at node 7. The branch and bound diagram reflecting these results is shown in [Figure C-8](#). This version of the branch and bound diagram indicates that the optimal integer solution, x_1

$x_1 = 1, x_2 = 6$, has been reached at node 6. The value of 1,000 at node 6 is the maximum, or upper bound, integer value that can be obtained. It is also the recomputed lower bound because it is the maximum integer solution achieved to this point. Thus, it is not possible to achieve any higher value by further branching from node 6. A comparison of the node 6 solution with those at nodes 2, 5, and 7 shows that a better solution is not possible. The upper bound at node 2 is 1,000, which is the

same as that obtained at node 6; thus, node 2 can result in no improvement. The solutions at nodes 5 and 7 are infeasible (and thus further branching will result in only infeasible solutions). By the process of elimination, the integer solution at node 6 is optimal.

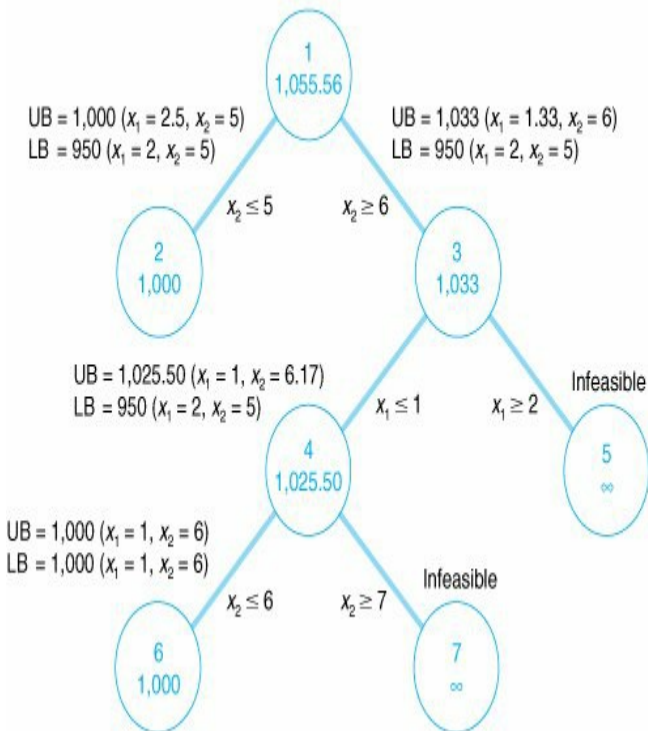
[Page C-9]

**Figure C-8. The
branch and bound
diagram with optimal**

solution at node 6

$$UB = 1,055.56 \ (x_1 = 2.22, x_2 = 5.56)$$

$$LB = 950 \ (x_1 = 2, x_2 = 5)$$



An optimal integer solution is reached when a feasible integer solution is achieved at a node that has an upper bound greater than or equal to the upper bound at any other ending node.

In general, the optimal integer

solution is reached when a feasible integer solution is generated at a node and the upper bound at that node is greater than or equal to the upper bound at any other *ending* node (i.e., a node at the end of a branch).

In the context of the original example, this solution indicates that if the machine shop owner purchases one press and six lathes, a daily increase in profit of \$1,000 will result.

The steps of the branch and bound method for determining

an optimal integer solution for a maximization model (with $\leq$ constraints) can be summarized as follows.

The steps of the branch and bound method.

- 1.** Find the optimal solution to the linear programming model with the integer restrictions relaxed.

2. At node 1 let the relaxed solution be the upper bound and the *rounded-down* integer solution be the lower bound.
3. Select the variable with the greatest fractional part for branching. Create two new constraints for this variable reflecting the partitioned integer values. The result will be a new $\leq$ constraint and a new $\geq$ constraint.
4. Create two new nodes, one for the $\geq$ constraint and one for the $\leq$ constraint.

5. Solve the *relaxed* linear programming model with the new constraint added at each of these nodes.
6. The relaxed solution is the upper bound at each node, and the *existing* maximum integer solution (at any node) is the lower bound.
7. If the process produces a feasible integer solution with the greatest upper bound value of any ending node, the optimal integer solution has been reached. If a feasible integer

solution does not emerge,
branch from the node with
the greatest upper bound.

8. Return to step 3.

For a minimization model,
relaxed solutions are rounded
up, and upper and lower
bounds are reversed.

*The branch and
bound method can be
used for mixed
integer problems,
except only variables
with integer
restrictions are*

*rounded down to
achieve the initial
lower bound and only
integer variables are
branched on.*

Solution of the Mixed Integer Model

Mixed integer linear
programming problems can also
be solved using the branch and

bound method. The same basic steps that were applied to the total integer model in the previous section are used for a mixed integer model with only a few differences.

First, at node 1 only those variables with integer restrictions are rounded down to achieve the *lower bound*. Second, in determining which variable to branch from, we select the greatest fractional part from among only those variables that must be integer. All other steps remain the same. The optimal solution is

reached when a feasible solution is generated at a node that has integer values for those variables requiring integers and that has reached the maximum upper bound of all ending nodes.

Solution of the 01 Integer Model

The 01 integer model can also be solved using the branch and bound method. First, the 01 restrictions for variables must be reflected as model

constraints, $x_j \leq 1$. As an example, consider the following 01 integer model for selecting recreational facilities following from [chapter 5](#) in the text.

A community council must decide which recreation facilities to construct in its community. Four new recreation facilities have been proposed: a swimming pool, a tennis center, an athletic field, and a gymnasium. The council wants to construct facilities that will maximize the expected daily usage by the residents of

the community subject to land and cost limitations. The expected daily usage and cost and land requirements for each facility follow.

[Page C-10]

Recreation Facility	Expected Usage (people/day)	Cost (\$)	Land Requirements (acres)
Swimming pool	300	35,000	4
Tennis center	90	10,000	2

Athletic field	400	25,000	7
Gymnasium	150	90,000	3

The community has a \$120,000 construction budget and 12 acres of land. Because the swimming pool and tennis center must be built on the same part of the land parcel, however, only one of these two facilities can be constructed. The council wants to know which of the recreation

facilities to construct in order to maximize the expected daily usage. The model for this problem is formulated as follows.

$$\text{maximize } Z = 300x_1 + 90x_2 + 400x_3 + 150x_4$$

subject to

$$\$35,000x_1 + 10,000x_2 + 25,000x_3 + 90,000x_4 \leq \$120,000 \text{ (capital budget)}$$

$$4x_1 + 2x_2 + 7x_3 + 3x_4 \leq 12 \text{ acres (space available)}$$

$$x_1 + x_2 \leq 1 \text{ facility}$$

$$x_1, x_2, x_3, x_4 = 0 \text{ or } 1$$

where

$$Z = \text{expected daily usage (people per day)}$$

x_1 = construction of a swimming pool

x_2 = construction of a tennis center

x_3 = construction of an athletic field

x_4 = construction of a gymnasium

In this model, the decision variables can have a solution value of either *zero* or *one*. If a facility is not selected for construction, the decision

variable representing it will have a value of zero. If a facility is selected, its decision variable will have a value of one.

The last constraint, $x_1 + x_2 \leq 1$, reflects the *contingency* that either the swimming pool (x_1) or the tennis center (x_2) can be constructed, but not both. In order for the sum of x_1 and x_2 to be less than or equal to one, either of the variables can have a value of one, or both variables can equal zero. This is also referred to as a *mutually*

exclusive constraint.

To apply the branch and bound method, the following four constraints have to be added to the model in place of the single restriction $x_1, x_2, x_3, x_4 = 0$ or 1.

$$x_1 \leq 1$$

$$x_2 \leq 1$$

$$x_3 \leq 1$$

$$x_4 \leq 1$$

The branch and bound method can be used for 01 integer problems by adding " ≤ 1 " constraints for each variable.

The only other change in the normal branch and bound method is at step 3. Once the variable x_j with the greatest fractional part has been determined, the two new constraints developed from this

variable are $x_j = 0$ and $x_j = 1$. These two new constraints will form the two branches at each node.

[Page C-11]

Another method for solving 01 integer problems is implicit enumeration. In implicit enumeration, obviously infeasible solutions are eliminated and the remaining solutions are evaluated (i.e., enumerated) to see which is the best. This approach will be demonstrated using our

original 01 example model for selecting a recreational facility (i.e., without the $x_j \leq 1$ constraints).

In implicit enumeration all feasible solutions are evaluated to see which is best.

The *complete enumeration* (i.e., the list of all possible solution sets) for this model is as

follows.

<i>Solution</i>	<i>x_1</i>	<i>x_2</i>	<i>x_3</i>	<i>x_4</i>	<i>Feasibility</i>	<i>Z Value</i>
1	0	0	0	0	Feasible	0
2	1	0	0	0	Feasible	300
3	0	1	0	0	Feasible	90
4	0	0	1	0	Feasible	400
5	0	0	0	1	Feasible	150
6	1	1	0	0	Infeasible	∞

7	1	0	1	0	Feasible	700
8	1	0	0	1	Infeasible	∞
9	0	1	1	0	Feasible	490
10	0	1	0	1	Feasible	240
11	0	0	1	1	Feasible	550
12	1	1	1	0	Infeasible	∞
13	1	0	1	1	Infeasible	∞
14	1	1	0	1	Infeasible	∞

15	0	1	1	1	Infeasible	∞
16	1	1	1	1	Infeasible	∞

Solutions 6, 12, 14, and 16 can be immediately eliminated because they violate the third constraint, $x_1 + x_2 \leq 1$.

Solutions 8, 13, and 15 can also be eliminated because they violate the other two constraints. This leaves eight possible solution sets

(assuming that solution 1 i.e., choosing none of the recreational facilities can be eliminated) for consideration. After evaluating the objective function value of these eight solutions, we find the best solution to be 7, with $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$. Within the context of the example, this solution indicates that a swimming pool (x_1) and an athletic field (x_3) should be constructed and that these facilities will generate an expected usage of 700 people per day.

The process of eliminating infeasible solutions and then evaluating the feasible solutions to see which is best is the basic principle behind implicit enumeration. However, implicit enumeration is usually done more systematically, by evaluating solutions with branching diagrams like those used in the branch and bound method, rather than by sorting through a complete enumeration as in this previous example.

Problems

Consider the following linear programming model

$$\text{maximize } Z = 5x_1 + 4x_2$$

subject to

$$3x_1 + 4x_2 \leq 10$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

1.

[Page C-12]

- 1.** Solve this model using the branch and bound method.
- 2.** Demonstrate the solution partitioning graphically.

Solve the following linear programming model using the branch and bound method.

2. minimize $Z = 3x_1 + 6x_2$
subject to
 $7x_1 + 3x_2 \geq 40$
 $x_1, x_2 \geq 0$ and integer

A tailor makes wool tweed sport coats and wool slacks. He is able to get a shipment of 150 square yards of wool cloth from Scotland each month to make coats and slacks, and he has 200 hours of his own labor to make them each month. A coat requires 3 square yards of wool and 10 hours to make, and a pair of pants requires 5 square yards of wool and 4 hours to make. He earns \$50 in profit from

each coat he makes and \$40 from each pair of slacks. He wants to know

- 3.** how many coats and slacks to produce to maximize profit.
 - 1.** Formulate an integer linear programming model for this problem.
 - 2.** Determine the integer solution to this problem using the branch and bound method. Compare this solution with the solution without integer restrictions and indicate if the rounded-down solution would have been optimal.

A jeweler and her apprentice make silver pins and necklaces by hand. Each week they have 80 hours of labor and 36 ounces of silver available. It requires 8 hours of labor and 2

ounces of silver to make a pin, and 10 hours of labor and 6 ounces of silver to make a necklace. Each pin also contains a small gem of some kind. The demand for pins is no more than six per week. A pin earns the jeweler \$400 in profit, and a necklace earns \$100. The jeweler wants to know

4. how many of each item to make each week in order to maximize profit.

- 1.** Formulate an integer programming model for this problem.
- 2.** Solve this model using the branch and bound method. Compare this solution with the solution without integer restrictions and indicate if the rounded-down solution would have been optimal.

A glassblower makes glass decanters and glass trays on a weekly basis. Each item requires 1 pound of glass, and the glassblower has 15 pounds of glass available every week. A glass decanter requires 4 hours of labor, a glass tray requires only 1 hour of labor, and the glassblower works 25 hours a week. The profit from a decanter is \$50, and the profit from a tray is \$10. The glassblower wants to determine the total number of

5. decanters (x_1) and trays (x_2) that he needs to produce in order to maximize his profit.

- 1.** Formulate an integer programming model for this problem.
- 2.** Solve this model using the branch and bound method.
- 3.** Demonstrate the solution

partitioning graphically.

The Livewright Medical Supplies Company has a total of 12 salespeople it wants to assign to three regions the South, the East, and the Midwest. A salesperson in the South earns \$600 in profit per month for the company, a salesperson in the East earns \$540, and a salesperson in the Midwest earns \$375. The southern region can have a maximum assignment of 5 salespeople. The company has a total of \$750 per day available for expenses for all 12 salespeople. A salesperson in the South has average expenses of \$80 per day, a salesperson in the East has average expenses of \$70 per day, and a salesperson in the Midwest has average daily expenses of \$50. The company wants to determine the

6.

number of salespeople to assign to each region to maximize profit.

[Page C-13]

- 1.** Formulate an integer programming model for this problem.
- 2.** Solve this model using the branch and bound method.

Helen Holmes makes pottery by hand in her basement. She has 20 hours available each week to make bowls and vases. A bowl requires 3 hours of labor, and a vase requires 2 hours of labor. It requires 2 pounds of special clay to make a bowl and 5 pounds to produce a vase; she is able to acquire 35 pounds of clay per week. She sells her bowls for \$50 and her vases for

\$40. She wants to know how many of each item to make each week in order to maximize her revenue.

1. Formulate an integer programming model for this problem.
2. Solve this model using the branch and bound method. Compare this solution with the solution with integer restrictions and indicate if the rounded-down solution would have been optimal.

Lauren Moore has sold her business for \$500,000 and wants to invest in condominium units (which she intends to rent) and land (which she will lease to a farmer). She estimates that she will receive an annual return of \$8,000 for each condominium and \$6,000 for

- each acre of land. A condominium unit costs \$70,000, and land is \$30,000 per acre. A condominium will cost her
- 8.** \$1,000 per unit and an acre of land \$2,000 for maintenance and upkeep. Lauren wants to know how much to invest in condominiums and land in order to maximize her annual return.
- 1.** Formulate a mixed integer programming model for this problem.
 - 2.** Solve this model using the branch and bound method.

The owner of the Consolidated Machine Shop has \$10,000 available to purchase a lathe, a press, a grinder, or some combination thereof. The following 01 integer linear programming model has been developed for determining which of

- the three machines (lathe, x_1 ; press, x_2 ; grinder, x_3) should be purchased in order to maximize the annual profit.

$$\text{maximize } Z = 1,000x_1 + 700x_2 + 800x_3 \text{ (profit, \$)}$$

subject to

$$5,000x_1 + 6,000x_2 + 4,000x_3 \leq 10,000 \text{ (cost, \$)}$$

$$x_1, x_2, x_3 = 0 \text{ or } 1$$

Solve this model using the branch and bound method.

Solve the following mixed integer linear programming model using the branch and bound method.

10.

maximize $Z = 5x_1 + 6x_2 + 4x_3$

subject to

$$5x_1 + 3x_2 + 6x_3 \leq 20$$

$$x_1 + 3x_2 \leq 12$$

$$x_1, x_3 \geq 0$$

$$x_2 \geq 0 \text{ and integer}$$

- 11.** Solve [problem 9](#) using the implicit enumeration method.

Consider the following linear programming model.

maximize $Z = 20x_1 + 30x_2 + 10x_3 + 40x_4$
subject to

12.
$$\begin{aligned} 2x_1 + 4x_2 + 3x_3 + 7x_4 &\leq 10 \\ 10x_1 + 7x_2 + 20x_3 + 15x_4 &\leq 40 \\ x_1 + 10x_2 + x_3 &\leq 10 \\ x_1, x_2, x_3, x_4 &= 0 \text{ or } 1 \end{aligned}$$

[Page C-14]

- 1.** Solve this problem using the implicit enumeration method.
- 2.** What difficulties would be encountered with the implicit enumeration method if this problem were expanded to contain five or more variables and more constraints?

Module D. Nonlinear Programming Solution Techniques

[Page D-2]

Most mathematical techniques for solving nonlinear programming problems are very complex. In this module two of the more well known but simpler mathematical methods will be demonstrated the substitution method and the method of Lagrange multipliers.

The Substitution Method

The least complex method for solving nonlinear programming problems is referred to as *substitution*. This method is restricted to models that contain only equality constraints, and typically only one of these. The method involves solving the constraint equation for one variable in terms of another. This new expression is then substituted into the objective function,

effectively eliminating the constraint. In other words, a constrained optimization model is transformed into an unconstrained model.

*In the **substitution method** the constraint equation is solved for one variable in terms of another and then substituted into the objective function.*

For an example of the substitution method we will use a profit analysis model. This is a nonlinear model that we introduced in [chapter 10](#) of the text. The demand function is a constraint. The nonlinear programming model is formulated as

$$\text{maximize } Z = v p - c_f v - c_v v$$

subject to

$$v = 1,500 - 24.6p$$

The objective function in this model is nonlinear, because

both v (volume) and p (price) are variables and multiplying them (i.e., vp) creates a curvilinear relationship.

The constraint has already been solved for one variable (v) in terms of another (p); thus, we can substitute this expression directly into the objective function. This results in the following unconstrained function.

$$Z = 1,500p - 24.6p^2 + c_f - 1,500c_v + 24.6pc_v$$

By substituting the constant values for c_f (\$10,000) and c_v (\$8), we obtain

$$Z = 1,696.8p - 24.6p^2 - 22,000$$

Next, we solve this problem by differentiating the function Z and setting it equal to zero.

$$\frac{\partial Z}{\partial p} = 1,696.8 - 49.2p$$

$$0 = 1,696.8 - 49.2p$$

$$49.2p = 1,696.8$$

$$p = \$34.49$$

We will present another example as a further illustration of a nonlinear programming problem and the substitution method. In this example the Beaver Creek Pottery Company produces bowls (x_1) and mugs (x_2). We will assume that the profit contribution for each product *declines* as the quantity of each item produced increases. Thus, for bowls the per unit profit contribution is now expressed according to the relationship

$$\$4 - 0.1x_1$$

For mugs, the profit contribution per unit is

$$\$5 - 0.2x_2$$

(These relationships express the fact that production costs for each product increase as the number of units sold increases.)

These profit relationships are on a per unit basis. Thus, the total profit contribution from each product is determined by multiplying these relationships by the number of units produced. For bowls, the profit

contribution is

$$(4 - 0.1x_1)x_1$$

or

$$4x_1 - 0.1x_1^2$$

For mugs, the profit contribution is

$$(5 - 0.2x_2)x_2$$

or

$$5x_2 - 0.2x_2^2$$

Total profit is the sum of these

two terms.

$$Z = \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

In this model, we will consider only a labor constraint, and we will treat it as an equality rather than an inequality.

$$x_1 + 2x_2 = 40 \text{ hr}$$

The complete nonlinear programming model is as follows.

$$\text{maximize } Z = \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

subject to

$$x_1 + 2x_2 = 40$$

The first step in the substitution method is to solve the constraint equation for one variable in terms of another. We will arbitrarily decide to solve for x_1 as follows.

$$x_1 = 40 - 2x_2$$

Now wherever x_1 appears in the nonlinear objective function, we will substitute the expression $40 - 2x_2$.

$$Z = 4(40 - 2x_2) - 0.1(40 - 2x_2)^2 + 5x_2 - 0.2x_2^2$$

$$= 160 - 8x_2 - 0.1(1,600 - 1,600x_2 + 4x_2^2) + 5x_2 - 0.2x_2^2$$

$$= 160 - 8x_2 - 160 + 160x_2 - 0.4x_2^2 + 5x_2 - 0.2x_2^2$$

$$= 13x_2 - 0.6x_2^2$$

This is an unconstrained optimization function, and we can solve it by differentiating it and setting it equal to zero.

$$\begin{aligned}\frac{\partial Z}{\partial x_2} &= 13 - 1.2x_2 \\ 0 &= 13 - 1.2x_2 \\ 1.2x_2 &= 13 \\ x_2 &= 10.8 \text{ mugs}\end{aligned}$$

To determine x_1 , we can substitute x_2 into the constraint equation.

$$\begin{aligned}x_1 + 2x_2 &= 40 \\ x_1 + 2(10.8) &= 40 \\ x_1 &= 18.4 \text{ bowls}\end{aligned}$$

Substituting the values of x_1 and x_2 into the original objective function gives the total profit, as follows.

$$\begin{aligned} Z &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 \\ &= \$4(18.4) - 0.1(18.4)^2 + 5(10.8) - 0.2(10.8)^2 \\ &= \$70.42 \end{aligned}$$

Both of the examples presented in this section for solving nonlinear programming problems exhibit the limitations of this approach. The objective functions were not very complex (i.e., the highest order of a variable was a power of two in the second example),

there were only two variables, and the single constraint in each example was an equation. This method becomes very difficult if the constraint becomes complex. An alternative solution approach that is not quite as restricted is the method of Lagrange multipliers.

The Method of Lagrange Multipliers

The method of Lagrange multipliers is a general mathematical technique that can be used for solving constrained optimization problems consisting of a nonlinear objective function and one or more linear or nonlinear constraint equations. In this method, the constraints as multiples of a Lagrange multiplier, λ , are subtracted

from the objective function.

*In the method of **Lagrange multipliers**, constraints as multiples of a multiplier, λ , are subtracted from the objective function, which is then differentiated with respect to each variable and solved.*

To demonstrate this method, we will use our modified pottery company example developed in the preceding section. This model was formulated as

$$\text{maximize } Z = 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

subject to

$$x_1 + 2x_2 = 40$$

The first step is to transform the nonlinear objective function into a *Lagrangian function*. This

is accomplished by transforming the constraint equation as follows.

$$x_1 + 2x_2 - 40 = 0$$

Next, this expression is multiplied by λ the *Lagrangian multiplier*, and subtracted from the objective function to form the Lagrangian function L.

$$L = 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 - \lambda(x_1 + 2x_2 - 40)$$

Because the constraint equation now equals zero, the subtraction of the constraint, multiplied by λ , from the objective function does not affect the value of the function. (We will explain the exact meaning of λ after a solution has been determined.)

Now we must determine the partial derivatives of the Lagrangian function with respect to each of the three variables, x_1 , x_2 , and λ .

*The **Lagrangian***

function is
differentiated with
respect to each
variable, and the
resulting equations
are solved
simultaneously.

$$\frac{\partial L}{\partial x_1} = 4 - 0.2x_1 - \lambda$$

$$\frac{\partial L}{\partial x_2} = 5 - 0.4x_2 - 2\lambda$$

$$\frac{\partial L}{\partial \lambda} = -x_1 - 2x_2 + 40$$

These three equations are all set equal to zero and solved simultaneously to determine the values of x_1 , x_2 , and λ .

$$\begin{aligned}4 - 0.2x_1 - \lambda &= 0 \\5 - 0.4x_2 - 2\lambda &= 0 \\-x_1 - 2x_2 + 40 &= 0\end{aligned}$$

To solve these equations simultaneously, we multiply the first equation by 2 and add it to the second equation, which eliminates λ .

$$\begin{array}{r} -8 + 0.4x_1 + 2\lambda = 0 \\ 5 - 0.4x_2 - 2\lambda = 0 \\ \hline -3 + 0.4x_1 - 0.4x_2 = 0 \end{array}$$

This new equation and the original preceding third equation represent two equations with two unknowns (x_1 and x_2). We multiply the preceding third equation by 0.4 and add it to the new equation in order to eliminate x_1 .

$$\begin{array}{r} -0.4x_1 - 0.8x_2 + 16 = 0 \\ 0.4x_1 - 0.4x_2 - 3 = 0 \\ \hline -1.2x_2 + 13 = 0 \end{array}$$

The resulting equation is solved for x_2 as follows.

$$-1.2x_2 + 13 = 0$$

$$-1.2x_2 = -13$$

$$x_2 = 10.8 \text{ mugs}$$

Substituting this value back into the previous equations gives the values of both x_1 and λ .

$$-x_1 - 2x_2 + 40 = 0$$

$$-x_1 - 2(10.8) = -40$$

$$x_1 = 18.4 \text{ bowls}$$

$$5 - 0.4x_2 - 2\lambda = 0$$

$$5 - 0.4(10.8) = 2\lambda$$

$$\lambda = 0.33$$

Substituting the values for x_1 and x_2 into the original objective function yields the

total profit.

$$\begin{aligned} Z &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 \\ &= \$4(18.4) - 0.1(18.4)^2 + 5(10.8) - 0.2(10.8)^2 \\ &= \$70.42 \end{aligned}$$

[Page D-6]

This result can also be obtained by using the Lagrangian function, L , and multiplier, λ .

$$\begin{aligned} L &= \$4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2 - \lambda(x_1 + 2x_2 - 40) \\ &= \$4(18.4) - 0.1(18.4)^2 + 5(10.8) - 0.2(10.8)^2 - 0.33(0) \\ &= \$70.42 \end{aligned}$$

To summarize, we have $x_1 = 18.4$ bowls, $x_2 = 10.8$ mugs, Z

= \$70.42, and $\lambda = 0.33$. This is the same answer obtained previously using the substitution method. However, unlike the substitution method, the Lagrange multiplier approach can be used to solve nonlinear programming problems with more complex constraint equations and *inequality constraints*. In addition, it can encompass problems with more than two variables. We will not pursue any examples, though, that demonstrate the complexities involved. The Lagrange

multiplier method must be altered to compensate for inequality constraints and additional variables, and the resulting mathematics are very difficult. Even though the Lagrange multiplier method is more flexible than the substitution method, it is practical for solving only small problems. As the size of the problem increases, the mathematics become overwhelmingly difficult.

The Meaning of λ

The Lagrange multiplier, λ , in nonlinear programming problems is analogous to the dual variables in a linear programming problem. It reflects the approximate change in the objective function resulting from a unit change in the quantity (right-hand-side) value of the constraint equation. For our example, we will increase the quantity value in the constraint equation from 40 to 41 hours of labor. (You will recall from the preceding calculations that $\lambda = 0.33$.)

The **Lagrange multiplier**, λ , reflects a change in the objective function from a unit change in the right-hand-side value of a constraint.

$$x_1 + 2x_2 = 41 \text{ or } x_1 + 2x_2 - 41 = 0$$

This constraint equation will result in the following Lagrangian function.

$$L = 4x_1 - 0.1x_1^2 + 5x_2 - 0.2x_2^2$$

$$\lambda(x_1 + 2x_2 - 41)$$

Solving this problem the same way we solved the original model gives the following solution.

$$x_1 = 18.7$$

$$x_2 = 11.2$$

$$\lambda = 0.27$$

$$Z = \$70.75$$

This value for Z is \$0.33 greater than the previous Z

value of \$70.42. Thus, a one unit increase in the right-hand side of the constraint equation results in a λ increase in the objective function. More specifically, a unit increase in a resource (labor) results in a \$0.33 increase in profit. Thus, we would be willing to pay \$0.33 for one additional hour of labor. This is the same interpretation as that given for a dual variable in linear programming.

In general, if λ is positive, the optimal objective function value will increase if the

quantity (absolute) value in the constraint equation is increased, and it will decrease if the quantity (absolute) value is decreased. On the other hand, if λ is negative, the optimal objective function value will increase if the quantity (absolute) value is decreased, and it will decrease if the quantity (absolute) value is increased.

Problems

The Hickory Cabinet and Furniture Company makes chairs. The fixed cost per month of making chairs is \$7,500, and the variable cost per chair is \$40. Price is related to demand according to the following linear equation.

1. $v = 400 - 1.2p$

Develop the nonlinear profit function for this company and determine the price that will maximize profit, the optimal volume, and the maximum profit per month.

Graphically illustrate the profit curve developed in [problem 1](#).

2. Indicate the optimal price and the maximum profit per month.

The Rainwater Brewery produces beer. The annual fixed cost is \$150,000, and the variable cost per barrel is \$16. Price is related to demand according to the following linear equation.

3. $v = 75,000 - 1,153.8p$

Develop the nonlinear profit function for the brewery and determine the price that will maximize profit, the optimal volume, and the maximum profit per year.

The Rolling Creek Textile Mill makes denim. The monthly fixed cost is \$8,000, and the variable cost per yard of denim is \$0.35. Price is related to demand according to the following linear equation.

4.

$$v = 17,000 - 5,666p$$

Develop the nonlinear profit function for the textile mill and determine the optimal price, the optimal volume, and the maximum profit per month.

The Grady Tire Company recaps tires. The weekly fixed cost is \$2,500, and the variable cost per tire is \$9. Price is related to demand according to the following linear equation.

5.

$$v = 200 - 4.75p$$

Develop the nonlinear profit function for the tire company and determine the optimal price, the optimal volume, and the maximum profit per week.

Andy Mendoza makes handcrafted dolls, which he sells at craft fairs. He is considering massproducing the dolls to sell in stores. He estimates that the initial investment for plant and equipment will be \$25,000, while labor, material, packaging, and shipping will be about \$10 per doll.

6. He has determined that sales volume is related to price according to the following linear equation.

$$v = 4,000 - 80p$$

Develop the nonlinear profit function for Andy and determine the price that will maximize profit, the optimal volume, and the maximum profit per month.

The Rainwater Brewery produces beer, which it sells to distributors in barrels. The brewery incurs a monthly fixed cost of \$12,000, and the variable cost per barrel is \$17. The brewery has developed the following profit function and demand constraint.

7. maximize $Z = vp - \$12,000 - 17v$

subject to

$$v = 800 - 15p$$

Solve this nonlinear programming model for the optimal price (p) using the substitution method.

The Beaver Creek Pottery Company has developed the following nonlinear programming model to determine the optimal number of bowls (x_1) and mugs (x_2) to produce each day.

8. maximize $Z = \$7x_1 - 0.3x_1^2 + 8x_2 - 0.4x_2^2$

subject to

$$4x_1 + 5x_2 = 100 \text{ hr}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

The Evergreen Fertilizer Company produces two types of fertilizers, Fastgro and Super Two. The company has developed the following nonlinear programming model to determine the optimal number of bags of Fastgro (x_1) and Super Two (x_2) that they must produce each day to maximize profit, given a constraint for available potassium.

9.

$$\text{maximize } Z = \$30x_1 - 2x_1^2 + 25x_2 - 0.5x_2^2$$

subject to

$$3x_1 + 6x_2 = 300 \text{ lb}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

The Rolling Creek Textile Mill produces denim and brushed-cotton cloth. The company has developed the following nonlinear programming model to determine the optimal number of yards of denim (x_1) and brushed cotton (x_2) to produce each day to maximize profit, subject to a labor constraint.

10.

$$\text{maximize } Z = \$10x_1 - 0.02x_1^2 + 12x_2 - 0.03x_2^2$$

subject to

$$0.2x_1 + 0.1x_2 = 40 \text{ hr}$$

Determine the optimal solution to this nonlinear programming model using the substitution method.

- 11.** Solve [problem 8](#) using the method of Lagrange multipliers.
- 12.** Solve [problem 9](#) using the method of Lagrange multipliers.
- 13.** Solve [problem 10](#) using the method of Lagrange multipliers.

The Riverwood Paneling Company makes two kinds of wood paneling, Colonial and Western. The company has developed the following nonlinear programming model to determine the optimal number of sheets of Colonial paneling (x_1) and Western paneling (x_2) to produce to maximize profit, subject to a labor

constraint.

14. maximize $Z = \$25x_1 - 0.08x_1^2 + 30x_2 - 1.2x_2^2$

subject to

$$x_1 + 2x_2 = 40 \text{ hr}$$

[Page D-9]

Determine the optimal solution to this nonlinear programming model using the method of Lagrange multipliers.

- 15.** Interpret the meaning of λ , the Lagrange multiplier, in [problem 14](#).

Module E. Game Theory

[Game Theory](#)

[Types of Game Situations](#)

[A Pure Strategy](#)

[A Mixed Strategy](#)

[Problems](#)

Game Theory

In [Chapter 12](#) in the text, on decision analysis, we discussed methods to aid the *individual* decision maker. All the decision situations involved one decision maker. There were no *competitors* whose decisions might alter the decision maker's analysis of a decision situation. However, many situations do, in fact, involve several decision makers who compete with one another to

arrive at the best outcome. These types of competitive decision-making situations are the subject of **game theory**. Although the topic of game theory encompasses a different type of decision situation than does decision analysis, many of the fundamental principles and techniques of decision making apply to game theory as well. Thus, game theory is, in effect, an extension of decision analysis rather than an entirely new topic area.

Game theory

addresses decision situations with two or more decision makers in competition.

Anyone who has played card games or board games is familiar with situations in which competing participants develop plans of action to win. Game theory encompasses similar situations in which competing decision makers develop plans of action to win. In addition,

game theory consists of several mathematical techniques to aid the decision maker in selecting the plan of action that will result in the best outcome. In this module we will discuss some of those techniques.

Types of Game Situations

Competitive game situations can be subdivided into several categories. One classification is based on the number of competitive decision makers, called *players*, involved in the game. A game situation consisting of two players is referred to as a *two-person game*. When there are more than two players, the game situation is known as an *n-person game*.

A two-person game encompasses two players.

Games are also classified according to their outcomes in terms of each player's gains and losses. If the sum of the players' gains and losses equals zero, the game is referred to as a *zero-sum game*. In a two-person game, one player's gains represent another's losses. For example, if one

player wins \$100, then the other player loses \$100; the two values sum to zero (i.e., +\$100 and -\$100).

Alternatively, if the sum of the players' gains and losses does not equal zero, the game is known as a *non-zero-sum game*.

In a zero-sum game, one player's gains represent another's exact losses.

The *two-person, zero-sum game* is the one most frequently used to demonstrate the principles of game theory because it is the simplest mathematically. Thus, we will confine our discussion of game theory to this form of game situation. The complexity of the *n-person* game situation not only prohibits us from demonstrating it but also restricts its application in real-world situations.

The Two-Person, Zero-

Sum Game

Examples of competitive situations that can be organized into two-person, zero-sum games include (1) a union negotiating a new contract with management; (2) two armies participating in a war game; (3) two politicians in conflict over a proposed legislative bill, one attempting to secure its passage and the other attempting to defeat it; (4) a retail firm trying to increase its market share with a new product and a competitor

attempting to minimize the firm's gains; and (5) a contractor negotiating with a government agent for a contract on a project.

[Page E-3]

The following example will demonstrate a two-person, zero-sum game. A professional athlete, Biff Rhino, and his agent, Jim Fence, are renegotiating Biff's contract with the general manager of the Texas Buffaloes, Harry Sligo. The various outcomes of

this game situation can be organized into a payoff table similar to the payoff tables used for decision analysis. The payoff table for this example is shown in [Table E.1](#).

Table E.1. Payoff Table for Two-Person, Zero-Sum Game

		General Manager Strategy		
Athlete/Agent Strategy		<i>A</i>	<i>B</i>	<i>C</i>
	1	\$50,000	\$35,000	\$30,000

2

60,000

40,000

20,000

The payoff table for a two-person game is organized so that the player who is trying to maximize the outcome of the game is on the left and the player who is trying to minimize the outcome is on the top. In [Table E.1](#) the athlete and agent want to maximize the athlete's contract, and the general manager hopes to minimize the athlete's contract.

In a sense, the athlete is an offensive player in the game, and the general manager is a defensive player. In game theory, it is assumed that the payoff table is known to both the offensive player and the defensive player an assumption that is often unrealistic in real-world situations and thus restricts the actual application of this technique.

In a game situation, it is assumed that the payoff table is known to all players.

A **strategy** is a plan of action that a player follows. Each player in a game has two or more strategies, only one of which is selected for each playing of a game. In [Table E.1](#) the athlete and his agent have two strategies available, 1 and 2, and the general manager has three strategies, A, B, and C. The values in the table are the payoffs or outcomes associated with each player's strategies.

*A **strategy** is a plan of action that a player follows.*

For our example, the athlete's strategies involve different types of contracts and the threat of a holdout and/or of becoming a free agent. The general manager's strategies are alternative contract proposals that vary with regard to such items as length of contract, residual payments,

no-cut/no-trade clauses, and off-season promotional work. The outcomes are in terms of dollar value. If the athlete selects strategy 2 and the general manager selects strategy C, the outcome is a \$20,000 gain for the athlete and a \$20,000 loss for the general manager. This outcome results in a zero sum for the game (i.e., $+\$20,000 - \$20,000 = 0$). The amount \$20,000 is known as the *value of the game*.

The value of the

game is the offensive player's gain and the defensive player's loss in a zero-sum game.

The purpose of the game for each player is to select the strategy that will result in the best possible payoff or outcome, regardless of what the opponent does. The best strategy for each player is known as the *optimal strategy*.

Next, we will discuss methods for determining strategies.

The best strategy for each player is his or her optimal strategy.

A Pure Strategy

When each player in a game adopts a single strategy as an optimal strategy, the game is a *pure strategy* game. The value of a pure strategy game is the same for both the offensive player and the defensive player. In contrast, in a *mixed strategy* game, the players adopt a mixture of strategies if the game is played many times.

In a pure strategy

game, each player adopts a single strategy as an optimal strategy.

[Page E-4]

A pure strategy game can be solved according to the **minimax decision criterion**. According to this principle, each player plays the game to minimize the maximum possible losses. The offensive

player selects the strategy with the largest of the minimum payoffs (called the *maximin* strategy), and the defensive player selects the strategy with the smallest of the maximum payoffs (called the *minimax* strategy). In our example involving the athlete's contract negotiation process, the athlete will select the maximin strategy from strategies 1 and 2, and the general manager will select the minimax strategy from strategies A, B, and C. We will first discuss the athlete's decision, although in game

theory the decisions are actually made *simultaneously*.

*With the **minimax decision criterion**, each player seeks to minimize maximum possible losses; the offensive player selects the strategy with the largest of the minimum payoffs, and the defensive player selects the strategy with the smallest of the maximum payoffs.*

To determine the maximin strategy, the athlete first selects the minimum payoff for strategies 1 and 2, as shown in [Table E.2](#). The maximum of these minimum values indicates the optimal strategy and the value of the game for the athlete.

Table E.2. Payoff Table with Maximin Strategy

Athlete/Agent Strategy	General Manager Strategy			← Maximum of minimum payoffs
	A	B	C	
1	\$50,000	\$35,000	\$30,000	
2	60,000	40,000	20,000	

The value \$30,000 is the maximum of the minimum values for each of the athlete's strategies. Thus, the optimal strategy for the athlete is strategy 1. The logic behind this decision is as follows. If the athlete selected strategy 1, the general manager could be

expected to select strategy C, which would minimize the possible loss (i.e., a \$30,000 contract is better *for the manager* than a \$50,000 or \$35,000 contract).

Alternatively, if the athlete selected strategy 2, the general manager could be expected to select strategy C for the same reason (i.e., a \$20,000 contract is better for the manager than a \$60,000 or \$40,000 contract). Now, because the athlete has anticipated how the general manager will respond to each strategy, he realizes

that he can negotiate either a \$30,000 or a \$20,000 contract. The athlete selects strategy 1 in order to get the larger possible contract of \$30,000, given the actions of the general manager.

Simultaneously, the general manager applies the minimax decision criterion to strategies A, B, and C. First, the general manager selects the maximum payoff for each strategy, as shown in [Table E.3](#). The minimum of these maximum values determines the optimal strategy and the value of the

game for the general manager.

Table E.3. Payoff Table with Minimax Strategy

Athlete/Agent Strategy	General Manager Strategy			← Minimum of maximum values
	A	B	C	
1	\$50,000	\$35,000	\$30,000	
2	60,000	40,000	20,000	

The value \$30,000 is the

minimum of the maximum values for each of the strategies of the general manager. Thus, the optimal strategy for the general manager is C. The logic of this decision is similar to that of the athlete's decision. If the general manager selected strategy A, the athlete could be expected to select strategy 2 with a payoff of \$60,000 (i.e., the athlete will choose the better of the \$50,000 and \$60,000 contracts). If the general manager selected strategy B, then the athlete

could be expected to select strategy 2 for a payoff of \$40,000. Finally, if the general manager selected strategy C, the athlete could be expected to select strategy 1 for a payoff of \$30,000. Because the general manager has anticipated how the athlete will respond to each strategy, he realizes that either a \$60,000, \$40,000, or \$30,000 contract could possibly be awarded. Thus, the general manager selects strategy C, which will result in the minimum contract of \$30,000. In general, the

manager considers the worst outcome that could result if a particular strategy were followed. Under the minimax criterion, the general manager will select the strategy that ensures that he loses only the minimum of the maximum amounts that could be lost.

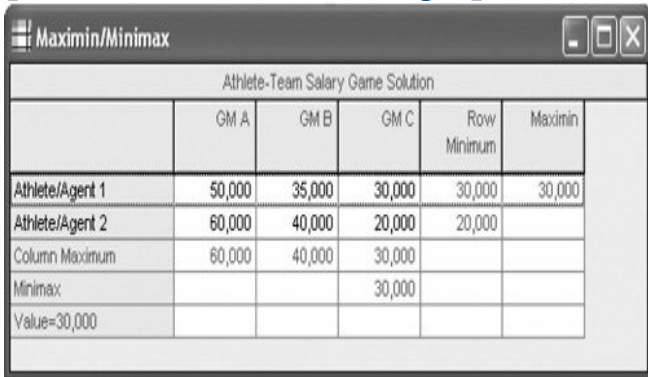
[Page E-5]

Solution of Game Theory Problems with QM for Windows

QM for Windows, which we use in the text to solve decision analysis problems, also has a module for solving game theory problems. The QM for Windows solution for our athlete/agent example is shown in [Exhibit E.1](#). Notice that QM for Windows indicates the row minimums and column maximums, and it provides the maximin solution in the upper-right-hand corner of the solution table.

Exhibit E.1.

[\[View full size image\]](#)



The screenshot shows a window titled "Maximin/Minimax" with a table titled "Athlete-Team Salary Game Solution". The table has 6 columns: Athlete/Agent, GM A, GM B, GM C, Row Minimum, and Maximin. The rows represent the choices of Athlete/Agent 1 and Athlete/Agent 2, along with the Column Maximum, Minimax value, and the final Value=30,000.

	GM A	GM B	GM C	Row Minimum	Maximin
Athlete/Agent 1	50,000	35,000	30,000	30,000	30,000
Athlete/Agent 2	60,000	40,000	20,000	20,000	
Column Maximum	60,000	40,000	30,000		
Minimax			30,000		
Value=30,000					

Dominant Strategies

We could have reduced the choices of the general manager

if we had noticed that strategy *C dominates* strategies A and B. Dominance occurs when all the payoffs for one strategy are better than the corresponding payoffs for another strategy. In [Table E.3](#) the values \$30,000 and \$20,000 are both lower than the corresponding payoffs of \$50,000 and \$60,000 for strategy A and the corresponding payoffs of \$35,000 and \$40,000 for strategy B. Because strategy C dominates A and B, these two latter strategies can be eliminated from consideration

altogether, as shown in [Table E.4](#). If this had been done earlier, strategy C could have been selected automatically, without applying the minimax criterion. Thus, the most efficient approach is to first examine the payoff table for dominance in order to possibly reduce its size.

A strategy is dominated, and can be eliminated, if all its payoffs are worse than the corresponding

payoffs for another strategy.

Table E.4. Payoff Table with Dominated Strategies Eliminated

**General Manager
Strategy**

**Athlete/Agent
Strategy**

C

1

\$30,000

The fact that the optimal strategy for each player in this game resulted in the *same pay-off* game value of \$30,000 is what classifies it as a pure strategy game. In other words, because strategy 1 is optimal for the athlete and strategy C is optimal for the general manager, a contract for \$30,000 will be awarded to the athlete. Because the outcome

of \$30,000 results from a pure strategy, it is referred to as an **equilibrium point** (or sometimes as a *saddle point*). A point of equilibrium is a value that is simultaneously the *minimum of a row* and the *maximum of a column*, as is the payoff of \$30,000 in [Table E.3](#).

[Page E-6]

In a pure strategy game, the optimal strategy for each player results in the same payoff, called

*an **equilibrium**, or
saddle, point.*

It is important to realize that the minimax criterion results in the optimal strategy for each player as long as each player uses this criterion. If one of the players does not use this criterion, the solution of the game will not be optimal. If we assume that both players are logical and rational, however, we can assume that this

criterion will be employed.

*The equilibrium point
in a game is
simultaneously the
minimum of a row
and the maximum of
a column.*

If an equilibrium point exists, it makes the determination of optimal strategies relatively easy because no complex mathematical calculations are

necessary. However, as mentioned earlier, if a game does not involve a pure strategy, it is a mixed strategy game. We will discuss mixed strategy games next.

The minimax criterion will result in the optimal strategies only if both players use it.

A Mixed Strategy

A mixed strategy game occurs when each player selects an optimal strategy and the two strategies do not result in an equilibrium point (i.e., the same outcome) when the maximin and minimax decision criteria are applied.

A mixed strategy game occurs when each player selects an optimal strategy

*that does not result
in an equilibrium
point when the
minimax criterion is
used.*

The following example will demonstrate a mixed strategy game. The Coloroid Camera Company (which we will refer to as company I) is going to introduce a new camera into its product line and hopes to capture as large an increase in

its market share as possible. In contrast, the Camco Camera Company (which we will refer to as company II) hopes to minimize Coloroid's market share increase. Coloroid and Camco dominate the camera market, and any gain in market share for Coloroid will result in a subsequent identical loss in market share for Camco. The strategies for each company are based on their promotional campaigns, packaging, and cosmetic differences between the products. The payoff table, which includes the strategies

and outcomes for each company (I = Coloroid and II = Camco), is shown in [Table E.5](#). The values in [Table E.5](#) are the *percentage* increases or decreases in market share for company I.

Table E.5. Payoff Table for Camera Companies

		Company II Strategy		
Company I Strategy		A	B	C
	1	9	7	2

2	11	8	4
3	4	1	7

The first step is to check the payoff table for any dominant strategies. Doing so, we find that strategy 2 dominates strategy 1, and strategy B dominates strategy A. Thus, strategies 1 and A can be eliminated from the payoff table, as shown in [Table E.6](#).

Table E.6. Payoff Table with Strategies 1 and A Eliminated

Company II Strategy

Company I Strategy

B

C

2

8

4

3

1

7

[Page E-7]

Time Out: For John von Neumann and Oskar Morgenstern

John von Neumann first introduced the topic of game theory in 1928 at age 24 while on the faculty of the University of Berlin. He had received his doctorate in mathematics from the University of Budapest only 2 years earlier. He continued to develop the theory during the next 15 years. In 1944, while he was a member of the Institute for Advanced Study at Princeton University, as well as a consultant with the Navy Bureau of Ordinance and the Los Alamos Laboratory (working on the atomic bomb), his work culminated in the publication of the book *Theory of Games and Economic Behavior* with

Oskar Morgenstern. Morgenstern was an economist who had arrived at Princeton in 1938 from his native Austria. Besides providing a mathematical basis for how strategies are developed in game situations, von Neumann and Morgenstern's work also influenced other researchers, such as George Dantzig, in his development of linear programming, and Richard Bellman, in the development of dynamic programming.

Next, we apply the maximin decision criterion to the strategies for company I, as shown in [Table E.7](#). The

minimum value for strategy 2 is 4%, and the minimum value for strategy 3 is 1%. The maximum of these two minimum values is 4%; thus, strategy 2 is optimal for company I.

Table E.7. Payoff Table with Maximin Criterion

Company I Strategy	Company II Strategy		← Minimum of the maximum values
	B	C	
2	8	4	
3	1	7	

Next, we apply the minimax decision criterion to the strategies for company II in [Table E.8](#). The maximum value for strategy B is 8%, and the maximum value for strategy C is 7%. Of these two maximum values, 7% is the minimum; thus, the optimal strategy for

company II is C.

Table E.8. Payoff Table with Minimax Criterion

Company I Strategy	Company II Strategy	
	B	C
2	⑧	4
3	1	⑦

← Minimum of the maximum values

Table E.9 combines the results

of the application of the maximin and minimax criteria by the companies.

Table E.9. Company I and II Combined Strategies

Company I Strategy		
Company II Strategy	<i>B</i>	<i>C</i>
2	8	④
3	1	⑦

From [Table E.9](#) we can see that the strategies selected by the companies do not result in an equilibrium point. Therefore, this is not a pure strategy game. In fact, this condition will not result in any strategy for either firm. Company I maximizes its market share percentage increase by selecting strategy 2. Company II selects strategy C to minimize company I's market share. However, as soon as company I noticed that

company II was using strategy C, it would switch to strategy 3 to increase its market share to 7%. This move would not go unnoticed by company II, which would immediately switch to strategy B to reduce company I's market share to 1%. This action by company II would cause company I to immediately switch to strategy 2 to maximize its market share increase to 8%. Given the action of company I, company II would switch to strategy C to minimize company I's market share increase to 4%. Now you

will notice that the two companies are right back where they started. They have completed a closed loop, as shown in [Table E.10](#), which could continue indefinitely if the two companies persisted.

In a mixed strategy game, a player switches decisions in response to the decision of the other player and they eventually return to the initial decisions, resulting in a closed

loop.

Table E.10. Payoff Table with Closed Loop

Company I Strategy	Company II Strategy	
	<i>B</i>	<i>C</i>
2	8	4
3	1	7



Several methods are available for solving mixed strategy games. We will look at one of them, which is analytical the *expected gain and loss method*.

Expected Gain and Loss Method

The *expected gain and loss method* is based on the principle that in a mixed strategy game, a plan of strategies can be developed by

each player so that the *expected gain* of the maximizing player or the *expected loss* of the minimizing player will be the same, regardless of what the opponent does. In other words, a player develops a plan of mixed strategies that will be employed, regardless of what the opposing player does (i.e., the player is indifferent to the opponent's actions). As might be expected from its name, this method is based on the concept of expected values.

In the expected gain and loss method, a plan of strategies is determined by each player so that the expected gain of one equals the expected loss of the other.

The mixed strategy game for the two camera companies, described in the previous section, will be used to demonstrate this method. First,

we will compute the *expected gain* for company I. Company I arbitrarily assumes that company II will select strategy B. Given this condition, there is a probability of p that company I will select strategy 2 and a probability of $1 - p$ that company I will select strategy 3. Thus, if company II selects B, the expected gain for company I is

$$8p + 1(1 - p) = 1 + 7p$$

Next, company I assumes that company II will select strategy C. Given strategy C, there is a probability of p that company I

will select strategy 2 and a probability of $1 - p$ that company I will select strategy 3. Thus, the expected gain for company I, given strategy C, is

$$4p + 7(1 - p) = 7 - 3p$$

Previously, we noted that this method was based on the idea that company I would develop a plan that would result in the same expected gain, regardless of the strategy that company II selected. Thus, if company I is indifferent to whether company II selects strategy B or C, we equate the expected gain from

each of these strategies:

$$1 + 7p = 7 + 3p$$

[Page E-9]

and

$$10p = 6$$

$$p = 6/10 = .60$$

Recall that p is the probability of using strategy 2, or the *percentage of time* strategy 2 would be employed. Thus, company I's plan is to use strategy 2 for 60% of the time and to use strategy 3 the

remaining 40% of the time. The expected gain (i.e., market share increase for company I) can be computed using the payoff of either strategy B or C because the gain will be the same, regardless. Using the payoffs from strategy B:

$$EG \text{ (company I)} = .60(8) + .40(1) = 5.2\% \text{ market share increase}$$

To check this result, we compute the expected gain if strategy C is used by company II:

$EG(\text{company I}) = .60(4) + .40(7) = 5.2\%$ market share increase

Now we must repeat this process for company II to develop its mixed strategy except that what was company I's expected *gain* is now company II's expected *loss*. First, we assume that company I will select strategy 2. Thus, company II will employ strategy B for p percent of the time and C the remaining $1-p$ percent of the time. The expected *loss* for company II, given strategy 2, is

$$8p + 4(1 - p) = 4 + 4p$$

Next, we compute the expected loss for company II, given that company I selects strategy 3:

$$1p + 7(1 - p) = 7 - 6p$$

Equating these two expected losses for strategies 2 and 3 will result in values for p and $1-p$:

$$4 + 4p = 7 - 6p$$

$$10p = 3$$

$$p = 3/10 = .30$$

and

$$1 \quad p = .70$$

Because p is the probability of employing strategy B, company II will employ strategy B 30% of the time and, thus, strategy C will be used 70% of the time. The actual expected loss, given strategy 2 (which is the same as that for strategy 3), is computed as

$$\begin{aligned} EL(\text{company II}) &= .30(8) + .70(4) \\ &= 5.2\% \text{ market share loss} \end{aligned}$$

The mixed strategies for the two companies are summarized next:

Company I

Strategy 2: 60% of the time

Strategy 3: 40% of the time

Company II

Strategy B: 30% of the time

Strategy C: 70% of the time

In the expected gain and loss method, the probability that each strategy will be used by each player is computed.

The expected gain for company I is a 5.2% increase in market share, and the expected loss for company II is also a 5.2% market share. Thus, the mixed strategies for each company have resulted in an equilibrium point such that a 5.2% *expected* gain for company I results in a simultaneous 5.2% *expected* loss for company II.

It is also interesting to note

that each company has improved its position over the one arrived at using the maximin and minimax strategies. Recall from [Table E.4](#) that the payoff for company I was only a 4% increase in market share, whereas the mixed strategy yields an expected gain of 5.2%. The outcome for company II from the minimax strategy was a 7% loss; however, the mixed strategies show a loss of only 5.2%. Thus, each company puts itself in a better situation by using the mixed strategy

approach.

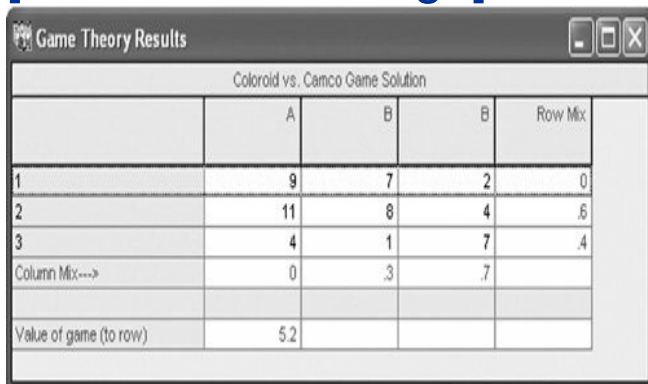
This approach assumes that the game is repetitive and will be played over a period of time so that a strategy can be employed a certain percentage of that time. For our example, it can be logically assumed that the marketing of the new camera by company I will require a lengthy time frame. Thus, each company could employ its mixed strategy.

The QM for Windows solution of this mixed strategy game for the two camera companies is

shown in [Exhibit E.2](#).

Exhibit E.2.

[\[View full size image\]](#)



The screenshot shows a software window titled "Game Theory Results" with standard window controls (minimize, maximize, close) in the top right corner. The main content area is titled "Coloroid vs. Camco Game Solution". It contains a table with 5 columns: an unlabeled column for row indices, and columns labeled A, B, B, and Row Mix. The table has 6 rows. The first three rows represent the game matrix for players 1, 2, and 3. The fourth row shows the column player's mixed strategy (0, .3, .7). The fifth row is empty. The sixth row shows the value of the game to the row player as 5.2.

	A	B	B	Row Mix
1	9	7	2	0
2	11	8	4	.6
3	4	1	7	.4
Column Mix--->	0	.3	.7	
Value of game (to row)	5.2			

Problems

The Army is conducting war games in Europe. One simulated encounter is between the Blue and Red Divisions. The Blue Division is on the offensive; the Red Division holds a defensive position. The results of the war game are measured in terms of troop losses. The following payoff table shows Red Division troop losses for each battle strategy available to each division:

Red Division Strategy

1. Blue

Division Strategy	<i>A</i>	<i>B</i>	<i>C</i>
1	1,800	2,000	1,700
2	2,300	900	1,600

Determine the optimal strategies for both divisions and the number of troop losses the Red Division can expect to suffer.

The Baseball Players' Association has voted to go on strike if a settlement is not reached with the owners within the next month. The players' representative, Melvin Mulehead, has

two strategies (containing different free agent rules, pension formulas, etc.); the owners' representative, Roy Stonewall, has three counterproposals. The financial gains, in millions of dollars, from each player strategy, given each owner strategy, are shown in the following payoff table:

[Page E-11]

2.

Owner Strategy

**Player
Strategy**

A

B

C

1

\$15

\$9

\$11

- 1.** Determine the initial strategy for the players and for the owners.
- 2.** Is this a pure or a mixed strategy game? Explain.

Mary Washington is the incumbent congresswoman for a district in New Mexico, and Franklin Truman is her opponent in the upcoming election. Because Truman is seeking to unseat Washington, he is on the offensive, and she hopes to minimize his gains in the polls. The following payoff table shows the possible percentage point gains for Truman, given the political strategies available to each politician:

Mary Washington Strategy

**3. Franklin
Truman**

A

B

1

7

3

2

6

10

- 1.** Determine the optimal political strategy for each politician and the percentage gain in the polls Franklin Truman can expect.
- 2.** Solve this problem by using the

computer.

Edgar Allan Melville is a successful novelist who is negotiating a contract for a new novel with his publisher, Potboiler Books, Inc. The novelist's contract strategies encompass various proposals for royalties, movie rights, advances, and the like. The following payoff table shows the financial gains for the novelist from each contract strategy:

		Publisher Strategy		
Novelist Strategy		<i>A</i>	<i>B</i>	<i>C</i>
4.	1	\$80,000	\$120,000	\$90,000

2	130,000	90,000	80,000
---	---------	--------	--------

3	110,000	140,000	100,000
---	---------	---------	---------

- 1.** Does this payoff table contain any dominant strategies?
- 2.** Determine the strategy for the novelist and the publisher and the gains and losses for each.

Two major soft drink companies are located in the Southeast the Cooler Cola Company and Smoothie Soft Drinks, Inc. Cooler Cola is the market leader, and Smoothie has developed several marketing strategies to gain a larger percentage of the market now

belonging to Cooler Cola. The following payoff table shows the gains for Smoothie and the losses for Cooler, given the strategies of each company:

[Page E-12]

Cooler Cola Strategy

5. Smoothie Strategy

A

B

C

1

10

9

3

2

4

7

5

3

6

8

4

Determine the mixed strategy for each company and the expected market share gains for Smoothie and losses for Cooler Cola.

Tech is playing State in a basketball game. Tech employs two basic offenses the shuffle and the overload; State uses three defenses the zone, the man-to-man. The points Tech expects to score (estimated from past games), using each offense against each State defense, are given in the following payoff table:

State Defense

6.	Tech Offense	Zone	<i>Man- to- Man</i>	<i>Combination</i>
	Shuffle	72	60	83
	Overload	58	91	72

Determine the mixed strategy for each team and the points Tech can expect to score. Interpret the strategy probabilities.

Module F. Markov Analysis

[Page F-2]

Markov analysis, like decision analysis, is a probabilistic technique. However, Markov analysis is different in that it does not provide a recommended decision. Instead, Markov analysis provides probabilistic information about a decision situation that can aid the decision maker in making a

decision. In other words, Markov analysis is not an optimization technique; it is a *descriptive* technique that results in probabilistic information.

Markov analysis is specifically applicable to systems that exhibit probabilistic movement from one state (or condition) to another, over time. For example, Markov analysis can be used to determine the probability that a machine will be running one day and broken down the next or that a customer will change brands of

cereal from one month to the next. This latter type of example referred to as the "brand-switching" problem will be used to demonstrate the principles of Markov analysis in the following discussion.

The Characteristics of Markov Analysis

Markov analysis can be used to analyze a number of different decision situations; however, one of its most popular applications has been the analysis of customer brand switching. This is basically a marketing application that focuses on the loyalty of customers to a particular product brand, store, or supplier. Markov analysis

provides information on the probability of customers' switching from one brand to one or more other brands. An example of the brand-switching problem will be used to demonstrate Markov analysis.

The brand-switching problem analyzes the probability of customers' changing brands of a product over time.

A small community has two gasoline service stations, Petroco and National. The residents of the community purchase gasoline at the two stations on a monthly basis. The marketing department of Petroco surveyed a number of residents and found that the customers were not totally loyal to either brand of gasoline. Customers were willing to change service stations as a result of advertising, service, and other factors. The marketing department found that if a

customer bought gasoline from Petroco in any given month, there was only a .60 probability that the customer would buy from Petroco the next month and a .40 probability that the customer would buy gas from National the next month.

Likewise, if a customer traded with National in a given month, there was an .80 probability that the customer would purchase gasoline from National in the next month and a .20 probability that the customer would purchase gasoline from Petroco. These

probabilities are summarized in [Table F.1](#).

Table F.1. Probabilities of Customer Movement per Month

This Month	Next Month	
	PETROCO	NATIONAL
Petroco	.60	.40
National	.20	.80

This example contains several important assumptions. *First*, notice that in [Table F.1](#) the probabilities in each row sum to one because they are mutually exclusive and collectively exhaustive. This means that if a customer trades with Petroco one month, the customer *must* trade with either Petroco or National the next month (i.e., the customer will not give up buying gasoline, nor will the customer trade with both in one month). *Second*, the probabilities in [Table F.1](#) apply

to every customer who purchases gasoline. *Third*, the probabilities in [Table F.1](#) will not change over time. In other words, regardless of when the customer buys gasoline, the probabilities of trading with one of the service stations the next month will be the values in [Table F.1](#). The probabilities in [Table F.1](#) will not change in the future if conditions remain the same.

Markov assumptions:
(1) *the probabilities of moving from a*

*state to all others
sum to one, (2) the
probabilities apply to
all system
participants, and (3)
the probabilities are
constant over time.*

It is these properties that make this example a Markov process. In Markov terminology, the service station a customer trades at in a given month is referred to as a *state of the*

system. Thus, this example contains two states of the system a customer will purchase gasoline at either Petroco or National in any given month. The probabilities of the various states in [Table F.1](#) are known as *transition probabilities*. In other words, they are the probabilities of a customer's making the transition from one state to another during one time period. [Table F.1](#) contains four transition probabilities.

*The **state** of the*

system is where the system is at a point in time.

A transition probability *is the probability of moving from one state to another during one time period.*

The properties for the service station example just described define a Markov process. They are summarized in Markov terminology as follows:

- *Property 1:* The transition probabilities for a given beginning state of the system sum to one.
- *Property 2:* The probabilities apply to all participants in the system.
- *Property 3:* The transition probabilities are constant

over time.

- *Property 4:* The states are independent over time.

Summary of Markov properties.

Markov Analysis Information

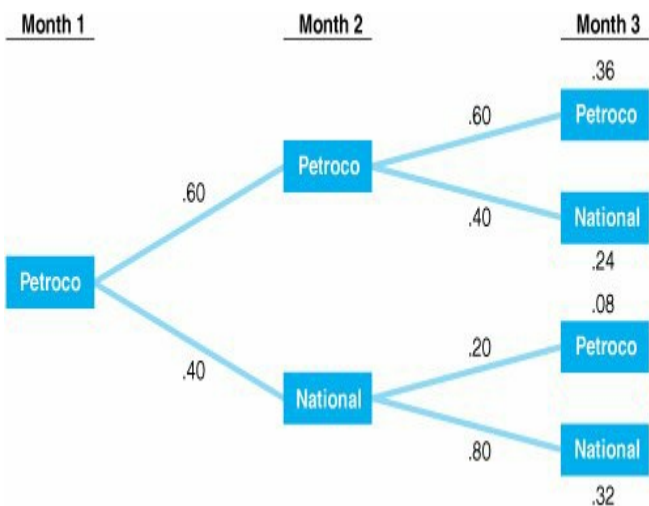
Now that we have defined a

Markov process and determined that our example exhibits the Markov properties, the next question is "What information will Markov analysis provide?" The most obvious information available from Markov analysis is the probability of being in a state at some future time period, which is also the sort of information we can gain from a *decision tree*.

For example, suppose the service stations wanted to know the probability that a customer would trade with it in month 3, given that the

customer trades with it this month (1). This analysis can be performed for each service station by using decision trees, as shown in [Figures F.1](#) and [F.2](#).

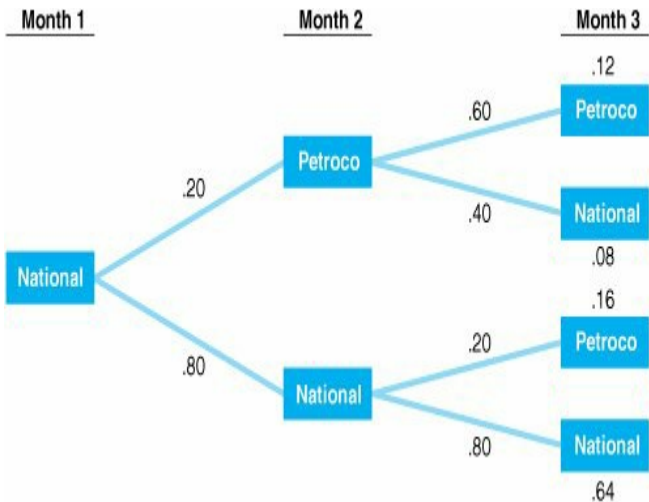
Figure F.1.
Probabilities of future states, given that a customer trades with Petroco this month



[Page F-4]

Figure F.2.

Probabilities of future states, given that a customer trades with National this month



To determine the probability of a customer's trading with Petroco in month 3, given that the customer initially traded with Petroco in month 1, we must add the two branch probabilities in [Figure F.1](#) associated with Petroco:

$.36 + .08 = .44$, the probability of a customer's trading with Petroco in month 3

Likewise, to determine the probability of a customer's purchasing gasoline from

National in month 3, we add the two branch probabilities in [Figure F.1](#) associated with National:

$.24 + .32 = .56$, the probability of a customer's trading with National in month 3

This same type of analysis can be performed under the condition that a customer initially purchased gasoline from National, as shown in [Figure F.2](#). Given that National is the starting state in month 1, the probability of a customer's purchasing gasoline from

National in month 3 is

$$.08 + .64 = .72$$

and the probability of a customer's trading with Petroco in month 3 is

$$.12 + .16 = .28$$

Notice that for each starting state, Petroco and National, the probabilities of ending up in either state in month 3 sum to one:



Time Out: For Andrey A. Markov

Andrey Markov, a Russian mathematician, was born in 1856. His early research focused on number theory, which later developed into probability theory. His work focused on the probability of mutually dependent events, and it was in this area that he was able to prove the central limit theorem. He also introduced the concept of chained events that formed the basis for Markov chains and what we now refer to as Markov analysis.

**Probability of Trade in Month
3**

Starting State	PETROCO	NATIONAL	SUM
Petroco	.44	.56	1.00
National	.28	.72	1.00

Although the use of decision trees is perfectly logical for this

type of analysis, it is time consuming and cumbersome. For example, if Petroco wanted to know the probability that a customer who trades with it in month 1 will trade with it in month 10, a rather large decision tree would have to be constructed. Alternatively, the same analysis performed previously using decision trees can be done by using *matrix algebra* techniques.

The future probabilities of being in a state can be

*determined by using
matrix algebra.*

The Transition Matrix

The probabilities of a customer's moving from service station to service station within a 1-month period, presented in tabular form in [Table F.1](#), can also be presented in the form of a rectangular array of numbers called a *matrix*, as follows:

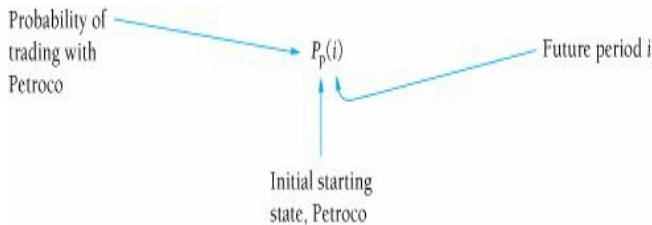
	<i>First Month</i>	<i>Next Month</i>
		Petroco National
$T =$	Petroco National	$\begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$

Because we previously defined these probabilities as transition probabilities, we will refer to the preceding matrix, T , as a *transition matrix*. The present states of the system are listed on the left of the transition matrix, and the future states in the next time period are listed across the top. For example, there is a .60 probability that a customer who traded with Petroco in month 1 will trade with Petroco in month 2.

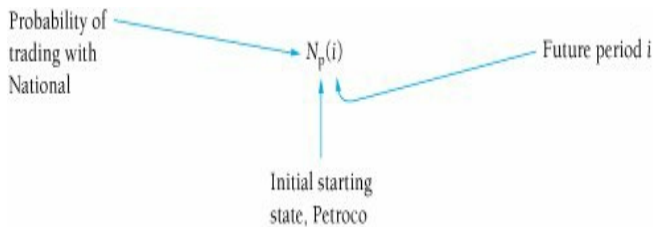
***A transition matrix
includes the***

*transition
probabilities for each
state of nature.*

Several new symbols are needed for Markov analysis using matrix algebra. We will define the probability of a customer's trading with Petroco in period i , given that the customer initially traded with Petroco, as



Similarly, the probability of a customer's trading with National in period i , given that a customer initially traded with Petroco, is



For example, the probability of a customer's trading at National in month 2, given that the customer initially traded with Petroco, is

$$N_p(2)$$

The probabilities of a customer's trading with Petroco and National in a future period, i , given that the customer

traded initially with National,
are defined as

$$P_n(i) \quad \text{and} \quad N_n(i)$$

(When interpreting these symbols, always recall that the subscript refers to the starting state.)

*Petroco as the initial
starting state.*

If a customer is presently

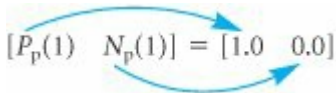
trading with Petroco (month 1), the following probabilities exist:

$$P_p(1) = 1.0$$

$$N_p(1) = 0.0$$

In other words, the probability of a customer's trading at Petroco in month 1, given that the customer trades at Petroco, is 1.0.

These probabilities can also be arranged in matrix form, as follows:


$$\begin{bmatrix} P_p(1) & N_p(1) \end{bmatrix} = \begin{bmatrix} 1.0 & 0.0 \end{bmatrix}$$

This matrix defines the starting conditions of our example system, given that a customer initially trades at Petroco, as in the decision tree in [Figure F.1](#). In other words, a customer is originally trading with Petroco in month 1. We can determine the subsequent probabilities of a customer's trading at Petroco or National in month 2 by multiplying the preceding matrix by the transition matrix, as follows:

$$\begin{aligned}\text{month 2: } [P_p(2) \quad N_p(2)] &= [1.0 \quad 0.0] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.60 \quad .40]\end{aligned}$$

Computing probabilities of a customer trading at either station in future months, using matrix multiplication.

These probabilities of .60 for a customer's trading at Petroco and .40 for a customer's

trading at National are the same as those computed in the decision tree in [Figure F.1](#). We use the same procedure for determining the month 3 probabilities, except we now multiply the transition matrix by the month 2 matrix:

$$\begin{aligned} \text{month 3: } [P_p(3) \quad N_p(3)] &= [.60 \quad .40] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.44 \quad .56] \end{aligned}$$

These are the same probabilities we computed by using the decision tree analysis in [Figure F.1](#). However, whereas it would be cumbersome to

determine additional values by using the decision tree analysis, we can continue to use the matrix approach as we have previously:

[Page F-7]

$$\begin{aligned}\text{month 4: } [P_p(4) \quad N_p(4)] &= [.44 \quad .56] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.38 \quad .62]\end{aligned}$$

The state probabilities for several subsequent months are as follows:

$$\text{month} \qquad N_p(5)] =$$

$$5: \quad [P_p(5) \quad N_p(5)] = [.35 \quad .65]$$

$$\text{month 6:} \quad [P_p(6) \quad N_p(6)] = [.34 \quad .66]$$

$$\text{month 7:} \quad [P_p(7) \quad N_p(7)] = [.34 \quad .66]$$

$$\text{month 8:} \quad [P_p(8) \quad N_p(8)] = [.33 \quad .67]$$

$$\text{month 9:} \quad [P_p(9) \quad N_p(9)] = [.33 \quad .67]$$

Notice that as we go farther and farther into the future, the changes in the state probabilities become smaller and smaller, until eventually there are no changes at all. At that point every month in the future will have the same probabilities. For this example, the state probabilities that result after some future month, i , are

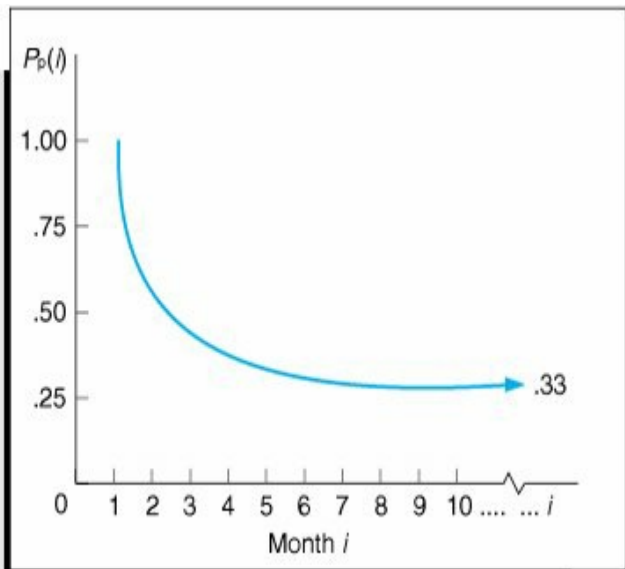
$$[P_p(i) = \begin{matrix} N_p(i) \\ [.33 \quad .67] \end{matrix}]$$

In future periods, the state probabilities become constant.

This characteristic of the state probabilities approaching a constant value after a number of time periods is shown for $P_p(i)$ in [Figure F.3](#).

Figure F.3. The

**probability $P_p(i)$ for
future values of i**



$N_p(i)$ exhibits this same characteristic as it approaches a value of .67. This is a potentially valuable result for the decision maker. In other words, the service station owner can now conclude that after a certain number of months in the future, there is a .33 probability that the customer will trade with Petroco if the customer initially traded with Petroco.

Computing future

*state probabilities
when the initial
starting state is
National.*

This same type of analysis can be performed given the starting condition in which a customer initially trades with National in month 1. This analysis, shown as follows, corresponds to the decision tree in [Figure F.2](#).

Given that a customer initially

trades at the National station,
then

$$\begin{bmatrix} P_n(1) & N_n(1) \end{bmatrix} = \begin{bmatrix} 0.0 & 1.0 \end{bmatrix}$$

Using these initial starting-state probabilities, we can compute future-state probabilities as follows:

$$\begin{aligned} \text{month 2: } \begin{bmatrix} P_n(2) & N_n(2) \end{bmatrix} &= \begin{bmatrix} 0.0 & 1.0 \end{bmatrix} \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= \begin{bmatrix} .20 & .80 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{month 3: } [P_n(3) \quad N_n(3)] &= [.20 \quad .80] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= [.28 \quad .72] \end{aligned}$$

These are the same values obtained by using the decision tree analysis in [Figure F.2](#).

Subsequent state probabilities, computed similarly, are shown next:

$$\begin{aligned} \text{month 4: } [P_n(4) \quad N_n(4)] &= \quad .69] \\ &[.31 \end{aligned}$$

$$\begin{aligned} \text{month} \quad N_n(5)] &= \quad .68] \end{aligned}$$

$$5: \quad [P_n(5) \quad N_n(5)] = .32$$

$$\text{month 6:} \quad [P_n(6) \quad N_n(6)] = .67$$

$$\text{month 7:} \quad [P_n(7) \quad N_n(7)] = .67$$

$$\text{month 8:} \quad [P_n(8) \quad N_n(8)] = .67$$

$$\text{month 9:} \quad [P_n(9) \quad N_n(9)] = .67$$

As in the previous case in which Petroco was the starting state, these state probabilities also become constant after several periods. However, notice that the eventual state probabilities (i.e., .33 and .67) achieved when National is the starting state *are exactly the same* as the previous state probabilities achieved when Petroco was the starting state. In other words, the probability of ending up in a particular state in the future is not dependent on the starting state.

The probability of ending up in a state in the future is independent of the starting state.

Steady-State Probabilities

The probabilities of .33 and .67 in our example are referred to as **steady-state probabilities**. The steady-state probabilities are average probabilities that the system will be in a certain state after a large number of transition periods. This does not mean the system stays in one state. The system will continue to move from state to state in future time periods; however,

the average *probabilities* of moving from state to state for all periods will remain constant in the long run. In a Markov process, after a number of periods have passed, the probabilities will approach steady state.

Steady-state probabilities are average, constant probabilities that the system will be in a state in the future.

For our service station example, the steady-state probabilities are

probability of a customer's trading at Petroco after a number of
.33 = months in the future, regardless of where the customer traded in month 1

probability of a customer's trading at National after a number of
.67 = months in the future, regardless of where the customer traded in month 1

Notice that in the determination of the preceding steady-state probabilities, we considered each starting state separately. First, we assumed that a customer was initially trading at Petroco, and the steady-state probabilities were computed given this starting condition. Then we determined that the steady-state probabilities were the same, regardless of the starting condition. However, it was not necessary to perform these

matrix operations separately.
We could have simply combined
the operations into one matrix,
as follows:

$$\begin{aligned}\text{month 2: } \begin{bmatrix} P_p(2) & N_p(2) \\ P_n(2) & N_n(2) \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{month 3: } \begin{bmatrix} P_p(3) & N_p(3) \\ P_n(3) & N_n(3) \end{bmatrix} &= \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= \begin{bmatrix} .44 & .56 \\ .28 & .72 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\text{month 4: } \begin{bmatrix} P_p(4) & N_p(4) \\ P_n(4) & N_n(4) \end{bmatrix} &= \begin{bmatrix} .44 & .56 \\ .28 & .72 \end{bmatrix} \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix} \\ &= \begin{bmatrix} .38 & .62 \\ .31 & .69 \end{bmatrix}\end{aligned}$$

until eventually we arrived at the steady-state probabilities:

$$\text{month 9: } \begin{bmatrix} P_p(9) & N_p(9) \\ P_n(9) & N_n(9) \end{bmatrix} = \begin{bmatrix} .33 & .67 \\ .33 & .67 \end{bmatrix}$$

Direct Algebraic Determination of Steady- State Probabilities

In the previous section, we computed the state probabilities for approximately eight periods (i.e., months)

before the steady-state probabilities were reached for both states. This required quite a few matrix computations. Alternatively, it is possible to solve for the steady-state probabilities directly, without going through all these matrix operations.

Notice that after eight periods in our previous analysis, the state probabilities did not change from period to period (i.e., from month to month). For example,

$$\text{month 8:} \quad [P_p(8) \quad N_p(8)] = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

$$\text{month 9:} \quad [P_p(9) \quad N_p(9)] = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

At some point in the future, the state probabilities remain constant from period to period.

Thus, we can also say that after a number of periods in the future (in this case, eight), the state probabilities in period i equal the state probabilities in period $i + 1$. For our example, this means that

$$[P_p(8) \ N_p(8)] = [P_p(9) \ N_p(9)]$$

In fact, it is not necessary to designate which period in the future is actually occurring. That is,

$$[P_p \ N_p] = [P_p \ N_p]$$

given steady-state conditions.

After steady state is reached, it is not necessary to designate the time period.

These probabilities are for some period, i , in the future once a steady state has already been reached. To determine the state probabilities for period $i + 1$, we would normally do the following computation:

$$[P_p(i+1) \quad N_p(i+1)] = [P_p(i) \quad N_p(i)] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$

However, we have already stated that once a steady state has been reached, then

$$[P_p(i+1) \quad N_p(i+1)] = [P_p(i) \quad N_p(i)]$$

and it is not necessary to designate the period. Thus, our computation can be rewritten as

$$[P_p \quad N_p] = [P_p \quad N_p] \begin{bmatrix} .60 & .40 \\ .20 & .80 \end{bmatrix}$$

Performing matrix operations results in the following set of equations:

$$P_p = .6P_p + .2 N_p$$

$$N_p = .4P_p + .8 N_p$$

Steady-state probabilities can be computed by developing a set of equations, using matrix operations, and solving them simultaneously.

Recall that the transition probabilities for a row in the transition matrix (i.e., the state probabilities) must sum to one:

$$P_p + N_p = 1.0$$

[Page F-10]

which can also be written as

$$N_p = 1.0 - P_p$$

Substituting this value into our first foregoing equation ($P_p =$

$.6P_p + .2 N_p$) results in the following:

$$P_p = \frac{.6P_p + .2(1.0P_p)}{.2 + .4P_p} = .6P_p + .2.2P_p =$$

$$.6P_p = .2$$

$$P_p = .2/.6 = .33$$

and

$$N_p = 1.0 P_p = 1.0 .33 = .67$$

These are the steady-state probabilities we computed in our previous analysis:

$$[P_p \ N_p] = [.33 \ .67]$$

Application of the Steady-State Probabilities

The steady-state probabilities indicate not only the probability of a customer's trading at a particular service station in the long-term future but also the *percentage of customers* who will trade at a service station

during any given month in the long run. For example, if there are 3,000 customers in the community who purchase gasoline, then in the long run the following *expected* number will purchase gasoline at each station on a monthly basis:

$$\text{Petroco: } P_p(3,000) = .33(3,000)$$

$$= 990 \text{ customers}$$

$$\text{National: } N_p(3,000) = .67(3,000)$$

= 2,010 customers

Steady-state probabilities can be multiplied by the total system participants to determine the expected number in each state in the future.

Now suppose that Petroco has decided that it is getting less than a reasonable share of the market and would like to increase its market share. To accomplish this objective, Petroco has improved its service substantially, and a survey indicates that the transition probabilities have changed to the following:

$$T = \begin{array}{cc} & \begin{array}{cc} \text{Petroco} & \text{National} \end{array} \\ \begin{array}{c} \text{Petroco} \\ \text{National} \end{array} & \left[\begin{array}{cc} .70 & .30 \\ .20 & .80 \end{array} \right] \end{array}$$

In other words, the improved service has resulted in a

smaller probability (.30) that customers who traded initially at Petroco will switch to National the next month.

Now we will recompute the steady-state probabilities, based on this new transition matrix:

$$[P_p \quad N_p] = [P_p \quad N_p] \begin{bmatrix} .70 & .30 \\ .20 & .80 \end{bmatrix}$$

$$P_p = .7P_p + .2N_p$$

$$N_p = .3P_p + .8N_p$$

Using the first equation and the fact that $N_p = 1.0 - P_p$, we have

$$P_p = .7P_p + .2(1.0 P_p) = .7P_p + .2$$

$$.2P_p$$

$$.5P_p = .2$$

$$P_p = .2/.5 = .4$$

[Page F-11]

and thus

$$N_p = 1 P_p 1 .4 .6$$

This means that out of the

3,000 customers, Petroco will now get 1,200 customers (i.e., $.40 \times 3,000$) in any given month in the long run. Thus, improvement in service will result in an increase of 210 customers per month (if the new transition probabilities remain constant for a long period of time in the future). In this situation Petroco must evaluate the trade-off between the cost of the improved service and the increase in profit from the additional 210 customers. For example, if the improved service costs \$1,000

per month, then the extra 210 customers must generate an increase in profit greater than \$1,000 to justify the decision to improve service.

This brief example demonstrates the usefulness of Markov analysis for decision making. Although Markov analysis does not yield a recommended decision (i.e., a solution), it does provide information that will help the decision maker to make a decision.

*Markov analysis
results in probabilistic
information, not a
decision.*

Determination of Steady States with QM for Windows

QM for Windows has a Markov analysis module, which is extremely useful when the

dimensions of the transition matrix exceed two states. The algebraic computations required to determine steady-state probabilities for a transition matrix with even three states are lengthy; for a matrix with more than three states, computing capabilities are a necessity. Markov analysis with QM for Windows will be demonstrated using the service station example in this section.

[Exhibit F.1](#) shows our example input data for the Markov analysis module in QM for

Windows. Note that it is not necessary to enter a number of transitions to get the steadystate probabilities. The program automatically computes the steady state. "Number of Transitions" refers to the number of transition computations you might like to see.

Exhibit F.1.

[\[View full size image\]](#)

Number of Variables		Initial Value	
x	0	Enter when the possibility of staying in process is the number of elements that start a process. Any non-negative value is permissible.	
Market Share Analysis			
	Initial	Process	Final
Process	0	5	4
Initial	0	2	8

[Exhibit F.2](#) shows the solution with the steady-state transition matrix for our service station example.

Exhibit F.2.



Markov Analysis Results



Market Share Analysis Steady state transition matrix

	Petroco	National
Petroco	.3334	.6667
National	.3334	.6667
Ending number (given initial)	0	0
Steady State probability	.3334	.6667

Additional Examples of Markov Analysis

Although analyzing brand switching is probably the most popular example of Markov analysis, this technique does have other applications. One prominent application relates to the breakdown of a machine or system (such as a computer system, a production operation, or an electrical system). For example, a particular production machine could be

assigned the states "operating" and "breakdown." The transition probabilities could then reflect the probability of a machine's either breaking down or operating in the next time period (i.e., month, day, or year).

*A machine
breakdown example.*

As an example, consider a machine that has the following

daily transition matrix:

$$T = \begin{array}{c|cc} & \begin{array}{c} \text{Day 1} \\ \text{Operate} \\ \text{Breakdown} \end{array} & \begin{array}{c} \text{Day 2} \\ \text{Operate} \\ \text{Breakdown} \end{array} \\ \hline \begin{array}{c} \text{Operate} \\ \text{Breakdown} \end{array} & \begin{bmatrix} .90 & .10 \\ .70 & .30 \end{bmatrix} \end{array}$$

The steady-state probabilities for this example are

.88 = steady-state probability of the machine's operating

.12 = steady-state probability of the machine's breaking down

Now if management decides that the long-run probability of

.12 for a breakdown is excessive, it might consider increasing preventive maintenance, which would change the transition matrix for this example. The decision to increase maintenance would be based on the cost of the increase versus the value of the increased production output gained from having fewer breakdowns.

Thus far in our discussion of Markov analysis, we have considered only examples that consisted of two states. This has been partially a matter of

convenience because 2×2 matrices are easier to work with than matrices of a higher magnitude. However, examples that contain a larger number of states are analyzed in the same way as our previous examples. For example, consider the Carry-All Rental Truck Firm, which serves three states Virginia, North Carolina, and Maryland. Trucks are rented on a daily basis and can be rented and returned in any of the three states. The transition matrix for this example follows:

Rented

Returned

	Virginia	Maryland	North Carolina
Virginia	.60	.20	.20
$T =$ Maryland	.30	.50	.20
North Carolina	.40	.10	.50

*A rental truck firm
with three states.*

The steady-state probabilities for this example are determined by using the same algebraic approach presented earlier, although the

mathematical steps are more lengthy and complex. Instead of solving three simultaneous equations, we solve four.

The steady-state probabilities for this example are

Virginia	Maryland	North Carolina
[.471	.244	.285]

Thus, in the long run, these percentages of Carry-All trucks will end up in the three states. If the company had 200 trucks, then it could expect to have the following number of trucks available in each state at any time in the future:

Virginia	Maryland	North Carolina
[94	49	57]

Special Types of Transition Matrices

In some cases the transition matrix derived from a Markov problem is not in the same form as those in the examples shown in this chapter. Some matrices have certain characteristics that alter the normal methods of Markov analysis. Although a detailed analysis of these special cases is beyond the scope of this chapter, we will give examples

of them so that they can be easily recognized.

In the transition matrix

$$T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} .40 & .60 & 0 \\ .30 & .70 & 0 \\ 1.0 & 0 & 0 \end{bmatrix} \end{matrix}$$

state 3 is a *transient state*.

Once state 3 is achieved, the system will never return to that state. Both states 1 and 2 contain a 0.0 probability of going to state 3. The system will move out of state 3 to state 1 (with a 1.0 probability) but will never return to state 3.

Once the system leaves a transient state, it will never return.

The following transition matrix is referred to as cyclic:

$$T = \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix} \end{matrix}$$

A transition matrix is cyclic when the system moves back

and forth between states.

The system will simply cycle back and forth between states 1 and 2, without ever moving out of the cycle.

Finally, consider the following transition matrix for states 1, 2, and 3:

$$T = \begin{matrix} & \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} .30 & .60 & .10 \\ .40 & .40 & .20 \\ 0 & 0 & 1.0 \end{bmatrix} \end{matrix}$$

State 3 in this transition matrix is referred to as an *absorbing*, or trapping, state. Once state 3 is achieved, there is a 1.0 probability that it will be achieved in succeeding time periods. Thus, the system in effect ends when state 3 is achieved. There is no movement from an absorbing state; the item is trapped in that state.

Once the system moves into an absorbing state, it is trapped and cannot leave.

The Bad Debt Example

A unique and popular application of an absorbing state matrix is the bad debt example. In this example, the states are the months during

which a customer carries a debt. The customer may pay the debt (i.e., a bill) at any time and thus achieve an absorbing state for payment. However, if the customer carries the debt longer than a specified number of periods (say, 2 months), the debt will be labeled "bad" and will be transferred to a bill collector. The state "bad debt" is also an absorbing state. Through various matrix manipulations, the portion of accounts receivable that will be paid and the portion that will become

bad debts can be determined.
(Because of these matrix manipulations, the debt example is somewhat more complex than the Markov examples presented previously.)

[Page F-14]

The debt example will be demonstrated by using the following transition matrix, which describes the accounts receivable for the A-to-Z Office Supply Company:

$$T = \begin{matrix} & \begin{matrix} p & 1 & 2 & b \end{matrix} \\ \begin{matrix} p \\ 1 \\ 2 \\ b \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ .70 & 0 & .30 & 0 \\ .50 & 0 & 0 & .50 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

In this absorbing state transition matrix, state p indicates that a debt is paid, states 1 and 2 indicate that a debt is 1 or 2 months old, respectively, and state b indicates that a debt becomes bad. Notice that once a debt is paid (i.e., once the item enters state p), the probability of moving to state 1, 2, or b is zero. If the debt is 1 month old,

there is a .70 probability that it will be paid in the next month and a .30 probability that it will go to month 2 unpaid. If the debt is in month 2, there is a .50 probability that it will be paid and a .50 probability that it will become a bad debt in the next time period. Finally, if the debt is bad, there is a zero probability that it will return to any previous state.

An absorbing state has a transition probability of one.

The next step in analyzing this Markov problem is to rearrange the transition matrix into the following form:

$$T = \begin{array}{c} \begin{array}{cc} p & b \end{array} \\ \begin{array}{cc} 1 & 2 \end{array} \end{array} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline .70 & 0 & 0 & .30 \\ .50 & .50 & 0 & 0 \end{array} \right]$$

We have now divided the transition matrix into four parts, or submatrices, which we will label as follows:

$$T = \left[\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array} \right]$$

where

$$I = \begin{matrix} & p & b \\ \begin{matrix} p \\ b \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{matrix}$$

= an identity matrix

$$0 = \begin{matrix} & 1 & 2 \\ \begin{matrix} p \\ b \end{matrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{matrix}$$

= a matrix of zeros

$$R = \begin{matrix} & p & b \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} .70 & 0 \\ .50 & .50 \end{bmatrix} \end{matrix}$$

= a matrix containing the transition probabilities of the debt's being absorbed in the next period

$$Q = \frac{1}{2} \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & .30 \\ 0 & 0 \end{bmatrix} \end{matrix}$$

= a matrix containing the transition probabilities for movement between both nonabsorbing states

The matrix labeled I is an *identity matrix*, so called because it has ones along the diagonal and zeros elsewhere in the matrix.

The first matrix operation to be performed determines the *fundamental matrix*, F , as follows:

$$F = (I - Q)^{-1}$$

The notation to raise the $(I - Q)$ matrix to the -1 power indicates what is referred to as the *inverse* of a matrix. The fundamental matrix is computed by taking the inverse of the difference between the identity matrix, I , and Q . For our example, the fundamental matrix is computed as follows:

$$\begin{aligned}
 F &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 30 \\ 0 & 0 \end{bmatrix} \right)^{-1} \\
 &= \begin{bmatrix} 1 & - .30 \\ 0 & 1 \end{bmatrix}^{-1} \\
 &\quad \begin{matrix} 1 & 2 \\ 1 & 2 \end{matrix} \\
 &= \frac{1}{2} \begin{bmatrix} 1 & .30 \\ 0 & 1 \end{bmatrix}
 \end{aligned}$$

The fundamental matrix indicates the expected number of times the system will be in any of the nonabsorbing states before absorption occurs (for our example, before a debt becomes bad or is paid). Thus, according to F , if the customer is in state 1 (1 month late in paying the debt), the expected

number of times the customer would be 2 months late would be .30 before the debt is paid or becomes bad.

Next we will multiply the fundamental matrix by the R matrix created when the original transition matrix was partitioned:

$$\begin{aligned}
 F \times R &= \begin{matrix} & \begin{matrix} 1 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} 1 & .30 \\ 0 & 1 \end{bmatrix} \end{matrix} \times \begin{matrix} & \begin{matrix} p & b \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} .70 & 0 \\ .50 & .50 \end{bmatrix} \\
 &= \begin{matrix} & \begin{matrix} p & b \end{matrix} \\ \begin{matrix} 1 \\ 2 \end{matrix} & \begin{bmatrix} .85 & .15 \\ .50 & .50 \end{bmatrix}
 \end{aligned}$$

The $F \times R$ matrix reflects the

probability that the debt will eventually be absorbed, given any starting state. For example, if the debt is presently in the first month, there is an .85 probability that it will eventually be paid and a .15 probability that it will result in a bad debt.

Now, suppose the A-to-Z Office Supply Company has accounts receivable of \$4,000 in month 1 and \$6,000 in month 2. To determine what portion of these funds will be collected and what portion will result in bad debts, we multiply a matrix

of these dollar amounts by the $F \times R$ matrix:

[Page F-16]

$$\begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ & \begin{array}{cc} p & q \end{array} \\ \text{determination of accounts receivable} = & \begin{array}{cc} 1 & 2 \end{array} \begin{bmatrix} .85 & .15 \\ .50 & .50 \end{bmatrix} \\ & \begin{array}{cc} p & q \end{array} \\ & = \begin{bmatrix} 6,400 & 3,600 \end{bmatrix} \end{array}$$

Thus, of the total \$10,000 owed, the office supply company can expect to receive \$6,400, and \$3,600 will become bad debts. The debt example can be analyzed even further than we have done

here, although the
mathematics become
increasingly difficult.

Excel Solution of the Debt Example

All the matrix operations performed manually in the previous section for the debt example can be accomplished by using Excel. [Exhibit F.3](#) shows an Excel spreadsheet with the matrix operations for our A-to-Z Office Supply Company accounts receivable example.

Exhibit F.3.

[\[View full size image\]](#)

Microsoft Excel - Exhibit F.3.xls

File Edit View Insert Format Tools Data Window Help Add-Ins

Type a question for help

100%

Sum of Selection

1

2

3

4

5

6

7

8

9

10

11

12

13

14

15

16

17

18

19

20

21

22

A B C D E F G H I J K L M N O P Q R

1 Accounts Receivable Example

2

3

4 (I) $I-Q =$

1	0
0	1

 $-$

0	0.3
0	0

 $=$

1	-0.3
0	1

5

6

7 $F =$

1	0.3
0	1

8

9

10

11 $R =$

0.7	0
0.5	0.5

12

13

14 (2) $FR =$

0.35	0.15
0.50	0.50

15

16

17

18 Accounts receivable =

1	2
4000	8000

19

20

21 (3) Paid/Not debts =

0	0
6400	3600

22

In step 1 the fundamental matrix, F , is developed. First, the Q matrix is subtracted from the identity matrix, I . This can be done by inputting the numeric values from our example in the matrices set up in cells **C4:D5** and **F4:G5**, and subtracting the array in **F4:G5** from the array in **C4:D5**. This subtraction can be conveniently accomplished by covering cells **I4:J5** with the cursor, embedding the formula = (**C4:D5**) (**F4:G5**) in cell **I4**; then with the "Ctrl" and "Shift"

keys pressed down, we press "Enter."

Next, we take the inverse of the matrix in cells **I4:J5**. This is done by covering cells **C7:D8** with the cursor and entering the formula shown on the formula bar at the top of the spreadsheet in cell C7. Then with the "Ctrl" and "Shift" keys pressed down, we press "Enter."

In step 2 the *FR* matrix is computed by entering the matrix operation formulas for multiplying two matrices in cells **C14:D15**. For example,

the formula **=C7* C11 + D7* C12** is entered in cell C14, which results in the value 0.85 in cell C14.

Finally, in step 3 the matrix values indicating the amounts of paid and bad debts for our example are computed, using the formulas for multiplying two matrices. For example, the formula in cell C21 is **=C14* C18 + C15* D18**, resulting in the value 6,400.

Problems

A town has three gasoline stations, Petroco, National, and Gascorp. The residents purchase gasoline on a monthly basis. The following transition matrix contains the probabilities of the customers' purchasing a given brand of gasoline next month:

<i>This Month</i>		<i>Next Month</i>		
		Petroco	National	Gascorp
1.	Petroco	$\begin{bmatrix} .5 & .3 & .2 \\ .1 & .7 & .2 \\ .1 & .1 & .8 \end{bmatrix}$		
	National			
	Gascorp			

Using a decision tree, determine the probabilities of a customer's purchasing

each brand of gasoline in month 3, given that the customer purchases National in the present month. Summarize the resulting probabilities in a table.

- Discuss the properties that must exist for the transition matrix in [Problem 1](#) to be considered a Markov process.

The only grocery store in a community stocks milk from two dairies, Creamwood and Cheesedale. The following transition matrix shows the probabilities of a customer's purchasing each brand of milk next week, given that he or she purchased a particular brand this week:

	<i>This Week</i>	<i>Next Week</i>
	Creamwood	Cheesedale
Creamwood	$\left[\begin{array}{cc} .7 & .3 \end{array} \right]$	
Cheesedale	$\left[\begin{array}{cc} .4 & .6 \end{array} \right]$	

Given that a customer purchases Creamwood milk this week, use a decision tree to determine the probability that he or she will purchase Cheesedak milk in week 4.

[Page F-18]

- Given the transition matrix in [Problem 3](#), use matrix multiplication methods to
4. determine the state probabilities for month 3, given that a customer initially purchases Petroco gasoline.

- Determine the state probabilities in
5. [Problem 3](#) by using matrix multiplication methods.

A manufacturing firm has developed a

transition matrix containing the probabilities that a particular machine will operate or break down in the following week, given its operating condition in the present week:

	<i>Next Week</i>	
	<i>Operate</i>	<i>Break Down</i>
<i>This Week</i>		
Operate	.4	.6
Break Down	.8	.2

6.

1. Assuming that the machine is operating in week 1, determine the probabilities that the machine will operate or break down in week 2, week 3, week 4, week 5, and week 6. b.
2. Determine the steady-state probabilities for this transition matrix algebraically and indicate the percentage of future weeks in which the machine will break down.

A city is served by two newspapers the *Tribune* and the *Daily News*. Each Sunday readers purchase one of the newspapers at a stand. The following transition matrix contains the probabilities of a customer buying a particular newspaper in a week given the newspaper purchased the previous Sunday:

7. This Sunday	Next Sunday	
	<i>Tribune</i>	<i>Daily News</i>
<i>Tribune</i>	.65	.35
<i>Daily News</i>	.45	.55

Determine the steady-state probabilities for this transition matrix algebraically and explain what they mean.

The Hergeshiemer Department Store wants to analyze the payment behavior

of customers who have outstanding accounts. The store's credit department has determined the following bill payment pattern for credit customers from historical records:

	<i>Present Month</i>	<i>Next Month</i>
	Pay	Not Pay
Pay	$\left[\begin{array}{c} .9 \end{array} \right]$	$\left[\begin{array}{c} .1 \end{array} \right]$
Not Pay	$\left[\begin{array}{c} .8 \end{array} \right]$	$\left[\begin{array}{c} .2 \end{array} \right]$

1. If a customer did not pay his or her bill in the present month, what is the probability that the bill will not be paid in any of the next 3 months?
2. Determine the steady-state probabilities for this transition matrix and explain what they mean.

A rural community has two television stations, and each Wednesday night the local viewers watch either the *Wednesday Movie* or a show called *Western Times*. The following transition matrix contains the probabilities of a viewer's watching one of the shows in a week, given that he or she watched a particular show the preceding week:

[Page F-19]

This Week

Next Week

Movie

Western

9.

Movie	$\begin{bmatrix} .75 & .25 \end{bmatrix}$
Western	$\begin{bmatrix} .45 & .55 \end{bmatrix}$

- Determine the steady-state probabilities for this transition matrix algebraically.
- If the community contains 1,200

television sets, how many will be tuned to each show in the long run?

- 3.** If a prospective local sponsor wanted to pay for commercial time on one of the shows, which show would more likely be selected?

In [Problem 3](#), assume that 600 gallons of milk are sold weekly, regardless of the brand purchased.

- 1.** How many gallons of each brand of milk will be purchased in any given week in the long run?
 - 2.** The Cheesedale dairy is considering paying \$500 per week for a new advertising campaign that would alter the brand-switching probabilities as follows:
- 10.**

This Week

Next Week

	Creamwood	Cheesedale
Creamwood	$\left[\begin{array}{c} .6 \end{array} \right]$	$\left[\begin{array}{c} .4 \end{array} \right]$
Cheesedale	$\left[\begin{array}{c} .2 \end{array} \right]$	$\left[\begin{array}{c} .8 \end{array} \right]$

If each gallon of milk sold results in \$1.00 in profit for Cheesedale, should the dairy institute the advertising campaign?

The manufacturing company in [Problem 10](#) is considering a preventive maintenance program that would change the operating probabilities as follows:

This Week

Next Week

	Operate	Break Down
11. Operate	$\left[\begin{array}{c} .7 \end{array} \right]$	$\left[\begin{array}{c} .3 \end{array} \right]$
Break Down	$\left[\begin{array}{c} .9 \end{array} \right]$	$\left[\begin{array}{c} .1 \end{array} \right]$

The machine earns the company \$1,000 per week when it operates.

in profit each week it operates. The preventive maintenance program would cost \$8,000 per year. Should the company institute the preventive maintenance program?

In [Problem 7](#), assume that 20,000 newspapers are sold each Sunday, regardless of the publisher.

1. How many copies of the *Tribune* and the *Daily News* will be purchased in a given week in the long run?
2. The *Daily News* is considering a promotional campaign estimated to change the weekly reader probabilities as follows:

12.

*This Week**Next Sunday*

	<i>Tribune</i>	<i>Daily News</i>
<i>Tribune</i>	.5	.5
<i>Daily News</i>	.3	.7

[Page F-20]

The promotional campaign will cost \$1! per week. Each newspaper sold earns t *Daily News* \$0.05 in profit. Should the paper adopt the promotional campaign

The following transition matrix describe the accounts receivable process for the Ewing-Barnes Department Store:

$$T = \begin{matrix} & \begin{matrix} p & 1 & 2 & b \end{matrix} \\ \begin{matrix} p \\ 1 \\ 2 \\ b \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ .80 & 0 & .20 & 0 \\ .40 & 0 & 0 & .60 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

13.

The states p and b represent an account that is paid and a bad account, respectively. The numbers 1 and 2 represent the fact that an account is either 1 or 2 months overdue, respectively. After an account has been overdue for 2 months, it becomes a bad account and is transferred to the store overdue accounts section for collection. The company has sales of \$210,000 each month. Determine how much the company will be paid and how many of the debts will become bad debts in a 2-month period.

The department store in [Problem 13](#) will never be able to collect 20% of the bad accounts, and it costs the store an

additional \$0.10 per dollar collected to collect the remaining bad accounts. The store management is contemplating a new, more restrictive credit plan that would reduce sales to an estimated \$195,000 per month. However, the tougher credit plan would result in the following transition matrix for accounts receivable:

14.

$$T = \begin{matrix} & \begin{matrix} p & 1 & 2 & b \end{matrix} \\ \begin{matrix} p \\ 1 \\ 2 \\ b \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ .90 & 0 & .10 & 0 \\ .70 & 0 & 0 & .30 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

Determine whether the store should adopt the more restrictive credit plan or keep the existing one.

In Westvale, a small, rural town in Maine, virtually all shopping and business are

done in the town. The town has one farm and garden center that sells fertilizer to the local farmers and gardeners. The center carries three brands of fertilizer: Plant Plus, Crop Extra, and Gro-fast and every person in the town who uses fertilizer uses one of the three brands. The garden center has 9,000 customers for fertilizer each spring. An extensive market research study has determined that customers switch brands of fertilizer according to the following probability transition matrix:

<i>This Spring</i>	<i>Next Spring</i>		
	Plant Plus	Crop Extra	Gro-fast
Plant Plus	$\begin{bmatrix} .4 & .3 & .3 \\ .5 & .1 & .4 \\ .4 & .2 & .4 \end{bmatrix}$	.4	.3
Crop Extra		.5	.1
Gro-fast		.4	.2

- 15.** The number of customers presently using each brand of fertilizer is shown in the following table:

Fertilizer Brand	Customers
Plant Plus	3,000
Crop Extra	4,000
Gro-fast	2,000

- 1.** Determine the steady-state probabilities for the fertilizer brand
- 2.** Forecast the customer demand for

each brand of fertilizer in the long run and the changes in customer demand.

Westvale, a small community in Maine, has 7,000 bank patrons that do their banking business at one of three banks town, the American National Bank, the Bank of Westvale and the Commerce Bank. The following transition matrix shows the probability that a bank customer will trade with the same bank next month or move to one of the other banks.

16.	<i>Present Month</i>		<i>Next Month</i>		
		A	B	C	
	A	$\left[\begin{array}{ccc} .80 & .10 & .10 \\ .10 & .70 & .20 \\ .10 & .30 & .60 \end{array} \right]$			
	B				
	C				

Determine the steady-state probabilities and the number of customers expected to trade at each bank in the long run.

Students switch among the various colleges of a university according to the following probability transition matrix:

<i>This Fall</i>	<i>Next Fall</i>		
	Engineering	Liberal Arts	Business
Engineering	$\begin{bmatrix} .50 & .30 & .20 \\ .10 & .70 & .20 \\ .10 & .10 & .80 \end{bmatrix}$	.30	.20
Liberal Arts		.70	.20
Business		.10	.80

Assume that the number of students in each college of the university at the beginning of the fall quarter is as follows

Engineering	3,000
-------------	-------

17.

Liberal Arts

5,000

Business

2,000

- 1.** Forecast the number of students each college after the end of the third quarter, based on a four-quarter system.
- 2.** Determine the steady-state conditions for the university.

A rental firm in the Southeast serves three states—Virginia, North Carolina, and Maryland. The firm has 700 trucks that are rented on a weekly basis and can be rented in any of the three states. The

transition matrix for the movement of rental trucks from state to state is as follows:

18.

[Page F-22]

<i>Week n</i>	<i>Week n + 1</i>		
	Virginia	North Carolina	Maryland
Virginia	[	.30	.50
North Carolina		.20	.20
Maryland		.10	.50

Determine the steady-state probabilities and the number of trucks in each state the long run.

The Koher Company manufactures precision machine tools. It has a quality management program for maintaining

product quality that relies heavily on statistical process control techniques. A machine center operator takes a sample at the end of every hour to see if the process is within statistical control limits. If the process is not within the preestablished limits, it is out of control; if it is within the limits, the process is in control. If the process remains out of control for 2 hours, it is shut down by the operator. The following transition matrix provides probabilities that a machine center will be in control (C), will be out of control (O), or will shut down (S) in the next hour, given the process status in the current hour:

19.

This Hour

Next Hour

	C	O	S
C	.75	.25	0
O	.85	.10	.05
S	.80	.05	.15

1. Determine the steady-state probabilities for this transition matrix.
2. For a period of 2,000 hours of operating time (i.e., approximately a year), how many hours will a machine center process be in control, out of control, or shut down?

In [Problem 19](#) the Koher Company is considering a new operator training program that will alter the probability that a machine center process will be out of control, as follows:

20.

This Hour

Next Hour

	C	O	S
C	.85	.15	0
O	.90	.05	.05
S	.90	.10	.0

Determine the annual savings in the number of hours a process is out of control or shut down with this new program.

Whitesville, where State University is located, has three bookstores that sell textbooks: the State Bookstore, the Eagle Bookstore, and Books n' Things. The State Bookstore is operated by the university. Don Williams, the manager of the State Bookstore, is in the process of placing book orders with his book distributors for the next semester. There are 17,000 undergraduate students at

21.

State, and they all purchase their textbooks at one of the three stores. Students often change from which bookstore they buy from one semester to the next. Don has sampled a group of students and developed the following transition matrix for student movement between the stores:

[Page F-23]

<i>This Semester</i>	<i>Next Semester</i>		
	State	Eagle	B&T
State	$\begin{bmatrix} .42 & .34 & .24 \\ .57 & .25 & .18 \\ .33 & .26 & .41 \end{bmatrix}$		
$T =$ Eagle			
B&T			

In this semester, if 9,000 students bought their textbooks at the State Bookstore, 5,000 bought their books at the Eagle Bookstore, and 3,000 students bought their books at Books n' Things, how

many are likely to buy their books at these stores next semester? How many students will purchase their books from each store in the long-run future?

Don Williams, the manager of the State Bookstore from [Problem 21](#), would like to increase his volume of textbook business. He believes that if he reduces textbook prices by 10%, he could increase his sales volume. Individual student textbook purchases currently average \$175. He is not sure the increase in volume would make up the loss of revenue from the price cut; however, he does think that the additional customers would increase sales for other items, such as clothing, supplies, and computer software. The other stores cannot as easily implement a price cut because they do not carry all the other items the State Bookstore does. Don estimates that a price reduction would

alter the transition matrix for the movement of students between stores follows:

22.

	<i>This Semester</i>	<i>Next Semester</i>		
	State	Eagle	B&T	
$T =$	State	$\begin{bmatrix} .52 & .29 & .19 \\ .63 & .22 & .15 \\ .39 & .23 & .38 \end{bmatrix}$		
	Eagle			
	B&T			

However, the other bookstores frequently complain that the State Bookstore has an unfair competitive advantage because it is located on campus and does not charge state sales tax. The university is sensitive to these local business complaints and has indicated to Don that he should keep his total market share in the long run to about 50% or less of the total student body market share.

1. Should Don implement his price reduction strategy?

2. If Don does implement the price c
how much of an increase or
decrease in sales revenue could h
expect?

At 4:00 P.M. each weekday, the three l
television stations in Salem have no
network obligations and can schedule
whatever shows they choose. WAL
the O'Jays talk show, WBDJ run
the Josie Donald talk show, and WCXI
runs episodes of the Barney Fife show.
The transition matrix of probabilities th
a regular viewer will watch the same
show or change shows from one day t
the next is as follows:

23.

Day 1

Next Day

	O'rah	Josie	Barney
O'rah	.67	.23	.10
Josie	.36	.58	.06
Barney	.13	.16	.71

1. Determine the steady-state probabilities for this transition matrix.
2. If there are 27,000 regular viewers in the Salem market at 4:00 P.M., how many can be expected to watch each show in the long run?

[Page F-24]

The Josie Donald show in [Problem 23](#) is contemplating an advertising campaign increase its viewers, at a cost of \$25,000. Each viewer generates \$0.12

commercial revenue per day. The static manager knows that the effects of any campaign will last only 4 months. The revised transition matrix resulting from

24. the ad campaign is as follows:

<i>Day One</i>		<i>Next Day</i>		
		O'rah	Josie	Barney
$T =$	O'rah	$\begin{bmatrix} .58 & .34 & .08 \\ .30 & .65 & .05 \\ .11 & .31 & .58 \end{bmatrix}$		
	Josie			
	Barney			

Should the station undertake the ad campaign?

Klecko's Copy Center uses several cop machines that deteriorate rather rapidly in terms of the quality of copies produced as the volume of copies increases. Each machine is examined at the end of each day to determine the quality of the cop being produced, and the results of that

inspection are classified as follows:

Classification	Copy Quality	Maintenar Cost pe Day
1	Excellent	\$ 0
2	Acceptable	100
3	Marginal	400
4	Unacceptable	800

25.

The costs associated with each

classification are for maintenance and repair and redoing unacceptable copies. When a machine reaches classification 1 and copies are unacceptable, major maintenance is required (resulting in downtime), after which the machine resumes making excellent copies. The transition matrix showing the probability of a machine's being in a particular classification state after inspection is as follows:

<i>Day One</i>	<i>Day Two</i>				
	1	2	3	4	
1	$\begin{bmatrix} 0 & .8 & .1 & .1 \\ 0 & .6 & .2 & .2 \\ 0 & 0 & .5 & .5 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	0	.8	.1	.1
2		0	.6	.2	.2
3		0	0	.5	.5
4		1	0	0	0

Determine the expected daily cost of machine maintenance.

- 26.** Determine the steady-state probabilities for the transition matrix in [Problem 1](#).

When freshmen at Tech attend orientation, the university's president tells each freshman that one of the two students next to him or her will not graduate. The freshmen interpret this as meaning that two thirds of the entering freshmen will graduate. The following transition probabilities have been developed from data gathered by Tech's registrar. They show the probability of a student's moving from one class to the next during an academic year and eventually graduating (G) or dropping out (D):

27.

[Page F-25]

	F	So	J	Sr	D	G
F	.10	.70	0	0	.20	0
So	0	.10	.80	0	.10	0
J	0	0	.15	.75	.10	0
Sr	0	0	0	.15	.05	.80
D	0	0	0	0	1	0
G	0	0	0	0	0	1

1. Is the president's remark during orientation accurate?
2. What is the probability that a freshman will eventually drop out?
3. How many years can an entering freshman expect to remain at Tec

Libby Jackson is a carpenter who work for a large construction company that l a number of housing developments unc

way in the metropolitan Washington, D area. Each day Libby is assigned to one the company's developments, either hanging and finishing drywall, doing trim work, framing, or roofing. The following transition matrix describes the probability that she will move from a job one day the same job or another the next:

28.

		<i>Day 2</i>			
<i>Day 1</i>		Drywall	Trim	Framing	Roofing
$T =$	Drywall	.45	.23	.18	.14
	Trim	.31	.37	.23	.09
	Framing	.28	.15	.52	.05
	Roofing	.26	.32	.18	.24

1. Determine the steady-state probabilities for this transition matrix.
2. If Libby works 250 days during the year, how many days will she work at each job?

Frank Beamish, the head football coach at Tech, has had his staff scout State University for most of its games this season to get ready for the annual season-ending game. The Tech coaching staff has developed the following transition matrix of the probabilities that State will change its defense from one play to the next:

29.

	<i>This Play</i>	<i>Next Play</i>			
		54	63	Nickel	Blitz
$T =$	53	.48	.23	.19	.10
	63	.18	.55	.17	.10
	Nickel	.30	.40	.15	.15
	Blitz	.25	.50	.20	.05

- 1.** Determine the steady-state probabilities for this transition matrix.

- 2.** If Tech runs 115 plays during the game, how many of the plays will State be in each defense?
- 3.** If Tech splits its plays evenly between the pass and run, how many times during the game will State be blitzing when Tech is passing?

Case Problem

THE FRIENDLY CAR FARM

Buddy Friendly and his wife, Vera, own and manage the Friendly Car Farm, a large new and used car dealership. Buddy and Vera primarily stock family-oriented cars like vans and midsize sedans and pickup trucks. The profit margin on these types of cars is not much, but the Friendlys have a very high volume of business, which

offsets their low per-unit profit margin. They feel like they know their customer preferences, and they have tended to stock accordingly. However, for several years one of the automobile manufacturers the Friendlys deal with has repeatedly tried to get them to sell its new, high-priced sports car, the Zephyr AK2000. Buddy and Vera have finally agreed to stock the AK2000, and they have developed a tentative order policy as follows. If the number of Zephyrs they have

on the lot is one or less at the end of the month, they will order either two or three Zephyrs from the manufacturer so that they will never have more than three of the cars on the lot at the beginning of the month. (The new cars will arrive in the week following the order, at the first of the next month.) The Friendlys paid a statistical consultant from a nearby university to develop the following transition matrix:

$$T = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} .08 & .18 & .37 & .37 \\ .63 & .37 & 0 & 0 \\ .26 & .37 & .37 & 0 \\ .08 & .18 & .37 & .37 \end{bmatrix} \end{matrix}$$

(The consultant used a Poisson distribution with $\lambda =$ one sale per month to compute each transition probability a fact completely lost on the Friendlys.)

This transition matrix shows the probability of having either zero, one, two, or three cars in stock one month, given either zero, one, two, or three cars in stock the previous month. For

example, if the Friendlys have no Zephyrs in stock in one month, the probability is only .08 that they would have none in stock the next month. The reason this probability is so small is because if the stock level is zero, a new order is placed for three cars. Thus, .08 is actually the probability that three cars were ordered and they were sold in the next month.

The Friendlys pay the manufacturer for cars using loans from a local bank. The interest on these loans is part

of the inventory holding cost, which also includes maintenance costs while the car is on the lot. The inventory holding cost for the Zephyr is relatively high compared with that for other cars the Friendlys sell. In any month the holding cost for one Zephyr is \$75, for two Zephyrs it is \$175, and for three Zephyrs it is \$310. (Notice that the holding cost per car increases at an increasing rate because of accumulating interest charges.)

Determine the probability of

the Friendlys having zero, one, two, or three cars in stock in a month in the long run, the expected number of cars in stock in a month in the long-run future, and the average inventory holding cost per month.

Case Problem

DAVIDSON'S DEPARTMENT STORE

As a result of intense competition and an economic recession, Davidson's Department Store in Atlanta was forced to pay particularly close attention to its cash flow. Because of the poor economy, a number of Davidson's customers were not paying their bills upon receipt, delaying payment for several

months, and frequently not paying at all. In general, the Davidson's policy for accounts receivable was to allow a customer to be 2 months late on his or her bill before turning it over to a collection agency. However, it was not quite as simple as that.

Davidson's has approximately 10,000 open accounts at any time. The age of the account is determined by the oldest dollar owed. This means that a customer can have a balance for items bought in two different months, with the

overall account being listed as old as the earliest month of purchase. For example, suppose a customer has a balance of \$100 at the end of January, \$80 of which is for items bought in January and \$20 for items bought in November. This means the account is 2 months old at the end of January because the oldest amount on account is from November. If the customer subsequently pays \$20 on the bill in February, this cancels the November purchase. Then if the customer makes \$100

worth of purchases in February, the account is \$180, and it is 1 month old (since the oldest purchases were from January).

[Page F-27]

Carla Reata, Davidson's comptroller, analyzed the accounts receivable data for the store for an extended period. She summarized these data and developed some probabilities for the payment (or nonpayment) of bills. She determined that for current bills (in their first month of

billing), there is an .86 probability that the bills will be paid in the month and a .14 probability that they will be carried over to the next month and be 1 month late. If a bill is already 1 month late, there is a .22 probability that the oldest portion of the bill will be paid so that it will remain 1 month old, a .40 probability that the entire bill will be carried over so that it is 2 months old, and a .32 probability that the bill will be paid in the month. For bills 2 months old, there is a probability of .54 that the

oldest portion will be paid so that the bill remains 1 month old, a .16 probability that the next-oldest portion of the bill will be paid so that it remains 2 months old, a .18 probability that the bill will be paid in the month, and a .12 probability that the bill will be listed as a bad debt and turned over to a collection agency. If a bill is paid or listed as a bad debt, it will no longer move to any other billing status.

Under normal circumstances (i.e., not a holiday season), the store averages \$1,350,000 in

outstanding bills during an average month; \$750,000 of this amount is current, \$400,000 is 1 month old, and \$200,000 is 2 months old. The vice president of finance for the store wants Carla to determine how much of this amount will eventually be paid or end up as bad debts in a typical month. She also wants Carla to tell her if an average cash reserve of \$60,000 per month is enough to cover the expected bad debts that will occur each month. Perform this analysis for Carla.

[Page InsideFrontCover]

Have You Thought About Customizing This Book?

[The Prentice Hall Just-In-Time Program in Decision Science](#)

[You Can Customize Your Textbook With Chapters From Any Of The Following Prentice Hall Titles](#)

The Prentice Hall Just-In-Time Program in Decision Science

You can combine chapters from this book with chapters from any of the Prentice Hall titles listed on the following page to create a text tailored to your specific course needs. You can add your own material or cases from our extensive case collection. By taking a few minutes to look at what is

sitting on your bookshelf and the content available on our Web site, you can create your ideal textbook.

The Just-In-Time program offers:

- **Quality of Material to Choose From** In addition to the books listed, you also have the option to include any of the cases from Prentice Hall Custom Business Resources, which gives you access to cases

(and teaching notes where available) from Darden, Harvard, Ivey, NACRA, and Thunderbird. Most cases can be viewed online at our Web site.

- **Flexibility** Choose only that material you want, either from one title or several titles (plus cases) and sequence it in whatever way you wish.
- **Instructional Support** You have access to the text-specific CD-ROM that

accompanies the traditional textbook and desk copies of your JIT book.

- **Outside Materials** There is also the option to include up to 20% of the text from materials outside of Prentice Hall Custom Business Resources.
- **Cost Savings** Students pay only for material you choose. The base price is \$6.00, plus \$2.00 for case material, plus \$.09 per page. The text can be

shrink-wrapped with other Pearson textbooks for a 10% discount. Outside material is priced at \$.10 per page plus permission fees.

- **Quality of Finished Product** Custom cover and title page-including your name, school, department, course title, and section number. Paperback, perfect bound, black-and-white printed text. Customized table of contents. Sequential pagination

throughout the text.

Visit our Web site at
www.prenhall.com/custombusiness
and create your custom text
our bookbuildsite or download
order forms online.

You Can Customize Your Textbook With Chapters From Any Of The Following Prentice Hall Titles^[*]

[*] Selection of titles on the JIT program is subject to change.

Business Statistics

- Berenson/Levine/Krehbiel,
BASIC BUSINESS

STATISTICS, 10/e

- Groebner/Shannon/Fry/Smi
BUSINESS STATISTICS: A
DECISION- MAKING
APPROACH, 6/e
- Levine/Stephan/Krehbiel/Be
FOR MANAGERS USING
MICROSOFT EXCEL, 4/e
- Levine/Krehbiel/Berenson,
BUSINESS STATISTICS: A
FIRST COURSE, 4/e
- Newbold/Carlson/Thorne,
STATISTICS FOR BUSINESS

& ECONOMICS, 6/e

- Groebner/Shannon/Fry/Smi
A COURSE IN BUSINESS
STATISTICS, 4/e

Operations Management

- Anupindi/Chopra/Deshmukh
Miegham/Zemel,
MANAGING BUSINESS
PROCESS FLOWS, 2/e
- Bozarth/Handfield,
INTRODUCTION TO
OPERATIONS & SUPPLY

CHAIN MANAGEMENT

- Chopra/Meindl, SUPPLY CHAIN MANAGEMENT, 3e
- Foster, MANAGEMENT QUALITY: Interpreting the Supply Chain, 3/e
- Handfield/Nichols, Jr., SUPPLY CHAIN MANAGEMENT
- Heineke/Meile, GAMES AND EXERCISE FOR OPERATIONS MANAGEMENT

- Heizer/Render, OPERATIONS MANAGEMENT, 8/e
- Heizer/Render, PRINCIPLES OF OPERATIONS MANAGEMENT, 6/e
- Krajewski/Ritzman/Krajewski, OPERATIONS MANAGEMENT, 8/e
- Latona/Nathan, CASES AND READINGS IN PRODUCTION AND OPERATIONS MANAGEMENT

- Ritzman/Krajewski, FOUNDATIONS OF OPERATIONS MANAGEMENT
- Schmenner, PLANT AND SERVICE TOURS IN OPERATIONS MANAGEMENT, 5/e

Management Science/Spreadsheet Modeling

- Eppen/Gould/Schmidt/Moore

INTRODUCTORY MANAGEMENT SCIENCE, 5/e

- Render/Stair/Hanna,
QUANTITATIVE ANALYSIS
FOR MANAGEMENT, 9/e
- Balakrishnan/Render/Stair,
MANAGERIAL DECISION
MODELING WITH
SPREADSHEETS, 2/e
- Render/Greenberg/Stair,
CASES AND READINGS IN
MANAGEMENT SCIENCE, 2e

- Taylor, INTRODUCTION TO MANAGEMENT SCIENCE, 9/e

For more information, or to speak to a customer service representative, contact us at 1-800-777-6872.

www.prenhall.com/custombusiness

[Page InsidebackCover]

Site License Agreement and Limited Warranty

READ THIS LICENSE CAREFULLY BEFORE USING THIS PACKAGE. BY USING THIS PACKAGE, YOU ARE AGREEING TO THE TERMS AND CONDITIONS OF THIS LICENSE. IF YOU DO NOT AGREE, DO NOT USE THE PACKAGE. PROMPTLY RETURN THE UNUSED PACKAGE AND ALL ACCOMPANYING ITEMS TO

THE PLACE YOU OBTAINED.
THESE TERMS APPLY TO ALL LICENSED SOFTWARE ON THE DISK EXCEPT THAT THE TERMS FOR USE OF ANY SHAREWARE OR FREWARE ON THE DISKETTES ARE AS SET FORTH IN THE ELECTRONIC LICENSE LOCATED ON THE DISK:

1. GRANT OF LICENSE and OWNERSHIP: The enclosed computer programs and data ("Software") are licensed, not sold, to you by Prentice-Hall, Inc. ("We" or the "Company")

in consideration of your purchase or adoption of the accompanying Company textbooks and/or other materials, and your agreement to these terms. We reserve any rights not granted to you. You own only the disk(s) but we and/or licensors own the Software itself. This license allows you to install, use, and display the enclosed copy of the Software on individual computers in the computer lab designated for use by any students of a course requiring the accompanying Company

textbook and only for the as long as such textbook is a required text for such course, at a single campus or branch or geographic location of an educational institution, for academic use only, so long as you comply with the terms of this Agreement.

2. RESTRICTIONS: You may not transfer or distribute the Software or documentation to anyone else. Except for backup, you may not copy the documentation or the Software. You may not reverse engineer, disassemble, decompile, modify,

adapt, translate, or create derivative works based on the Software or the Documentation. You may be held legally responsible for any copying or copyright infringement that is caused by your failure to abide by the terms of these restrictions.

3. TERMINATION: This license is effective until terminated. This license will terminate automatically without notice from the Company if you fail to comply with any provisions or limitations of this license. Upon

termination, you shall destroy the Documentation and all copies of the Software. All provisions of this Agreement as to limitation and disclaimer of warranties, limitation of liability, remedies or damages, and our ownership rights shall survive termination.

4. LIMITED WARRANTY AND DISCLAIMER OF

WARRANTY: Company warrants that for a period of 60 days from the date you purchase this Software (or purchase or adopt the accompanying textbook), the

Software, when properly installed and used in accordance with the Documentation, will operate in substantial conformity with the description of the Software set forth in the Documentation, and that for a period of 30 days the disk(s) on which the Software is delivered shall be free from defects in materials and workmanship under normal use. The Company does not warrant that the Software will meet your requirements or that the operation of the Software will be uninterrupted or error-

free. Your only remedy and the Company's only obligation under these limited warranties is, at the Company's option, return of the disk for a refund of any amounts paid for it by you or replacement of the disk. THIS LIMITED WARRANTY IS THE ONLY WARRANTY PROVIDED BY THE COMPANY AND ITS LICENSORS, AND THE COMPANY AND ITS LICENSORS DISCLAIM ALL OTHER WARRANTIES, EXPRESS OR IMPLIED, INCLUDING WITHOUT LIMITATION, THE IMPLIED WARRANTIES OF

MERCHANTABILITY AND FITNESS FOR A PARTICULAR PURPOSE. THE COMPANY DOES NOT WARRANT, GUARANTEE OR MAKE ANY REPRESENTATION REGARDING THE ACCURACY, RELIABILITY, CURRENTNESS, USE, OR RESULTS OF USE, OF THE SOFTWARE.

5. LIMITATION OF REMEDIES AND DAMAGES:
IN NO EVENT, SHALL THE COMPANY OR ITS EMPLOYEES, AGENTS, LICENSORS, OR CONTRACTORS BE LIABLE FOR ANY INCIDENTAL, INDIRECT,

SPECIAL, OR CONSEQUENTIAL DAMAGES ARISING OUT OF OR IN CONNECTION WITH THIS LICENSE OR THE SOFTWARE, INCLUDING FOR LOSS OF USE, LOSS OF DATA, LOSS OF INCOME OR PROFIT, OR OTHER LOSSES, SUSTAINED AS A RESULT OF INJURY TO ANY PERSON, OR LOSS OF OR DAMAGE TO PROPERTY, OR CLAIMS OF THIRD PARTIES, EVEN IF THE COMPANY OR AN AUTHORIZED REPRESENTATIVE OF THE COMPANY HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES. IN NO

EVENT SHALL THE LIABILITY OF THE COMPANY FOR DAMAGES WITH RESPECT TO THE SOFTWARE EXCEED THE AMOUNTS ACTUALLY PAID BY YOU, IF ANY, FOR THE SOFTWARE OR THE ACCOMPANYING TEXTBOOK. SOME JURISDICTIONS DO NOT ALLOW THE LIMITATION OF LIABILITY IN CERTAIN CIRCUMSTANCES, THE ABOVE LIMITATIONS MAY NOT ALWAYS APPLY.

6. GENERAL: THIS AGREEMENT SHALL BE CONSTRUED IN ACCORDANCE

WITH THE LAWS OF THE UNITED STATES OF AMERICA AND THE STATE OF NEW YORK, APPLICABLE TO CONTRACTS MADE IN NEW YORK, AND SHALL BENEFIT THE COMPANY, ITS AFFILIATES AND ASSIGNEES. This Agreement is the complete and exclusive statement of the agreement between you and the Company and supersedes all proposals, prior agreements, oral or written, and any other communications between you and the company or any of its representatives relating to the

subject matter. If you are a U.S. Government user, this Software is licensed with "restricted rights" as set forth in subparagraphs (a)(d) of the Commercial Computer-Restricted Rights clause at FAR 52.227-19 or in subparagraphs (c)(1)(ii) of the Rights in Technical Data and Computer Software clause at DFARS 252.227-7013, and similar clauses, as applicable.

Should you have any questions concerning this agreement or if you wish to contact the Company for any reason,

please contact in writing:

Director, Media Production

Pearson Education

1 Lake Street

Upper Saddle River, NJ 07458

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Index

[**SYMBOL**] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

01 integer model 2nd 3rd 4th
capital budgeting problem
Excel 2nd 3rd 4th
facility location problem
fixed charge problem

QM for Windows

set covering problem

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

A priori probability

Absorbing state

Activities

Activity scheduling

backward pass

earliest finish time 2nd

earliest start time

forward pass 2nd

latest finish time

latest start time 2nd

shared slack

slack

Activity slack

Activity-on-Arrow (AOA) networks

Activity-on-Node (AON) networks

Adjusted exponential smoothing

Air Force 2nd

Alternate optimal solution

American Airlines

Analogue simulation

Analytical hierarchy process (AHP)

2nd 3rd

consistency index

Excel

normalized matrix

pairwise comparisons

preference scale

preference vector

random index

Saaty, Thomas

summary steps

synthesization

Arrival rate

Poisson

Artificial variable 2nd

Assignment model

balanced model

Excel

Excel QM

QM for Windows

unbalanced model

Assignment solution method

column reductions

dummy column

dummy row

multiple optimal solutions

opportunity cost table

optimal solution

prohibited assignment

row reductions

summary steps

unbalanced

Average error (bias)

Index

[SYMBOL] [A] [**B**] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Backward pass

Bad debt example [See
[debt example](#)]

Balanced transportation problem 2nd

Bank of America

Bar chart

Basic feasible solution 2nd 3rd

Basic variables

Battle of Britain

Bayes, Thomas

Bayes's rule

Bayesian analysis 2nd

Beer, Stafford

Bernoulli process

Beta distribution

mean

variance

Bias [See [average error](#)]

Binomial distribution

Blackett, P.M.S.

Blackett's Circus

Blend example (linear programming)

Bolder Boulder (10-K race)

Booz, Allen, Hamilton

Branch and bound method 2nd

Branches 2nd 3rd

Brand-switching problem

Break-even analysis 2nd 3rd

Excel 2nd

Excel QM

fixed costs 2nd

graphical solution

profit 2nd

profit analysis

QM for Windows 2nd

simulation

total cost

variable costs

volume 2nd 3rd

Break-even point

Break-even volume

Buffer stock

Bush, Vannevar

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Calling population

Capital budgeting example (01
integer programming)

Carrying costs 2nd 3rd

Causal forecasting methods

Central limit theorem

Charnes, Abraham

Chi-square table 2nd

Chi-square test

Classical optimization

Classical probability

Coefficient of determination

Coefficient of optimism

Coefficient of pessimism

Collectively exhaustive (events)

Complete enumeration

Computer simulation

Conant, James B.

Concurrent activities

Conditional constraint

Conditional probability 2nd 3rd

Confidence limits

Consistency index

Constant service times

Constrained optimization

Constraints 2nd

Constraints (in linear programming)

2nd 3rd

Contingency constraint 2nd

Continuous inventory system

Continuous probability distribution

2nd

Continuous random variable

Cooper, William W.

Coors

Corequisite constraint

Correlation

Cost variance

Covariance

CPM [See [critical path method](#)]

CPM/PERT 2nd 3rd

activity scheduling

backward pass

beta distribution

branches 2nd

concurrent activities

crashing

critical path

dummy activities

earliest finish time 2nd

earliest start time 2nd

forward pass 2nd

latest finish time 2nd

latest start time 2nd

linear programming formulation

Microsoft Project

most likely time estimate

nodes 2nd

optimistic time estimate

pessimistic time estimate

precedence relationships

probabilistic activity times

probability analysis

project variance

QM for Windows

shared slack

slack

Crash cost

Crash time

Crashing 2nd

crash costs

crash times

linear programming

normal activity costs

normal activity times

QM for Windows

Critical path 2nd 3rd 4th

Critical path method (CPM) 2nd 3rd
4th 5th 6th 7th 8th 9th

Crystal Ball

Cumulative error

Cumulative probability distribution

Cycles (in forecasting)

Cyclic (transition) matrix

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Dantzig, George B. 2nd

Dar Al-Hekma

Data

Data envelopment analysis (DEA)
example (linear programming)

Data warehouses

Debt example

Decision analysis

criteria

dominant decision

Excel 2nd

expected opportunity loss 2nd

expected value

payoff

payoff table

QM for Windows 2nd 3rd

states of nature

Treeplan 2nd

with additional information

with probabilities

without probabilities

Decision sciences

Decision strategies

Decision support systems (DSS)

Decision tree

Excel

QM for Windows

sequential

TreePlan

with posterior probabilities

Decision variables 2nd

Decision-making criteria

equal likelihood

expected opportunity loss

expected value

Hurwicz

maximax

maximin

minimax regret

minimin

Defense Contract Management

Agency

Degeneracy (in linear programming)

2nd

Degeneracy (in transportation problem)

Dell 2nd

Delphi method

Demand

Dependent demand

Dependent events 2nd

Dependent variable 2nd

Deterministic techniques 2nd

Deviational variables

Diet example (linear programming)

Dijkstra, E.W.

Directed branch

Discrete probability distribution

Discrete values

Doig, A.C.

Dominant decisions

Dominant strategies

Dual

Dual value 2nd 3rd 4th 5th 6th

Dummy activities

DuPont

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Earliest event times 2nd

Earliest finish time 2nd

Earliest start time 2nd

Earned value analysis (EVA)

Economic order quantity (EOQ)

assumptions of

basic model

carrying costs

Excel 2nd

Excel QM 2nd

gradual usage model

lead time

noninstantaneous receipt model

ordering cost

production lot size model

QM for Windows

reorder point

robustness

service level

time

total cost

with safety stocks

with shortages

Efficiency of sample information

Electronic data interchange (EDI)

Entergy Electric Systems

Entering nonbasic variable 2nd

Enterprise resource planning (ERP)

Equal likelihood criterion

Equation of a line 2nd

Equilibrium point

Erlang, A.K.

Events

collectively exhaustive

dependent

independent

mutually exclusive

Excel 2nd 3rd 4th 5th 6th 7th 8th 9th
10th 11th 12th 13th 14th 15th 16th
17th 18th 19th 20th 21st 22nd 23rd
24th 25th 26th 27th 28th 29th 30th
31st 32nd 33rd 34th 35th 36th 37th
38th 39th 40th 41st 42nd 43rd 44th
45th 46th 47th 48th 49th 50th 51st

analytical hierarchy process

(AHP)

assignment problem

break-even analysis 2nd

CPM/PERT

crashing

decision analysis

decision trees

EOQ analysis

expected value

forecasting 2nd 3rd

goal programming

integer programming 2nd 3rd 4th
5th 6th

inventory management 2nd 3rd
4th

linear programming 2nd 3rd 4th

5th 6th 7th 8th 9th 10th 11th 12th
13th 14th

linear regression

Markov analysis

maximal flow problem

multiple regression

nonlinear programming 2nd 3rd

4th

posterior probabilities

queuing analysis 2nd 3rd 4th

scoring model

sensitivity analysis 2nd 3rd 4th

shadow price

shortest route problem

simulation 2nd 3rd 4th

statistical analysis

time series forecasting

total integer programming

problem 2nd

transportation problem

transshipment problem

TreePlan

tutorial on

Excel QM 2nd 3rd 4th 5th 6th 7th
8th 9th

Expected gain and loss method

Expected opportunity loss 2nd

Expected value 2nd 3rd

Expected value given perfect
information

Expected value of perfect
information (EVPI) 2nd

Expected value of sample

information (EVSI)

Expected value without perfect information

Experiment (in probability)

Expert Choice

Exponential distribution 2nd

Exponential service times

Exponential smoothing

smoothing constant 2nd

Extranet

Extreme points (in linear programming)

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Facility location problem (01 integer programming)

Facility location problem (nonlinear programming)

Factorials

Feasible solution (in linear programming)

FedEx 2nd

Finite calling population

Finite queue

Fixed charge problem (01 integer programming)

Fixed costs 2nd

Fixed-order quantity system 2nd

Fixed-time period system 2nd

Ford Motor Company

Ford, L.R., Jr.

Forecast accuracy

Forecast error

Forecasting 2nd

accuracy

adjusted exponential smoothing

average error (bias)

cumulative error

cycles

error

Excel 2nd 3rd

Excel QM

exponential smoothing

linear regression

linear trend line

long-range forecasts

mean absolute deviation (MAD)

mean absolute percentage

deviation (MAPD)

mean squared error (MSE)

medium-range forecasts

moving averages

multiple regression

QM for Windows 2nd

qualitative methods 2nd

random variations

regression methods 2nd

seasonal adjustments

seasonal patterns

short-range forecasts

time series methods 2nd

trend

weighted moving averages

formulas

Forward pass 2nd

Frequency distribution

Fulkerson, D.R.

Functional relationship

Fundamental matrix

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[**G**] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Game theory

closed loop

dominant strategies

equilibrium point

expected gain and loss method

maximin strategy

minimax strategy

mixed strategy game

n-person game

optimal strategy

payoff table

pure strategy game

QM for Windows 2nd

saddle point

strategies

two-person game

value of the game

zero-sum game

Gantt chart 2nd

Gantt, Henry

GE Plastics

Georgia Power Company

Goal constraint

Goal programming

deviation variables

Excel

goals

graphical solution

model formulation

objective function

priorities

QM for Windows solution

satisfactory solution

Gomory, Ralph E.

Goodness-of-fit

Gradual usage model

Graphical solution 2nd 3rd 4th 5th
6th 7th

break-even analysis

goal programming

infeasible problem

integer programming model

linear programming

maximization model

minimization model

multiple optimal solutions

nonlinear programming 2nd

sensitivity analysis

slack variable

summary steps

surplus variables

unbounded problem

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Harris, Ford

Heery International

Hitchcock, Frank L.

Hurwicz criterion

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

IBM

Identity matrix

Implementation 2nd

Implicit enumeration

Independent demand

Independent events

Independent variable

Infeasible solution (in linear programming) 2nd

Instantaneous receipt

Institute for Operations Research and Management Sciences (INFORMS)

Integer programming 2nd 3rd

01integer model 2nd 3rd 4th

branch and bound method 2nd

capital budgeting example

complete enumeration

conditional constraint

contingency constraint 2nd

corequisite constraint

Excel 2nd 3rd 4th 5th 6th

facility location example

fixed charge problem

graphical solution

mixed integer model 2nd 3rd

multiple-choice constraint

mutually exclusive constraint 2nd

optimal solution

QM for Windows 2nd

relaxed solution

rounded down solution

set covering problem

total integer model 2nd

Interfaces

Internet

Intranet

Inventory

average level

buffer stock

carrying costs

costs 2nd

definition of

ordering costs 2nd

role of

safety stock 2nd

shortage costs

total cost

Inventory control

Inventory management 2nd

certain demand

continuous system 2nd

control systems

dependent demand

economic order quantity (EOQ)

elements of

Excel 2nd 3rd 4th

Excel QM

fixed-order quantity system

fixed-time period system 2nd

independent demand

periodic system 2nd

QM for Windows 2nd

quantity discounts

reorder point 2nd 3rd

role of

safety stocks 2nd

simulation

Inverse (of a matrix)

Investment example (linear programming)

Investment portfolio selection example (nonlinear programming)

Irregular types of linear programming problems

Israeli Army

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Joint probability 2nd
Joint probability table

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [**K**] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Kelley, James E., Jr.

Kellogg 2nd

Koopmans, T.C.

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

L.L. Bean

Lagrange multipliers method 2nd

Lagrangian function

Land, A.H.

Lands' End

LaPlace criterion

Latest finish time 2nd

Latest start time 2nd

Lead time 2nd 3rd

Least squares formula 2nd

Linear equation 2nd

Linear programming 2nd 3rd 4th 5th

6th 7th 8th

assignment problem

basic feasible solution

basic variables

blend example

characteristics

computer solution

constraints 2nd 3rd

CPM/PERT

crashing

data envelopment analysis

(DEA) example

decision variables 2nd

degeneracy

diet example

dual

dual value 2nd 3rd 4th 5th 6th

[See also [shadow price](#)]

Excel 2nd 3rd 4th 5th 6th 7th 8th

9th 10th 11th 12th 13th 14th 15th

extreme points

feasible solution

feasible solution area

graphical solution

infeasible solution 2nd

integer solutions 2nd

investment example

irregular types of problems

marginal value

marketing example

maximal flow problem

maximization model

minimization model

mixed constraint problem

model formulation

modeling examples

multiperiod scheduling example

multiple optimal solutions 2nd

objective function 2nd

optimal solution 2nd

primal

product mix example

project crashing

properties

QM for Windows 2nd 3rd 4th

sensitivity analysis 2nd

shadow price 2nd 3rd 4th [See

also [dual value](#)]

shortest route problem

simplex method 2nd

simplex tableau

slack variables

standard model form

surplus variables

transportation problem 2nd

transshipment problem

unbounded problem

Linear regression

Linear trend line

Long-range forecasts

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Malcolm, D.G.

Management science 2nd

definition

process

"Management Science Application"

(box) 2nd 3rd 4th 5th 6th 7th 8th 9th

10th 11th 12th 13th 14th 15th 16th
17th 18th 19th 20th 21st 22nd 23rd
24th 25th 26th 27th 28th 29th 30th
31st 32nd 33rd 34th 35th 36th 37th
38th 39th 40th 41st 42nd 43rd 44th
45th 46th 47th 48th

Abo, Finland

American Airlines

Bank of America

Bolder Boulder (10-K race)

Coors

Dar Al-Hekma

Defense Contact Management
Agency

Dell 2nd

DuPont

Entergy Electric Systems

FedEx 2nd

Florida Power Corporation

Ford Motor Company

GE Plastics

Georgia Power Company

Heery International

IBM

Israeli Army

Kellogg 2nd

L.L. Bean

Lands' End

Mars

Minnesota Nutrition Coordinating

Center

Mount Sinai Hospital (Toronto)

Nabisco

NASA

National Car Rental

NBC

New York Power Authority

Nu-kote International

Oglethorpe Power Corporation

Ohio University

Prudential Securities, Inc.

S.S. Central America

Santos, Ltd.

Soquimich (SA)

Taco Bell 2nd

Texaco

U.S. Coast Guard

UPS

Valley Metal Container (VMC)

Vermont Gas Systems

Walt Disney Company

Welch's

Winter Olympics (2002)

Yellow Freight Systems

Management scientist

Marginal probability 2nd

Marginal value 2nd

Marketing example (linear programming)

Markov analysis

absorbing state

cyclic (transition) matrix

debt example

Excel

properties

QM for Windows

steady-state probabilities

system state

transient state

transition matrix

transition probabilities

Markov process

Markov, Andrey A.

Markowitz, Harry

Mars

Matrix algebra

Maximal flow problem

directed branch

Excel

linear programming formulation

net flow

QM for Windows

solution steps

undirected branch

Maximax criterion

Maximin criterion

Maximin strategy

Mean

Mean absolute deviation (MAD)

Mean absolute percentage deviation (MAPD)

Mean squared error (MSE)

Medium-range forecasts

Microsoft Project

Middleton, Michael

Minimal spanning tree problem

QM for Windows

solution steps

Minimax regret criterion 2nd

Minimax strategy

Minimin criterion

Minimum cell cost method

Minnesota Nutrition Coordinating Center

Mixed constraint problem 2nd

Mixed integer model 2nd 3rd

Mixed strategy game 2nd

Model 2nd 3rd

constraints 2nd

formulation

linear programming

parameters

solution

variables 2nd

Modified distribution method (MODI)

summary steps

Monte Carlo process

Monte Carlo simulation

Morgenstern, Oskar

Most likely time estimate

Mount Sinai Hospital (Toronto)

Moving averages

Multicollinearity

Multicriteria decision making 2nd

analytical hierarchy process

(AHP)

goal programming

scoring model

Multiperiod scheduling example

(linear programming)

Multiple optimal solutions

(assignment problem)

Multiple optimal solutions (linear

programming) 2nd 3rd

Multiple optimal solutions
(transportation problem)

Multiple regression

Multiple-choice constraint

Multiple-server waiting line

Mutually exclusive constraint 2nd 3rd

Mutually exclusive events 2nd

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

N-person game

Nabisco

Naïve forecasting

NASA

National Car Rental

NBC

Net (branch) flow

Network

Network flow models 2nd

Excel 2nd

maximal flow

minimal spanning tree

QM for Windows 2nd 3rd

shortest route

New York Power Authority

Nodes 2nd 3rd

Non-zero-sum game

Nonbasic variable

Noninstantaneous receipt model

Nonlinear programming 2nd

constrained optimization

Excel 2nd 3rd 4th

Lagrange multipliers 2nd
substitution method

Nonnegativity constraints

Normal activity costs

Normal activity times

Normal distribution 2nd

CPM/PERT network

mean

reorder point

standard deviation

Normal table 2nd 3rd

Normalized matrix (AHP)

Northwest corner method

Nu-kote International

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Objective function 2nd 3rd

Objective probability

Oglethorpe Power Corporation

Ohio University

Online analytical processing (OLAP)

Operating characteristics (queuing)

2nd 3rd

Operations research 2nd

Opportunity loss

Opportunity loss table

Optimal solution (in linear programming) 2nd

Optimistic time estimate

Ordering cost 2nd 3rd

Organizational breakdown structure (OBS)

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Pairwise comparisons

Parameters 2nd

Payoff

Payoff table

Penalty cost (VAM)

Performance management

Periodic inventory system 2nd 3rd

with variable demand

Permanent set

Pessimistic time estimate 2nd

Pivot column

Pivot column tie

Pivot number

Pivot row

Pivot row tie (degeneracy)

Poisson arrival rate 2nd

Poisson distribution 2nd

Polaris missile project

Population mean

Population variance

Posterior probability 2nd

table

Precedence relationships 2nd

Preference scale

Preference vector

Primal 2nd

Prior probability

Priorities (in goal programming)

Probabilistic activity times

Probabilistic techniques

Probability

a priori

binomial distribution

chi-square test

classical

collectively exhaustive events

conditional 2nd 3rd

continuous distribution

cumulative distribution

dependent events 2nd

discrete distribution

events

Excel 2nd

expected value

experiment

frequency distribution

fundamentals of

goodness-of-fit

independent events

joint

joint probability table

marginal 2nd

mean

mutually exclusive events 2nd

normal distribution

objective

posterior 2nd

prior

probability distribution

relative frequency

revised

standard deviation

subjective 2nd

tree

unconditional

variance

Venn diagram

Probability distribution 2nd

Probability tree

Problem definition

Product mix (linear programming)
model 2nd

Production lot size model

Profit 2nd

Profit analysis [See [break-even analysis](#)]

Prohibited assignment

Prohibited transportation route 2nd

Project control

Project crashing [See [crashing](#)]

Project evaluation and review
technique (PERT) 2nd

Project management 2nd

control 2nd

cost management

CPM 2nd 3rd 4th 5th 6th 7th 8th

9th

crashing 2nd

critical path 2nd 3rd 4th

earned value analysis (EVA)

elements of

Gantt chart

organizational breakdown

structure (OBS)

performance management

PERT 2nd 3rd 4th 5th 6th 7th

8th 9th

planning

project manager

project network

project team

responsibility assignment matrix

(RAM)

scope statement

statement of work

time management

work breakdown structure

(WBS)

Project manager

Project planning

Project team

Project variance

Properties (of linear programming)

Proportionality

Prudential Securities, Inc.

Pseudorandom numbers

Pure strategy game

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [**Q**] [R] [S] [T] [U] [V] [W] [Y]
[Z]

QM for Windows 2nd 3rd 4th 5th 6th
7th 8th 9th 10th 11th 12th 13th 14th
15th 16th 17th 18th 19th 20th 21st
22nd 23rd 24th 25th 26th 27th 28th
29th 30th 31st

01 integer programming problem

assignment problem

break-even analysis 2nd

CPM/PERT

crashing

decision analysis

decision trees 2nd

EOQ analysis

EOQ with quantity discounts

expected value

forecasting 2nd

game theory 2nd

goal programming

integer programming 2nd

inventory 2nd

linear programming 2nd 3rd 4th

linear regression

Markov analysis

maximal flow problem

minimal spanning tree problem

mixed strategy game

queuing analysis 2nd 3rd 4th 5th

sensitivity analysis 2nd

shortest route problem

time series forecasting

transportation problem

Quadratic function

Qualitative forecasting methods 2nd

Quantitative analysis

Quantitative methods

Quantity discounts

carrying costs as a percentage
of price

constant carrying costs

Queue

Queue discipline

Queuing analysis 2nd

arrival rate

calling population

constant service times

cost trade-off

Excel 2nd 3rd 4th 5th

Excel QM 2nd 3rd 4th

exponential distribution

finite calling population

finite queue

multiple-server waiting line

operating characteristics 2nd 3rd

Poisson distribution

QM for Windows 2nd 3rd 4th

queue

queue discipline

service rate

simulation

single-server waiting line

steady-state

undefined service times

utilization factor

Queuing theory

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [**R**] [S] [T] [U] [V] [W] [Y]
[Z]

RAND

Random index

Random number table

Random numbers 2nd

Random variables

Random variations (in forecasts)

Regression methods 2nd

Regret 2nd 3rd

Regret table

Relative frequency

Reorder point 2nd 3rd

with variable demand

with variable demand and lead

time

with variable lead time

Responsibility assignment matrix
(RAM)

Revised probability

Rim requirements

Risk averters

Risk takers

Rounded down solution (in integer
programming)

Row operations

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [**S**] [T] [U] [V] [W] [Y]
[Z]

S.S. Central America

Saaty, Thomas

Saddle point

Safety stocks 2nd

Sample mean

Sample variance

Santos, Ltd.

Scheduling (LP) example

Scientific method

Scope statement

Scoring models

Seasonal adjustments

Seasonal factor

Seasonal patterns

Sensitivity analysis 2nd 3rd

Sensitivity analysis (in linear programming) 2nd

changes in constraint quantity values

changes in objective function coefficients

computer analysis

Excel 2nd 3rd

graphical analysis 2nd

Sequential decision trees

Server

Service level

Service rate

Service times

exponential distribution

Set covering problem (01 integer programming)

Shadow price 2nd 3rd 4th [See also [dual value](#)]

Shared slack

Short-range forecasts

Shortage costs 2nd 3rd 4th

Shortest route problem

Excel

linear programming formulation

permanent set

QM for Windows

solution steps

Simplex method 2nd

artificial variable 2nd

basic variables

degeneracy

entering nonbasic variable

infeasible problem

leaving basic variable

maximization problem

minimization problem

mixed constraint problem 2nd

multiple optimal solutions

negative quantity values

nonbasic variables

optimal solution

pivot column

pivot column tie

pivot number

pivot row

pivot row tie (degeneracy)

sensitivity analysis

summary steps

tableau

unbounded problem

Simplex tableau

Simulated time

Simulation 2nd

applications

continuous probability

distributions

Crystal Ball

environmental and resource

analysis

Excel 2nd 3rd 4th

finance

marketing

Monte Carlo process

of a queuing system 2nd

of an inventory system

production and manufacturing

pseudorandom numbers

public service operations

random numbers 2nd

statistical analysis

trials

verification

Simultaneous solution (of linear equations)

Single-server waiting line system

Slack

Slack variables

Slope of a line 2nd 3rd

Smoothing constant 2nd

Solver (Excel)

Soquimich (SA)

Spanning tree

Standard deviation 2nd

Standard model form (in linear programming)

Standard normal distribution

State (of a system)

Statement of work

States of nature

Statistical dependence

Statistical independence

Statistics

Steady-state

Stepping stone method

closed path

iteration

summary steps

Stigler, George 2nd

Strategies (in game theory)

Subjective probability 2nd

Suboptimal solution

Substitution method (in nonlinear programming)

Surplus variables

Synthesization

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Taco Bell 2nd

Texaco

Time management

"Time Out" (box) 2nd 3rd 4th 5th 6th
7th 8th 9th 10th 11th 12th 13th

Beer, Stafford

Blackett, P.M.S.

Bush, Vannevar

Charnes, Abraham

Conant, James B.

Cooper, William W.

Dantzig, George B.

Dijkstra, E.W.

Doig, A.C.

Erlang, Agner Krarup

Ford, L.R., Jr.

Fulkerson, D.R.

Gantt, Henry

Gomory, Ralph E.

Harris, Ford

Hitchcock, Frank L.

Ijiri, Yuji

Kelley, James E., Jr.

Koopmans, T.C.

Land, A.H.

Malcolm, D.G.

Markov, Andrey A.

Morgenstern, Oskar

pioneers in management science

Ulam, Stanislas

Von Neumann, John

Walker, Morgan R.

Time series methods 2nd

adjusted exponential smoothing

exponential smoothing

linear trend line

moving averages

seasonal adjustments

weighted moving averages

Total integer model 2nd

Total revenue

Transient state

Transition matrix

Transition probability

Transportation problem 2nd 3rd

balanced 2nd

degeneracy

entering nonbasic variable

Excel 2nd

Excel QM

minimum cell cost method

modified distribution method

(MODI)

multiple optimal solutions 2nd

northwest corner method

optimal solution

prohibited route 2nd

QM for Windows

stepping stone method

tableau

unbalanced transportation

problem 2nd

Vogel's approximation method

(VAM)

Transportation tableau

Transshipment model

TreePlan 2nd

Trend factor

Trends (in forecasting)

Trials 2nd

Two-person game

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

U.S. Coast Guard

Ulam, Stanislas

Unbalanced assignment problem

Unbalanced transportation problem

2nd

Unbounded problem (in linear programming)

Unconditional probability

Unconstrained optimization

Undefined service times

Uniform distribution 2nd

Universal product code (UPC)

UPS

Utiles

Utility

Utilization factor

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Valley Metal Container (VMC)

Value of the game

Variable

artificial

decision 2nd

dependent

deviational

independent

slack

surplus

Variable costs 2nd

Variance

Venn diagram

Verification (of simulation)

Vermont Gas Systems

Vogel's approximation method (VAM)

Von Neumann, John

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Waiting lines [See [queuing analysis](#)]

elements of

Walker, Morgan R.

Walt Disney Company

Weighted moving average

Weiss, Howard

Welch's

Work breakdown structure (WBS)

World War II

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Yellow Freight Systems

Index

[SYMBOL] [A] [B] [C] [D] [E] [F]
[G] [H] [I] [J] [K] [L] [M] [N] [O]
[P] [Q] [R] [S] [T] [U] [V] [W] [Y]
[Z]

Zero-sum game